On algebraic 1-motives related to Hodge cycles

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Abstract. The goal of this paper is to introduce *Hodge 1-motives* of algebraic varieties and to state a corresponding cohomological Grothendieck-Hodge conjecture, generalizing the classical Hodge conjecture to arbitrarily singular proper schemes.

We also construct generalized cycle class maps from the (Quillen) $\mathcal{K}$-cohomology groups $H^{p+i}(\mathcal{K}_p)$ to the sub-quotiens $W_{2p}H^{2p+i}/W_{2p-2}$ given by the weight filtration. However, in general, the image of this cycle map (as well as the image of the canonical map from motivic cohomology) is strictly smaller than the rational part of the Hodge filtration F^p on H^{2p+i}.

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Contents

0. Introduction

Let X be an algebraic $\mathbb{C}$-scheme. The singular cohomology groups $H^*(X,\mathbb{Z}(\cdot))$ carry a mixed Hodge structure, see [11, III]. Deligne theory of 1-motives (see [11, III]) is an algebraic framework in order to deal with *some* mixed Hodge structures extracted from $H^*(X,\mathbb{Z}(\cdot))$, *i.e.*, those having non-zero Hodge numbers in the set $\{(0,0),(0,-1),(-1,0),(-1,-1)\}$. Therefore, these cohomological invariants of algebraic varieties would be algebraically defined as 1-motives over arbitrary base fields or schemes. Note that a general theory of mixed motives can be regarded as an algebraic framework in order to deal with *all* mixed Hodge structures $H^*(X,\mathbb{Z}(\cdot))$.

A 1-motive M over a scheme S is given by an S-homomorphism of group schemes

$$M = [L \xrightarrow{u} G]$$

where G is an extension of an abelian scheme A by a torus T over S, and the group scheme L is, locally for the étale topology on S, isomorphic to a given finitely-generated free abelian group. There are Hodge, De Rham and ℓ-adic realizations (see [11, III] and [12]).

If X is a smooth proper $\mathbb{C}$-scheme then $H^i(X,\mathbb{Z}(j))$ is pure of weight $i-2j$. If $i=2p$ is even a natural 1-motive would be given by the lattice of Hodge cycles in $H^{2p}(X,\mathbb{Z}(p))$, *i.e.*, of those integral cohomology classes (modulo torsion) which are of type $(0,0)$. Classical Hodge conjecture claims that (over $\mathbb{Q}$) such a 1-motive would be obtained from classes of algebraic cycles on X only. For $i=2p+1$ odd the 1-motive corresponding to $H^{2p+1}(X,\mathbb{Z}(p+1))$ is given by the abelian variety associated to the largest sub-Hodge structure whose types are $(-1,0)$ or $(0,-1)$. Grothendieck-Hodge conjecture characterize (over $\mathbb{Q}$) this sub-Hodge structure as the coniveau $\geq p$ sub-space, *i.e.*, the abelian variety as the algebraic part of the intermediate jacobian.

Grothendieck-Hodge conjectures are concerned with the quest of an algebraic definition for the named 1-motives. In fact, the usual Hodge conjecture can be reformulated by saying that the Hodge realization of the algebraically defined $\mathbb{Q}$-vector space of codimension p algebraic cycles modulo numerical (or homological) equivalence is the 1-motivic part of $H^{2p}(X, \mathbb{Q}(p))$. Moreover, the 1-motivic part of $H^{2p+1}(X, \mathbb{Q}(p+1))$ would be the Hodge realization of the isogeny class of the universal regular quotient.

The main task of this paper is to define *Hodge 1-motives* of singular varieties and to state a corresponding cohomological Grothendieck-Hodge conjecture, by dealing with their Hodge realizations.

0.1. A short survey of the subject.

The classical Hodge conjecture along with a tantalizing overview can be found in [13]. Recall that Grothendieck corrected the general Hodge conjecture in [19]. The book of Lewis [27] is a very good compendium of methods and results.

Recall that Jannsen [23] formulated an homological version of the Hodge conjecture for singular varieties. Moreover, Bloch in a letter to Jannsen (see the Appendix A in [23] *cf.* Section 5), gave a counterexample to a naive cohomological Hodge conjecture for curves on a singular 3-fold. However, in the same letter, Bloch was guessing that the Hodge conjecture for divisors, *i.e.*, $F^1 \cap H^2(X, \mathbb{Z})$ is generated by c_1 of line bundles on X, holds true in the singular setting "because one has the exponential". Anyways, jointly with V. Srinivas, we gave a counterexample to this claim and questioned a reformulation of the Hodge conjecture for divisors in [3] by restricting to Zariski locally trivial cohomology classes, *i.e.*, let $L^pH^*(X, \mathbb{Z})$ be the filtration induced by the Leray spectral sequence along the canonical continuous map $X_{\mathrm{an}} \to X_{\mathrm{Zar}}$, is $F^1 \cap L^1H^2(X, \mathbb{Z})$ given by c_1 of line bundles on X ? Still, this reformulation doesn't hold in general, *e.g.*, see [7] where it is also proved for X normal.

From the work of Carlson (see [9] and [10]) and the theory of Albanese and Picard 1-motives [4] it now appears that the theory of 1-motives is a natural setting for a formulation of a cohomological version of the Hodge conjectures for singular varieties. For example, $F^1 \cap H^2(X, \mathbb{Z})$ is simply given by c_1 of simplicial line bundles on a smooth proper hypercovering $\pi : X_\bullet \to X$ via universal cohomological descent $\pi^* : H^2(X, \mathbb{Z}) \cong H^2(X_\bullet, \mathbb{Z})$. This Néron-Severi group $\mathrm{NS}(X_\bullet) \cong F^1 \cap H^2(X, \mathbb{Z})$ is actually independent of the choice of the smooth simplicial scheme. Furthermore, $\mathrm{NS}(X_\bullet)$ admits an algebraic definition as the quotient of the simplicial Picard group scheme $\mathbf{Pic}_{X_\bullet}$ by its connected component of the identity (*cf.* [4]). However, the 1-motivic part of $H^2(X, \mathbb{Z})$ is still larger than $\mathrm{NS}(X_\bullet)$. Therefore the largest algebraic part of $H^2(X, \mathbb{Z})$ will be detected from a honest 1-motive only (see [2]).

