A LETTER & SOME REMARKS ON THE MOTIVIC TOPOS

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Preface

All the relevant papers are available online, so that everyone can form its own view contributing to a better understanding of the research subject involving motivic toposes, $\mathbb{T}$-motives and Nori motives.

However, my email letter [*Motivic Topos*] of April 12, 2013 to L. Lafforgue is not public. Nevertheless, L. Lafforgue mentioned and commented it in his course (in the fall of 2015 at IHÉS) and he also partially reported its content in the introduction to the notes [*Catégories syntactiques pour les motifs de Nori*] which are available online. These notes are, to a large extent, a follow up of O. Caramello's writings and research programme, starting from the reconstruction of Nori motives she provided into the paper [*Syntactic categories for Nori motives*].

I therefore suppose that it is suitable to make available my letter in its integrality as a supplement to my paper [$\mathbb{T}$-*Motives*]. The reader shall find the letter here along with some remarks, comparing it to my paper, and a commented list of existing available material on the subject up to now.

This letter has been originated by some conversations with L. Lafforgue on the general theory of classifying toposes which has been pointed out to him by O. Caramello. The letter is my first draft of a *motivic topos* including some hints which are at the origin of an already ramified bunch of subjects. For example, I was guessing a possible reconstruction of Nori motives via topos theory. With respect to this guess, O. Caramello's reconstruction is a key result showing us that there is indeed a regular theory such that the (Barr) exact completion of its regular syntactic category is equivalent to the abelian category of Nori effective motives.

However, in the paper [*Definable categories & $\mathbb{T}$-motives*] a purely algebraic universal representation theorem is addressed, showing us that Nori's category can also be seen directly as a Serre quotient of Freyd's free abelian category on the preadditive category generated by a quiver. The link with the syntactic category is through the additive definable category generated by a model, as observed by M. Prest.

For the additive analogue of syntactic categories I suggest to read M. Prest's *Model theory in additive categories* in "Models, logics, and higher-dimensional categories" CRM Proc. Lecture Notes, Vol. 53, Amer. Math. Soc., Providence, RI, 2011.

As a reference for syntactic categories, classifying toposes and other key topics in topos theory I suggest and will make use of P. Johnstone's *Sketches of an Elephant: A Topos Theory Compendium* Vol. 1 & 2, Clarendon Press, Oxford Logic Guides Vol. 43 & 44, 2002.

In conclusion, I will follow here the point of view on motivic toposes which is explained in my paper [$\mathbb{T}$-*Motives*] and I do not pretend to be exhaustive. For the interested reader with a good logic background, I also suggest to read O. Caramello's [*Motivic Toposes*] which is providing a quite different and fascinating approach.

Date: July 1, 2016.

The letter

From: Luca Barbieri Viale
Subject: Motivic Topos
Date: 12 April 2013 at 16:37:49 GMT+2
To: Laurent Lafforgue

Dear Laurent,

as puzzled by your enthusiasm on "classifying topos" I think that it is feasible to work out a presentation of motives in the sense that I draft here below.

Let D be an oriented graph and let $T : D \to \mathrm{Ab}$ be a diagram of (finitely generated) abelian groups. Here T has to be thought as "homology theory" Note that I'm just adopting Nori's point of view in constructing "effective homological motives" where D = a suitable graph obtained from schemes (over a field k embedded in the complex numbers) and T = singular homology of pairs. Alternately, one can consider the category Sch_k (of pairs of) k-schemes and a functor $T : Sch_k \to \mathrm{Ab}$ satisfying a set of axioms (e.g. Bloch-Ogus axioms). In both cases I think we should be able to translate this T in a geometric theory of first order.

Assume that a classifying topos $E[T]$ for T-models exists such that $T-\mathrm{Mod}(E) = \mathrm{Hom}(E, E[T])$ for any topos E (over sets). That means we have a syntactic site on D (or the category associated to D) and an universal model providing the "motivic" site and the "motivic topos" for schemes. Now denote $A[T]$ the abelian category $\mathrm{Ab}(E[T])$ of internal abelian groups which is the abelian category of "mixed motives" for schemes. Any "model" i.e. morfism $f : E \to E[T]$ induces an exact functor $f^* : A[T] \to \mathrm{Ab}(E)$ which is the "realisation" functor (which is also faithful if f is a surjection)

Now I think that (by general non-sense) we also get in this way a result of Nori that there is a factorization $D \to A[T] \to \mathrm{Ab}$ of the given diagram T which is universal among all such factorisations through abelian categories (over Ab).