Note that the rank of the usual $\mathrm{NS}(X)$ (= the image of $\mathrm{Pic}(X)$ in $H^2(X, \mathbb{Z})$) is actually smaller than $\mathrm{NS}(X_\bullet)$, in general. Moreover $F^1 \cap W_0H^2(X, \mathbb{Q}) = 0$ thus $\mathrm{NS}(X_\bullet)_{\mathbb{Q}}$ is naturally a subspace of $H^2(X, \mathbb{Q})/W_0$.

Considering the Leray filtration $L^pH^{2p}(X, \mathbb{Q})$ a natural question formulated in [7] is if $F^p \cap L^pH^{2p}(X, \mathbb{Q})$ will be given by higher Chern classes. However, this would not be true without some extra hypothesis on X and does not tell enough about the algebraic part of all $H^{2p}(X, \mathbb{Q})$.

0.2. An outline of the conjectural picture.

Let X be a proper integral $\mathbb{C}$-scheme. Let $H \stackrel{\text{def}}{=} H^{2p+i}(X)$ be our mixed Hodge structure on $H^{2p+i}(X,\mathbb{Z})/(\text{torsion})$ for a fixed pair of integers $p \geq 0$ and $i \in \mathbb{Z}$. First remark that we always have an extension

$$0 \to \mathrm{gr}\,^W_{2p-1} H \to W_{2p}H/W_{2p-2}H \to \mathrm{gr}\,^W_{2p} H \to 0.$$

An extension always defines an extension class map

$$e^p : H^{p,p}_{\mathbb{Z}} \to J^p(H)$$

which is not, in general, a 1-motive. In fact, $J^p(H)$ is a complex torus which is not an abelian variety, in general. Recall that Carlson [9] studied abstract extensions of Hodge structures showing their geometric content for low-dimensional varieties. For higher dimensional schemes consider the largest abelian subvariety $A^p(H)$ of the torus $J^p(H)$. Denote $H^p(H)$ the group of Hodge cycles, that is the preimage in $H^{p,p}_{\mathbb{Z}}$ of $A^p(H)$ under the extension class map. Note that an example due to Srinivas shows that the group $H^p(H)$ of Hodge cycles in this sense can be strictly smaller than $H^{p,p}_{\mathbb{Z}}$ (see Section 5.2).

Define the *Hodge 1-motive* of the mixed Hodge structure H the so obtained 1-motive

$$e^p : H^p(H) \to A^p(H).$$

Conversely, Deligne's theory of 1-motives [11] grant us of a mixed Hodge structure H^h corresponding to this 1-motive (see Section 2.2 for details). According to Deligne's philosophy of 1-motives there should be an algebraically defined 1-motive whose Hodge realization is H^h. The algebraic definition (see Section 2.1) is predictable *via* Grothendieck-Hodge conjectures and Bloch-Beilinson motivic world as follows.

Assume X smooth. Consider the filtration F_a^* on the Chow group $CH^p(X)$ given by $F_a^0 = CH^p(X)$, $F_a^1 = CH^p(X)_{\mathrm{alg}}$ the sub-group of cycles algebraically equivalent to zero and $F_a^2 =$ the kernel of the Abel-Jacobi map. Thus the graded pieces are $\mathrm{gr}\,^0_{F_a} = NS^p(X)$, the Néron-Severi group, and $\mathrm{gr}\,^1_{F_a} = J_a^p(X) =$ the group of $\mathbb{C}$-points of an abelian subvariety of the intermediate jacobian. We then get an extension

$$0 \to J_a^p(X) \to CH^p(X)/F_a^2 \to NS^p(X) \to 0.$$

Note that Grothendieck-Hodge conjecture claims that $J_a^p(X) = A^p(H^{2p-1}(X))$ up to isogeny, *i.e.*, $J_a^p(X)$ is the largest abelian subvariety of the intermediate jacobian.

If X is not smooth then let $X_\bullet$ be a smooth proper simplicial scheme along with $\pi : X_\bullet \to X$, a universal cohomological descent morphism (*cf.* [18]). In zero characteristic, such $X_\bullet$ was firstly provided by the construction of hypercoverings in [11], then by that of cubical hyperresolutions in [16] where the dimensions of the components are bounded or by the method of hyperenvelopes given in [15]. In [14] such a simplicial scheme is provided in positive characteristics.

The above extension, given by the filtration F_a^* on the Chow groups of each component of the so obtained simplicial scheme $X_\bullet$, yields a short exact sequence of complexes. Let $(NS^p)^\bullet$ and $(J_a^p)^\bullet$ denote such complexes. By taking homology groups we then get boundary maps

$$\lambda_a^i : H^i((NS^p)^\bullet) \to H^{i+1}((J_a^p)^\bullet).$$

We conjecture that the boundary map λ_a^i behave well with respect to the extension class map e^p yielding a motivic cycle class map, *i.e.*, the following diagram

$$\begin{array}{ccc} H^i((NS^p)^\bullet) & \stackrel{\lambda_a^i}{\to} & H^{i+1}((J_a^p)^\bullet) \\ \downarrow & & \downarrow \\ H^{2p+i}(X)^{p,p} & \stackrel{e^p}{\to} & J^p(H^{2p+i}(X)) \end{array}$$

commutes. Note that all maps in the square are canonically defined. We guess that the image 1-motive (up to isogeny!) is the above Hodge 1-motive of the mixed Hodge structure (see Conjecture 2.3.4 for a full statement). In fact, we may expect J_a^p would be obtained as the universal regular quotient of $CH^p(X)_{\mathrm{alg}}$ and that the filtration F_a^* would be induced by the motivic filtration conjectured by Bloch, Murre and Beilinson. Accordingly we can sketch an algebraic definition of such Hodge 1-motives (see Section 2.1).

If X is singular one is then puzzled by the role of

$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(-p), H^{2p+i}(X))$$

where the Hom is taken in the abelian category of mixed Hodge structures. That is the integral part of F^p (the Hodge filtration on $H^{2p+i}(X,\mathbb{C})$) which is contained in the kernel of the extension class map e^p above. Note that $F^p \cap W_{2p-2}H^{2p+i}(X,\mathbb{Q}) = 0$ here. In the smooth case, such a target is usually reached by algebraic cycles. In order to obtain cycle class maps we may use local higher Chern classes and edge maps in coniveau spectral sequences (see [8] and [1]).

In the singular case, we show that such edge maps can be recovered by weight arguments. In order to do this we define *Zariski sheaves* of mixed Hodge structures, obtaining *infinite dimensional* mixed Hodge structures on their cohomology (see Section 3). The main example is given by the Zariski sheaf $\mathcal{H}_X^*$ associated to the presheaf $U \subset X \mapsto H^*(U)$ of mixed Hodge structures. On the smooth simplicial scheme $X_\bullet$ we also have a simplicial sheaf $\mathcal{H}_{X_\bullet}^*$ of mixed Hodge structures. Since $\pi : X_\bullet \to X$ yields $H^*(X) \cong \mathbb{H}^*(X_\bullet)$ we then obtain a local-to-global spectral sequence

$$L_2^{p,q} = \mathbb{H}^p(X_\bullet, \mathcal{H}_{X_\bullet}^q) \Rightarrow H^{p+q}(X)$$

in the category of infinite dimensional mixed Hodge structures. The sheaf $\mathcal{H}_{X_\bullet}^q$ has weights $\leq 2q$ and the same holds for its cohomology. There is an edge map (see Section 4)

$$s\ell^{p+i} : \mathbb{H}^{p+i}(X_\bullet, \mathcal{H}_{X_\bullet}^p)/W_{2p-2} \to W_{2p}H^{2p+i}(X)/W_{2p-2}.$$

We expect that the image of $s\ell^{p+i}$ is the mixed Hodge structure $H^{2p+i}(X)^h$ corresponding to the Hodge 1-motive.

Moreover, consider $\mathcal{K}$-cohomology groups $\mathbb{H}^*(X_\bullet, \mathcal{K}_p)$ where $\mathcal{K}_p$ are the simplicial sheaves associated to Quillen's higher K-theory. Recall that there are local higher Chern classes

$$c_p : \mathcal{K}_p \to \mathcal{H}^p_{X_\bullet}(p)$$

for each $p \geq 0$. We thus obtain a generalized cycle class map

$$c\ell^{p+i} : \mathbb{H}^{p+i}(X_\bullet, \mathcal{K}_p)_\mathbb{Q} \to W_{2p}H^{2p+i}(X, \mathbb{Q})/W_{2p-2}.$$

However the image of $c\ell^{p+i}$ is not $F^p \cap H^{2p+i}(X, \mathbb{Q})$, *i.e.*, the rational part of the Hodge filtration can be larger (see Section 5.1 where Bloch's counterexample is explained). The same applies to the canonical map

$$H^{2p+i}_m(X, \mathbb{Q}(p)) \to H^{2p+i}(X, \mathbb{Q}(p))$$

from motivic cohomology.

0.3. Towards Hodge mixed motives.

Any reasonable theory of mixed motives would include the theory of 1-motives, *i.e.*, it would be a fully faithful functor from the $\mathbb{Q}$-linear category of 1-motives to that of mixed motives. This is the case of the triangulated category of geometrical motives introduced by Voevodsky (see [39, 3.4], *cf.* [26] and [29]). Hanamura's construction (see [22] and [21]) doesn't apparently provide such a property as yet.

As remarked by Grothendieck [19, §2] and Deligne [13, §5] the Hodge conjecture yields nice properties of the Hodge realization of pure motives, *i.e.*, the usual Hodge conjecture means that the Hodge realization functor is fully faithful. It would be interesting to investigate such a property in the mixed case, *e.g.*, if this formulation of the Hodge conjecture provide such a property of mixed motives.

We remark that M. Saito recently observed (see [36, 2.5 (ii)] and [37]) that the canonical functor from arithmetic Hodge structures to mixed Hodge structures is not full. Even if the Hodge realization factors through arithmetic Hodge structures, this non-fullness doesn't imply the non-fullness of the Hodge realization of mixed motives (as noticed by M. Saito as well).

However, the first natural attempt to go further with Hodge mixed motives is to provide an intrinsic definition of such objects internally. In fact, since 1-motives provide mixed motives we may claim that such Hodge mixed motives exist and would be naturally defined over any field or base scheme.

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1. Filtrations on Chow groups

Following the general framework of mixed motives (*e.g.*, see [12], [26] and [24] for a full overview) we may expect the following picture for non-singular algebraic varieties over a field k (algebraically closed of characteristic zero for simplicity).

Let X be a smooth proper k-scheme. Bloch, Beilinson and Murre (*cf.* [24], [25] and [34]) conjectured the existence of a finite filtration F_m^* on Chow groups $CH^p(X) \stackrel{\text{def}}{=} \mathcal{Z}^p(X)/\equiv_{\text{rat}}$ of codimension p cycles modulo rational equivalence such that

- $F_m^0 CH^p(X) = CH^p(X)$,
- $F_m^1 CH^p(X)$ is given by $CH^p(X)_{\text{hom}}$, *i.e.*, by the sub-group of those codimension p cycles which are homologically equivalent to zero for some Weil cohomology theory,
- $F_m^* CH^p(X)$ should be functorial and compatible with the intersection pairing, and
- this filtration should be motivic, *e.g.*, $\mathrm{gr}^i_{F_m} CH^p(X)_{\mathbb{Q}}$ depends only on the Grothendieck motive $h^{2p-i}(X)$.

1.1. Regular homomorphisms.

Consider the sub-group $CH^p(X)_{\text{alg}}$ of those cycles in $CH^p(X)$ which are algebraically equivalent to zero, *i.e.*, $CH^p(X)_{\text{alg}} \stackrel{\text{def}}{=} \ker(CH^p(X) \to NS^p(X))$. Denote $CH^p(X)_{\text{ab}}$ the sub-group of $CH^p(X)_{\text{alg}}$ of those cycles which are abelian equivalent to zero, *i.e.*, $CH^p(X)_{\text{ab}}$ is the intersection of all kernels of regular homomorphisms from $CH^p(X)_{\text{alg}}$ to abelian varieties.