How it sounds!?

All the best, L.

Ps. I started to check the logic items with Silvio Ghilardi a colleague here in Milano which knows well model theory & also is not afraid of topos theory! I hope we can provide a more detailed version soon!

Some remarks

In the introduction of my paper [$\mathbb{T}$-*Motives*] I follow, expanding and refining it, exactly the picture drafted in this brief email: (co)homology theories as logical theories, their motivic topos and abelian categories of "mixed motives" inside the topos.

Obviously, we may object that this category $\mathrm{Ab}(E[T])$ is too big. A first refinement can be given by the fact that when considering internal abelian groups regarded as sheaves of abelian groups *additivity* is not granted: there is a similar situation for the Voevodsky setting of additive sheaves or presheaves with transfers! Therefore, additivity has to be required if we want to be coherent with Voevodsky setting.

Moreover, as in the Voevodsky setting, there are categories of geometric motives and other bigger categories of motivic complexes with infinite direct sums. In this setting, the abelian category of geometric or *constructible* effective motives, which I denote $\mathcal{A}[\mathbb{T}]$ in my paper [$\mathbb{T}$-*Motives*], is not the Grothendieck category $A[T]$ alluded in this email message. The passage from one to the other just involves the Ind-completion, on one side, and additivity, on the other side: *constructible* effective motives are just the "compact" or "finite presentation" objects of the category of effective motives, as usual. This is the framework explained in my paper [$\mathbb{T}$-*Motives*].

Notably, having a motivic topos is really enough in order to get effective "mixed motives" as we can just take abelian additive sheaves in it, as I recall below. Conversely, let's first see how a motivic topos can be provided by "mixed motives".

From mixed motives to motivic toposes. For any abelian category $\mathcal{A}$ a quasi-left-exact presheaf $F : \mathcal{A}^{op} \to \mathrm{Ab}$ of abelian groups can be defined by the following condition: given $p : A \twoheadrightarrow B$ epi the following sequence

$$0 \to F(B) \xrightarrow{p^*} F(A) \xrightarrow{d_0^*-d_1^*} F(A \times_B A)$$

is exact in Ab where

$$\begin{array}{ccc} A \times_B A & \xrightarrow{d_0} & A \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{p} & B \end{array}$$

Note that here $d_0^* - d_1^* \neq (d_0 - d_1)^*$ in general. As $\mathcal{A}$ is abelian is endowed with one natural topology J generated by epi = regular epi = descent = effective descent: here we declare that a cover is $p : A \twoheadrightarrow B$ (regular) epi. Note that this is a particular instance of the regular topology on a regular category. Then quasi-left-exact presheaves of abelian groups are just the sheaves of abelian groups for this topology. We then get that

$$\mathsf{QLex}(\mathcal{A}^{op}, \mathrm{Ab}) = \mathrm{Shv}(\mathcal{A}) \subset \mathrm{Pshv}(\mathcal{A}) := \mathrm{Fct}(\mathcal{A}^{op}, \mathrm{Ab})$$

Let $\mathsf{Add}(\mathcal{A}^{op}, \mathrm{Ab}) \subset \mathrm{Pshv}(\mathcal{A})$ be the *additive* presheaves. As $\mathcal{A}$ is abelian we have

$$\mathrm{Ind}(\mathcal{A}) = \mathsf{Lex}(\mathcal{A}^{op}, \mathrm{Ab}) \subset \mathsf{Add}(\mathcal{A}^{op}, \mathrm{Ab}) \subset \mathrm{Pshv}(\mathcal{A})$$

On the other hand

$$\mathrm{Ind}(\mathcal{A}) = \mathsf{Lex}(\mathcal{A}^{op}, \mathrm{Ab}) \subset \mathrm{Shv}(\mathcal{A}) = \mathsf{QLex}(\mathcal{A}^{op}, \mathrm{Ab})$$

and there are quasi-left-exact non left-exact presheaves. Just note that $F \in \mathsf{Lex}(\mathcal{A}^{op}, \mathrm{Ab})$ if and only if F is a quasi-left-exact additive presheaf of abelian groups.