Assume the existence of a universal regular homomorphism $\rho^p : CH^p(X)_{\text{alg}} \to A^p_{X/k}(k)$ to (the group of k-points) of an abelian variety $A^p_{X/k}$ defined over the base field k (*cf.* [30]). This is actually proved for $p = 1, 2, \dim(X)$ (see [32]). Note that $CH^p(X)_{\rm ab}$ is then a divisible group (see [6]).

Thus $CH^p(X)_{\rm alg} \subseteq CH^p(X)_{\rm hom}$ and there would be an induced functorial filtration F_a^* on $CH^p(X)$ such that

- $F_a^0 CH^p(X) = CH^p(X)$,
- $F_a^1 CH^p(X) = CH^p(X)_{\rm alg}$ is the sub-group of cycles algebraically equivalent to zero, and
- $F_a^2 CH^p(X) = CH^p(X)_{\rm ab}$, *i.e.*, is the kernel of the universal regular homomorphism ρ^p defined above.

Note that the existence of the abelian variety $A^p_{X/k}$ is not explicitly mentioned in the context of mixed motives but is a rather natural property after the case $k = \mathbb{C}$.

For X smooth and proper over $\mathbb{C}$ one obtains that the motivic filtration is such that *(i)* $F_m^1 CH^p(X) = CH^p(X)_{\rm hom}$ is the sub-group of cycles whose cycle class in $H^{2p}(X, \mathbb{Z}(p))$ is zero, and *(ii)* $F_m^2 CH^p(X)$ is contained in the kernel of the Abel-Jacobi map $CH^p(X)_{\rm hom} \to J^p(X)$ and $F_m^2 CH^p(X) \cap CH^p(X)_{\rm alg}$ is the kernel of the Abel-Jacobi map $CH^p(X)_{\rm alg} \to J^p(X)$. In this case, $CH^p(X)_{\rm ab}$ will be the kernel of the Abel-Jacobi map $CH^p(X)_{\rm alg} \to J^p(X)$, *i.e.*, *(iii)* $A^p_{X/\mathbb{C}}$ is the algebraic part of the intermediate jacobian. Is well known that the image $J^p_a(X)$ of $CH^p(X)_{\rm alg}$ into $J^p(X)$ yields a sub-torus of $J^p(X)$ which is an abelian variety: moreover, is known to be universal for $p = 1, 2, \dim(X)$ (see [32] and [33]).

In the following, for the sake of simplicity, the reader can indeed assume that $k = \mathbb{C}$ and *(i)–(iii)* are satisfied by the first two steps of the filtration F_m^i. In fact, for $k = \mathbb{C}$, S. Saito has obtained (up to torsion!) such a result (see [38, Prop. 2.1], *cf.* [28] and [25]). Moreover, in the following, the reader could also avoid reference to the motivic filtration by dealing with the first two steps of the "algebraic" filtration F_a^i defined above.

1.2. Extensions.

Let X be smooth and proper over k. For our purposes just consider the following extension

$$0 \to \mathrm{gr}\,^1_{F_m} CH^p(X) \to CH^p(X)/F_m^2 \to \mathrm{gr}\,^0_{F_m} CH^p(X) \to 0 \tag{1}$$

Note that $A^p_{X/k}(k)$ is contained in $\mathrm{gr}\,^1_{F_m} CH^p(X)$ (since $F_a^2 = F_m^2 \cap CH^p(X)_{\rm alg}$) and $\mathrm{gr}\,^0_{F_m} CH^p(X)$ has finite rank.

For $p = 1$ this extension is the usual extension associated to the connected component of the identity of the Picard functor, *i.e.*, $A^1_{X/k} = \mathrm{Pic}\,^0_{X/k}$ and $\mathrm{gr}\,^0_{F_m}$ is the Néron-Severi of X. If $p = \dim X$ then F_m^1 will be the kernel of the degree map and F_m^2 is the Albanese kernel, *i.e.*, $A^{\dim X}_{X/k}$ is the Albanese variety and $\mathrm{gr}\,^0_{F_m} = \mathbb{Z}^{\oplus c}$ where c is the number of components of X.

However, if $1 < p < \dim X$ then $CH^p(X)_{\rm alg} \neq CH^p(X)_{\rm hom}$ in general. Let $Grif^p(X)$ denote the quotient group, *i.e.*, the Griffiths group of X. Since

$\mathrm{gr}^1_{F_a} CH^p(X) = A^p_{X/k}(k)$ and $\mathrm{gr}^0_{F_a} CH^p(X) = NS^p(X)$ we then have a diagram with exact rows and columns

$$\begin{array}{ccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \to & F^2_m/F^2_a & \to & Grif^p(X) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & A^p_{X/k}(k) & \to & CH^p(X)/F^2_a & \to & NS^p(X) & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & \mathrm{gr}^1_{F_m} CH^p(X) & \to & CH^p(X)/F^2_m & \to & \mathrm{gr}^0_{F_m} CH^p(X) & \to 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{2}$$

Note that these extensions still fail to be of the same kind of the Pic extension. However, considering the extension

$$0 \to CH^p(X)_{\mathrm{alg}} \to CH^p(X) \to NS^p(X) \to 0$$

we may hope for a natural regular homomorphism to k-points of an abstract extension in the category of group schemes (locally of finite type over k)

$$0 \to A \to G \to N \to 0$$

where G is a commutative group scheme which is an extension of a discrete group of finite rank N, associated to the abelian group of codimension p cycles modulo numerical equivalence, by an abelian variety A, isogenous to the universal regular quotient (*cf.* [17]). If $k = \mathbb{C}$ it is easy to see that such extension $G(\mathbb{C})$ exists trascendentally.

2. Hodge 1-motives

Let k be a field, for simplicity, algebraically closed of characteristic zero. Consider the $\mathbb{Q}$-linear abelian category $1-\mathrm{Mot}_k$ of 1-motives over k with rational coefficients (see [11] and [4]). Denote $M_{\mathbb{Q}}$ the isogeny class of a 1-motive $M = [L \to G]$. The category $1-\mathrm{Mot}_k$ contains (as fully faithful abelian sub-categories) the tensor category of finite dimensional $\mathbb{Q}$-vector spaces as well as the semi-simple abelian category of isogeny classes of abelian varieties. The Hodge realization (see [11] and [4]) is a fully faithful functor

$$T_{\mathrm{Hodge}} : 1-\mathrm{Mot}_{\mathbb{C}} \hookrightarrow \mathrm{MHS} \qquad M_{\mathbb{Q}} \mapsto T_{\mathrm{Hodge}}(M_{\mathbb{Q}})$$

defining an equivalence of categories between $1-\mathrm{Mot}_{\mathbb{C}}$ and the abelian sub-category of mixed $\mathbb{Q}$-Hodge structures of type $\{(0,0),(0,-1),(-1,0),(-1,-1)\}$ such that gr^W_{-1} is polarizable. Under this equivalence a $\mathbb{Q}$-vector space corresponds to a $\mathbb{Q}$-Hodge structure purely of type $(0,0)$ and an isogeny class of an abelian variety corresponds to a polarizable $\mathbb{Q}$-Hodge structure purely of type $\{(0,-1),(-1,0)\}$.

2.1. Algebraic construction.

Let X be a proper scheme over k. We perform such a construction for simplicial schemes $X_\bullet$ coming from universal cohomological descent morphisms $\pi : X_\bullet \to X$ (*cf.* [11], [16], [15], [22] and [14]).

Let $X_\bullet$ be such a proper smooth simplicial scheme over the base field k. By functoriality, the filtration $F_m^j CH^p$ on each component X_i of $X_\bullet$ yields a complex

$$(F_m^j CH^p)^\bullet : \cdots \to F_m^j CH^p(X_{i-1}) \stackrel{\delta_{i-1}^*}{\to} F_m^j CH^p(X_i) \stackrel{\delta_i^*}{\to} F_m^j CH^p(X_{i+1}) \to \cdots$$

where δ_i^* is the alternating sum of the pullback along the face maps $\partial_i^k : X_{i+1} \to X_i$ for $0 \leq k \leq i+1$.

The complex of Chow groups $(CH^p)^\bullet$, induced from the simplicial structure as above, is filtered by sub-complexes:

$$0 \subseteq (F_m^p CH^p)^\bullet \subseteq \cdots \subseteq (F_m^1 CH^p)^\bullet \subseteq (F_m^0 CH^p)^\bullet = (CH^p)^\bullet.$$

Define $F_m^*(CH^p)^\bullet \stackrel{\text{def}}{=} (F_m^* CH^p)^\bullet$.

The extension (1) given by the filtration $F_m^* CH^p(X_i)$ on each component X_i of the simplicial scheme $X_\bullet$, for a fixed $p \geq 0$, yields the following short exact sequence of complexes

$$0 \to \operatorname{gr}^1_{F_m}(CH^p)^\bullet \to (CH^p)^\bullet / F_m^2 \to \operatorname{gr}^0_{F_m}(CH^p)^\bullet \to 0 \tag{3}$$

Note that $\operatorname{gr}^1_{F_m}(CH^p)^i$ contains the group of k-points of the abelian variety $A^p_{X_i/k}$ and, moreover $\operatorname{gr}^0_{F_m}(CH^p)^i_{\mathbb{Q}}$ is the finite dimensional vector space of codimension p cycles on X_i modulo homological (or numerical) equivalence.

From (3) we then get a long exact sequence of homology groups and, in particular, we obtain boundary maps

$$\lambda_m^i : H^i(\operatorname{gr}^0_{F_m}(CH^p)^\bullet) \to H^{i+1}(\operatorname{gr}^1_{F_m}(CH^p)^\bullet).$$

Denote $A^p_{X_\bullet/k}$ the complex of abelian varieties $A^p_{X_i/k}$. Since $A^p_{X_\bullet/k}(k)$ is a subcomplex of $\operatorname{gr}^1_{F_m}(CH^p)^\bullet$ we then get induced (functorial) maps on homology groups

$$\theta^i : H^{i+1}(A^p_{X_\bullet/k}(k)) \to H^{i+1}(\operatorname{gr}^1_{F_m}(CH^p)^\bullet).$$

Note that (3) is involved in a functorial diagram (2). The corresponding complex of Néron-Severi groups $(NS^p)^\bullet$ yield boundary maps

$$\lambda_a^i : H^i((NS^p)^\bullet) \to H^{i+1}(A^p_{X_\bullet/k}(k)). \tag{4}$$

These maps fit into the following commutative square

$$\begin{array}{ccc} H^i(\operatorname{gr}^0_{F_m}(CH^p)^\bullet) & \xrightarrow{\lambda_m^i} & H^{i+1}(\operatorname{gr}^1_{F_m}(CH^p)^\bullet) \\ \uparrow & & \uparrow \theta^i \\ H^i((NS^p)^\bullet) & \xrightarrow{\lambda_a^i} & H^{i+1}(A^p_{X_\bullet/k}(k)). \end{array}$$

Moreover, the kernel of θ^i is clearly equal to the image of the boundary map

$$\tau^i : H^i(\mathrm{gr}\,^1_{F_m}(CH^p)^\bullet/A^p_{X_\bullet/k}(k)) \to H^{i+1}(A^p_{X_\bullet/k}(k)).$$

Problem 2.1.1. *Is the image of the connected component of the identity of $H^{i+1}(A^p_{X_\bullet/k})$ under θ^i an abelian variety, e.g., is $\tau^i = 0$ up to a finite group ?*

This is clearly the case if $p = 1$ (see below for the case $k = \mathbb{C}$). For $k = \mathbb{C}$ this question is related to Griffiths Problem E in [17] asking a description of the "invertible points" of the intermediate jacobians (also *cf.* Mumford-Griffiths Problem F in [17]).

Assume that the above question has a positive answer and denote $H^{i+1}(A^p_{X_\bullet/k})^\dagger$ the so obtained abelian variety. We then obtain an algebraically defined 1-motive as follows.

Let $H^i(\mathrm{gr}\,^0_{F_m}(CH^p)^\bullet)^\dagger$ be the sub-group of those elements in $H^i(\mathrm{gr}\,^0_{F_m}(CH^p)^\bullet)$ which are mapped to $H^{i+1}(A^p_{X_\bullet/k})^\dagger$ under the boundary map λ^i_m above.