Furthermore, the topos of <u>all</u> J-sheaves on $\mathcal{A}$ is the classifying topos $\mathcal{E}[\mathbb{T}^r]$ of a geometric theory $\mathbb{T}^r$ whose models are J-continuous left exact functors: as J is generated by singleton covering families the theory $\mathbb{T}^r$ is <u>regular</u> (see Johnstone's [D3.1 Remark 3.1.13] and [D3.3 Theorem 3.3.1]). This regular theory $\mathbb{T}^r$ just axiomatises "realisation" functors, *i.e.* exact functors on $\mathcal{A}$. We thus get a classifying topos $\mathcal{E}[\mathbb{T}^r]$ associated to the regular theory $\mathbb{T}^r$ such that

$$\mathcal{A} \hookrightarrow \mathrm{Ind}(\mathcal{A}) = \mathrm{Shv}(\mathcal{A}) \cap \mathsf{Add}(\mathcal{A}^{op}, \mathrm{Ab}) \hookrightarrow \mathrm{Ab}(\mathcal{E}[\mathbb{T}^r]) = \mathrm{Shv}(\mathcal{A}) \hookrightarrow \mathcal{E}[\mathbb{T}^r]$$

There are natural equivalences

$$\mathbb{T}^r\text{-}\mathsf{Mod}(\mathcal{F}) \cong \mathsf{Hom}(\mathcal{F}, \mathcal{E}[\mathbb{T}^r])$$

for any topos $\mathcal{F}$ and

$$\mathbb{T}^r\text{-}\mathsf{Mod}(\mathcal{B}) \cong \mathsf{Ex}(\mathcal{A}, \mathcal{B})$$

for any abelian category $\mathcal{B}$.

Finally, let $\mathcal{A}$ be the category of constructible effective "mixed motives" attached to a suitable category $\mathcal{C}$ of "spaces" (*e.g.* $\mathcal{C} = \mathrm{Sch}_k$ the category of k-algebraic schemes). A relation with the category of "spaces" $\mathcal{C}$ shall be given by asking $\mathcal{A}$ to be a motivic site, *i.e.* a site of definition for a motivic topos. I simply mean that there should be a (co)homology theory $\mathbb{T}$ on $\mathcal{C}$ such that $\mathcal{E}[\mathbb{T}] \cong \mathcal{E}[\mathbb{T}^r]$. By the way we may ask more: for example, we may wish to have a motivic functor $M : \mathcal{C} \to D(\mathcal{A})$ which is determined by the theory $\mathbb{T}$ and providing the universal model.

From motivic toposes to mixed motives. Conversely, it is well known (see Johnstone's [D1.4 Proposition 1.4.12 (i)] and [D3.1 Remark 3.1.5]) that for any regular theory $\mathbb{T}$ we can use the (Barr) exact completion $\mathcal{A}[\mathbb{T}]$ of the regular syntactic category $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$ as a site of definition of its classifying topos $\mathcal{E}[\mathbb{T}]$. In fact, the toposes of <u>all</u> J-sheaves for J the previously mentioned regular topology are equivalent

$$\mathcal{E}[\mathbb{T}] := \underline{\mathrm{Shv}}(\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}, J) \cong \underline{\mathrm{Shv}}(\mathcal{A}[\mathbb{T}], J)$$

This $\mathcal{A}[\mathbb{T}]$ can be regarded as a category of <u>constructible</u> effective $\mathbb{T}$-motives inside a motivic topos for a (co)homology theory $\mathbb{T}$ even in the non-additive context, as it is clearly stated in [*Nori motives and the motivic topos*] where I firstly named $\mathbb{T}$-motives: see also [*Logical & categorical aspects of (co)homology theories*] and [$\mathbb{T}$-*Motives*].

Now this can be combined with the fact that the (Barr) exact completion of a regular <u>additive</u> category preserves additivity and then, using the theorem of Tierney (Abelian = Exact + Additive) is abelian! In particular, each syntactic regular additive category $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$ can be completed to $\mathcal{A}[\mathbb{T}]$ which is an abelian site of definition.

The Grothendieck category of abelian group objects $\mathrm{Ab}(\mathcal{E}[\mathbb{T}])$ is the same of the above category of sheaves of abelian groups $\mathrm{Shv}(\mathcal{A}[\mathbb{T}])$ for the named topology. Thus

$$\mathrm{Ind}(\mathcal{A}[\mathbb{T}]) = \mathrm{Ab}(\mathcal{E}[\mathbb{T}]) \cap \mathsf{Add}(\mathcal{A}[\mathbb{T}]^{op}, \mathrm{Ab})$$

as above. The deduction of effective $\mathbb{T}$-motives and constructible effective $\mathbb{T}$-motives from a motivic topos is formal.