Definition 2.1.2. Let $X_\bullet$ be such a smooth proper simplicial scheme over k. Denote

$$\Xi^{i,p} \stackrel{\mathrm{def}}{=} [H^i(\mathrm{gr}\,^0_{F_m}(CH^p)^\bullet)^\dagger \xrightarrow{\lambda^i_m} H^{i+1}(A^p_{X_\bullet/k})^\dagger]_{\mathbb{Q}}$$

the isogeny 1-motive obtained from the construction above. Call $\Xi^{i,p}$ the *Hodge 1-motive* of the simplicial scheme.

We expect that $\Xi^{i,p}$ is independent of the choice of $\pi : X_\bullet \to X$. The main motivation for questioning the existence of such a purely algebraic construction is given by the following analytic counterpart.

2.2. Analytic construction.

Let MHS be the abelian category of usual Deligne's mixed Hodge structures [11]. An object H of MHS is defined as a triple $H = (H_{\mathbb{Z}}, W, F)$ where $H_{\mathbb{Z}}$ is a finitely generated $\mathbb{Z}$-module, W is a finite increasing filtration on $H_{\mathbb{Z}} \otimes \mathbb{Q}$ and F is a finite decreasing filtration on $H_{\mathbb{Z}} \otimes \mathbb{C}$ such that W, F and $\overline{F}$ is a system of opposed filtrations.

Let $H \in$ MHS be a torsion free mixed Hodge structure with positive weights. Let W_*H denote the sub-structures defined by the intersections of the weight filtration and $H_{\mathbb{Z}}$. Let p be a fixed positive integer and assume that $\mathrm{gr}\,^W_{2p-1}H$ is polarizable.

Consider the following extension in the abelian category MHS

$$0 \to \mathrm{gr}\,^W_{2p-1}H \to \frac{W_{2p}H}{W_{2p-2}H} \to \mathrm{gr}\,^W_{2p}H \to 0 \tag{5}$$

Taking $\mathrm{Hom}\,(\mathbb{Z}(-p), -)$ we get the extension class map

$$e^p : \mathrm{Hom}\,(\mathbb{Z}(-p), \mathrm{gr}\,^W_{2p} H) \to \mathrm{Ext}\,(\mathbb{Z}(-p), \mathrm{gr}\,^W_{2p-1} H)$$

where $\mathrm{Hom}\,(\mathbb{Z}(-p), \mathrm{gr}\,^W_{2p} H) = H^{p,p}_{\mathbb{Z}}$ is the sub-structure of (p,p)-classes in $\mathrm{gr}\,^W_{2p} H$ and

$$\mathrm{Ext}\,(\mathbb{Z}(-p), \mathrm{gr}\,^W_{2p-1} H) \cong J^p(H) \overset{\mathrm{def}}{=} \frac{\mathrm{gr}\,^W_{2p-1} H_{\mathbb{C}}}{F^p + \mathrm{gr}\,^W_{2p-1} H_{\mathbb{Z}}}$$

is a compact complex torus. Note that (*cf.* [9])

$$\mathrm{Ext}\,(H^{p,p}_{\mathbb{Z}}, \mathrm{gr}\,^W_{2p-1} H) \cong \mathrm{Hom}\,(H^{p,p}_{\mathbb{Z}}, J^p(H)).$$

Thus $e^p \in \mathrm{Hom}\,(H^{p,p}_{\mathbb{Z}}, J^p(H))$ corresponds to a unique extension class

$$0 \to \mathrm{gr}\,^W_{2p-1} H \to H^e \to H^{p,p}_{\mathbb{Z}} \to 0 \tag{6}$$

which is the pull-back extension associated to $H^{p,p}_{\mathbb{Z}} \hookrightarrow \mathrm{gr}\,^W_{2p} H$ and (5). Moreover, since we always have $\mathrm{gr}\,^W_{2p-1} H \cap F^p = 0$ then

$$F^p \cap H^e_{\mathbb{Z}} = \ker(H^{p,p}_{\mathbb{Z}} \overset{e^p}{\to} J^p(H)).$$

Now, if $\mathrm{gr}\,^W_{2p-1} H$ is (polarizable) of level 1 then the torus $J^p(H)$ is an abelian variety and H^e is the Hodge realization of the 1-motive over $\mathbb{C}$ defined by the extension class map e^p above.

In general, let H' be the largest sub-structure of $W_{2p-1}H$ which is polarizable and purely of type $\{(p-1,p),(p,p-1)\}$ modulo $W_{2p-2}H$, *i.e.*, if

$$H^{2p-1}_a \overset{\mathrm{def}}{=} (H^{p-1,p} + H^{p,p-1})_{\mathbb{Z}}$$

is the polarizable sub-structure of $\mathrm{gr}\,^W_{2p-1} H$ of those elements which are purely of the above type, then H' is defined by the following pull-back extension

$$0 \to W_{2p-2}H \to H' \to H^{2p-1}_a \to 0,$$

along the canonical projection $W_{2p-1}H \twoheadrightarrow \mathrm{gr}\,^W_{2p-1} H$.

Let $H'' \subseteq W_{2p}H$ be defined by the following pull-back extension

$$0 \to W_{2p-1}H \to H'' \to H^{p,p}_{\mathbb{Z}} \to 0,$$

along the canonical projection $W_{2p}H \twoheadrightarrow \mathrm{gr}\,^W_{2p} H$. Thus, the extension (6) can be regarded as the push-out of such extension involving H'' along $W_{2p-1}H \twoheadrightarrow \mathrm{gr}\,^W_{2p-1} H$. Namely, we obtain that

$$\frac{H''}{W_{2p-2}H} = H^e$$

fitting in the extension

$$0 \to \frac{H'}{W_{2p-2}H} \to H^e \to \frac{H''}{H'} \to 0. \tag{7}$$

Let

$$h^p : \mathrm{Hom}\,(\mathbb{Z}(-p), \frac{H''}{H'}) \to \mathrm{Ext}\,(\mathbb{Z}(-p), \frac{H'}{W_{2p-2}H})$$

be the extension class map obtained from (7).

Proposition 2.2.1. *The map h^p above yields a 1-motive over $\mathbb{C}$ which is just the restriction of $e^p : H_{\mathbb{Z}}^{p,p} \to J^p(H)$ to the largest abelian subvariety in $J^p(H)$. In particular:* $\ker(h^p) = \ker(e^p)$.

Proof. Since we have that $H'/W_{2p-2}H = H_a^{2p-1}$ by construction we also get that $\mathrm{Ext}\,(\mathbb{Z}(-p), H_a^{2p-1})$ is the largest abelian sub-variety of $\mathrm{Ext}\,(\mathbb{Z}(-p), \mathrm{gr}\,^W_{2p-1}H) = J^p(H)$. Moreover, note that we also have induced extensions

$$0 \to \frac{H'}{W_{2p-2}H} \to \mathrm{gr}\,^W_{2p-1}H \to \frac{W_{2p-1}H}{H'} \to 0$$

and

$$0 \to \frac{W_{2p-1}H}{H'} \to \frac{H''}{H'} \to H_{\mathbb{Z}}^{p,p} \to 0$$

yielding, together with (6) and (7), the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}\,(\mathbb{Z}(-p), \frac{W_{2p-1}H}{H'}) & \to & \mathrm{Hom}\,(\mathbb{Z}(-p), \frac{H''}{H'}) & \to & \mathrm{Hom}\,(\mathbb{Z}(-p), H_{\mathbb{Z}}^{p,p}) & \to & \cdots \\
\| & & h^p \downarrow & & \downarrow e^p & & \| \\
\mathrm{Hom}\,(\mathbb{Z}(-p), \frac{W_{2p-1}H}{H'}) & \to & \mathrm{Ext}\,(\mathbb{Z}(-p), \frac{H'}{W_{2p-2}H}) & \to & \mathrm{Ext}\,(\mathbb{Z}(-p), \mathrm{gr}\,^W_{2p-1}H) & \to & \cdots
\end{array}$$

where the last group on the right part of the diagram is $\mathrm{Ext}\,(\mathbb{Z}(-p), \frac{W_{2p-1}H}{H'})$. Since $\mathrm{Hom}\,(\mathbb{Z}(-p), \frac{W_{2p-1}H}{H'}) = 0 = \mathrm{Hom}\,(\mathbb{Z}(-p), H_a^{2p-1})$ everything follows from a diagram chase. □

Definition 2.2.2. Let $A^p(H) \stackrel{\mathrm{def}}{=} \mathrm{Ext}\,(\mathbb{Z}(-p), H_a^{2p-1})$ denote the abelian part of the compact complex torus $J^p(H)$. Denote $H^p(H) \stackrel{\mathrm{def}}{=} \mathrm{Hom}\,(\mathbb{Z}(-p), H''/H')$ the group of Hodge cycles, *i.e.*, the sub-group of $H_{\mathbb{Z}}^{p,p}$ mapped to $A^p(H)$ under the extension class map e^p. Define

$$H^h \stackrel{\mathrm{def}}{=} T_{\mathrm{Hodge}}([H^p(H) \xrightarrow{h^p} A^p(H)])$$

the mixed Hodge structure corresponding to the 1-motive defined from (7) above. Call this 1-motive the *Hodge 1-motive* of H.

We remark that the mixed Hodge structure H^h clearly corresponds to the following extension

$$0 \to H_a^{2p-1} \to H^h \to H^p(H) \to 0,$$

obtained by pulling back $H^p(H) = F^p \cap (H''/H')_{\mathbb{Z}}$ along the induced projection $H^e \twoheadrightarrow H''/H'$ in (7). In particular $H^h \subseteq H^e$ and $F^p \cap H_{\mathbb{Z}} \subseteq F^p \cap H_{\mathbb{Z}}^h = F^p \cap H_{\mathbb{Z}}^e$.

2.3. Hodge conjecture for singular varieties.

Let X be a proper smooth $\mathbb{C}$-scheme. The coniveau or arithmetic filtration (*cf.* [19])

$$N^iH^j(X) \stackrel{\text{def}}{=} \ker(H^j(X) \to \varinjlim_{\operatorname{codim}_X Z \geq i} H^j(X - Z))$$

yields a filtration by Hodge sub-structures of $H^j(X)$. We have that $N^iH^j(X)$ is of level $j - 2i$ and

$$N^iH^j(X)_{\mathbb{Q}} \subseteq H^j(X, \mathbb{Q}) \cap F^iH^j(X). \tag{8}$$

Conjecture 2.3.1. (Grothendieck-Hodge conjecture [19]) *The left hand side of (8) is the largest sub-space of the right hand side, generating a sub-space of $H^j(X, \mathbb{C})$ which is a sub-Hodge structure.*

Let X be a proper (integral) $\mathbb{C}$-scheme. Recall that the weight filtration on $H^*(X, \mathbb{Q})$ is given by the canonical spectral sequence of mixed $\mathbb{Q}$-Hodge structures

$$E_1^{s,t} = H^t(X_s) \Rightarrow \mathbb{H}^{s+t}(X_\bullet)$$

for any smooth and proper hypercovering $\pi : X_\bullet \to X$ and universal cohomological descent $H^*(X) \cong \mathbb{H}^*(X_\bullet)$ (see [11]). In fact, the spectral sequence degenerates at E_2. Denote $(H^t)^\bullet$ the complexes $E_1^{\cdot,t}$ of E_1-terms. We clearly have

$$H^i((H^t)^\bullet) = \operatorname{gr}_t^W H^{t+i}(X).$$

Consider the complexes $(N^lH^t)^\bullet$ induced from the coniveau filtration $N^lH^t(X_i)$ on the components X_i of the simplicial scheme $X_\bullet$. We then have a natural map of Hodge structures

$$\nu^{i,l} : H^i((N^lH^t)^\bullet) \to \operatorname{gr}_t^W H^{t+i}(X).$$

Note that the image of $\nu^{i,l}$ is contained in the sub-space $\operatorname{gr}_t^W H^{t+i}(X, \mathbb{Q}) \cap F^l$.