Nori motives. As explained in [*Syntactic categories for Nori motives*] and [$\mathbb{T}$-*Motives*] the category $\mathsf{NM}^{\mathrm{eff}}$ of effective (co)homological Nori motives (as well as its Ind-completion) can be formally deduced from the regular theory $\mathbb{T}$ of "singular (co)homology". The abelian category $\mathsf{NM}^{\mathrm{eff}}$ is a motivic site, in the sense indicated above, along with a motivic functor $M : \mathrm{Sch}_k \to D(\mathsf{NM}^{\mathrm{eff}})$ which is indeed determined by the theory $\mathbb{T}$ as proven in [$\mathbb{T}$-*Motives*].

However, as far as we keep our interest confined to the additive context, as remarked jointly with M. Prest in [*Definable categories & $\mathbb{T}$-motives*], Nori's abelian category associated to a representation of a quiver is an instance of the universal abelian category which may be associated to any additive functor from the preadditive category generated by the quiver to abelian groups.

References

[*Motivic Topos*] Email letter to L. Lafforgue, April 12, 2013.

[*Cohomology theories in algebraic geometry & the motivic topos*] Notes of my first talk on the subject, March 13, 2014. In this talk I explained and expanded a little bit my email letter to L. Lafforgue by including Nori 1-motives. Actually, this was a lecture in the framework of a seminar "The problem of classifying cohomology theories: a topos-theoretic approach" which I organised jointly with L. Lafforgue and O. Caramello at the University of Milan. For the sake of a complete information, note that on September 8, 2014 I gave a second similar talk titled "The motivic topos" at the Conference "Around forms, cycles and motives" on the occasion of Albrecht Pfister's 80^{th} birthday (Mainz, September 8-12, 2014).

[*Syntactic categories for Nori motives*] Paper with O.Caramello & L. Lafforgue: arXiv:1506.06113v1 [math.AG], June 19, 2015. The construction and the results of this paper are due to O. Caramello and represents a first positive outcome of the previously mentioned general framework.

[*Motivic Toposes*] A research programme of O. Caramello: arXiv:1507.06271v1 [math.AG], July 22, 2015. This contains O. Caramello's original research programme on the subject proposing a new framework based on atomic two-valued toposes and homogeneous models.

[*Nori motives and the motivic topos*] Slides of my lecture at the Conference "K-theory, Cyclic Homology and Motives" a conference in celebration of C. A. Weibel's 65^{th} year, August 21, 2015. This extended report is in three parts: Nori Motives, Voevodsky Motives and Motivic Topos. It contains also reference to the previously mentioned reconstruction of Nori motives but settled in the framework of my email letter.

[*Catégories syntactiques pour les motifs de Nori*] Video on IHÉS YouTube channel and notes of L. Lafforgue's IHÉS lectures on September 22, October 6 & 13, 2015. In this course and in the notes my email letter is cited and commented. The course is devoted to explain O. Carmello's results and research programme on the subject.

[*Logical & categorical aspects of (co)homology theories*] Notes (in Italian) of my seminars in Milano, Fall Semester 2015-16. In these seminars I explained tasks and results of a preliminary version of my paper on $\mathbb{T}$-motives and in the notes I also consider some further hints in the non-additive context.

[$\mathbb{T}$-*Motives*] Video available on IHÉS YouTube channel of my talk at the conference "Topos à l'IHÉS" (November 23–27, 2015). The talk is also a follow up of my email letter starting from the general framework of a motivic topos, reporting the key result due to O. Caramello on Nori motives, introducing $\mathbb{T}$-motives and $\mathbb{T}$-motivic complexes.

[$\mathbb{T}$-*Motives*] My paper on arXiv:1602.05053 [math.AG], February 16, 2016.

[*Definable categories & $\mathbb{T}$-motives*] Paper with M. Prest: arXiv:1604.00153 [math.AG] April 1, 2016. Making use of Freyd's free abelian category on a preadditive category we show that $\mathbb{T}$-motives as well as Nori motives are given by a certain category of functors on definable categories.

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