Conjecture 2.3.2. *The image of $\nu^{i,l}$ is the largest sub-space of $\operatorname{gr}_t^W H^{t+i}(X, \mathbb{Q}) \cap F^l$ which is a sub-Hodge structure of $\operatorname{gr}_t^W H^{t+i}(X)$.*

It is reasonable to expect that such a statement will follow from the original Grothendieck-Hodge conjecture and abstract Hodge theory.

Grothendieck-Hodge conjecture (for coniveau p and degrees $2p, 2p+1$) can be reformulated as follows (*cf.* Grothendieck's remark on motives in [19]). Consider $\operatorname{gr}_{F_m}^0 CH^p(X)$ and $A_{X/k}^{p+1} \subseteq \operatorname{gr}_{F_m}^1 CH^{p+1}(X)$ (for $k = \mathbb{C}$ this is the algebraic part of $J^{p+1}(X)$) as 1-motives with rational coefficients. The Hodge realization of these algebraically defined 1-motives is $N^pH^{2p}(X)$ and $N^pH^{2p+1}(X)$ respectively.

Conjecture 2.3.3. *Let X be smooth and proper over $\mathbb{C}$. Then*

$$T_{\text{Hodge}}([\operatorname{gr}_{F_m}^0 CH^p(X) \to 0]_{\mathbb{Q}}) = H_{\mathbb{Q}}^{p,p}$$

and

$$T_{\mathrm{Hodge}}([0 \to A^{p+1}_{X/k}]_{\mathbb{Q}}) = (H^{p,p+1} + H^{p+1,p})_{\mathbb{Q}}.$$

Note that $\mathrm{gr}\,^0_{F_m} CH^p(X)$ would be better defined as $\mathcal{Z}^p(X)/\equiv_{\mathrm{num}}$, up to torsion.

Now apply to the mixed $\mathbb{Q}$-Hodge structure $H = H^{2p+i}(X)$ the construction performed in the previous section. For a fixed pair (i,p) of integers recall that (5) is an extension of $\mathrm{gr}\,^W_{2p} H^{2p+i}(X)$ by $\mathrm{gr}\,^W_{2p-1} H^{2p+i}(X)$, where:

$$H^{i+1}((H^{2p-1})^\bullet) = \mathrm{gr}\,^W_{2p-1} H^{2p+i}(X) = \frac{\ker(H^{2p-1}(X_{i+1}) \to H^{2p-1}(X_{i+2}))}{\mathrm{im}\,(H^{2p-1}(X_i) \to H^{2p-1}(X_{i+1}))}$$

and

$$H^{i}((H^{2p})^\bullet) = \mathrm{gr}\,^W_{2p} H^{2p+i}(X) = \frac{\ker(H^{2p}(X_{i}) \to H^{2p}(X_{i+1}))}{\mathrm{im}\,(H^{2p}(X_{i-1}) \to H^{2p}(X_{i}))}.$$

We then have that $J^p(H) = J^p(H^{i+1}((H^{2p-1})^\bullet))$ is isogenous to $H^{i+1}((J^p)^\bullet))$ where $(J^p)^\bullet$ is the complex of jacobians $J^p(X_i)$ of the components X_i.

Consider the complex $A^p_{X_\bullet/\mathbb{C}}$ of abelian sub-varieties given by the algebraic part of intermediate jacobians. The complex $A^p_{X_\bullet/\mathbb{C}}(\mathbb{C})$ is a sub-complex of the complex of compact tori $(J^p)^\bullet$. Therefore, the induced maps

$$H^{i+1}(A^p_{X_\bullet/\mathbb{C}}(\mathbb{C})) \to J^p(H)$$

are holomorphic mappings (which factor through θ^i) and whose image is isogenous to an abelian sub-variety of the maximal abelian sub-variety $A^p(H)$ of $J^p(H)$.

Moreover, the homology of the complex of (p,p)-classes is mapped to $H^{p,p}_{\mathbb{Q}} = F^p \cap H^i((H^{2p})^\bullet)$. Thus, there are canonical maps

$$H^i((NS^p)^\bullet) \to H^{p,p}_{\mathbb{Q}}$$

which factor through $H^i(\mathrm{gr}\,^0_{F_m}(CH^p)^\bullet)$.

We expect that the Hodge 1-motive of the simplicial scheme $X_\bullet$ would be canonically isomorphic to the Hodge 1-motive of $H^{2p+i}(X)$.

Conjecture 2.3.4. *Let X be a proper $\mathbb{C}$-scheme and let $\pi : X_\bullet \to X$ be a smooth and proper hypercovering. Let $H^{2p+i}(X)$ denote Deligne's mixed $\mathbb{Q}$-Hodge structure on $H^{2p+i}(X,\mathbb{Q})$, i.e., obtained by the universal cohomological descent isomorphism $H^{2p+i}(X,\mathbb{Q}) \cong \mathbb{H}^{2p+i}(X_\bullet,\mathbb{Q})$.*

1. *The following square*

$$\begin{array}{ccc} H^i((NS^p)^\bullet) & \stackrel{\lambda^i_a}{\to} & H^{i+1}(A^p_{X_\bullet/\mathbb{C}}(\mathbb{C})) \\ \downarrow & & \downarrow \\ H^{2p+i}(X)^{p,p} & \stackrel{e^p}{\to} & J^p(H^{2p+i}(X)) \end{array}$$

commutes, yielding a motivic "cycle class map".

2. *The image of the motivic "cycle class map" is the Hodge 1-motive of the $\mathbb{Q}$-Hodge structure $H^{2p+i}(X)$.*

3. *We have that*
$$T_{\mathrm{Hodge}}(\Xi^{i,p}) \cong H^{2p+i}(X)^h.$$

Remark 2.3.5. Note that if X is smooth and proper then $H^{2p+i}(X)$ is pure and $H^{2p+i}(X)^h \neq 0$ if and only if $i = 0, -1$ (p fixed). In this case, the above conjecture follows from the reformulation of Grothendieck-Hodge conjecture for $H^{2p}(X)$ and $H^{2p-1}(X)$.

3. Local Hodge theory

See [11] for notations, definitions and properties of mixed Hodge structures.

3.1. Infinite dimensional mixed Hodge structures.

Let MHS denote the abelian category of usual Deligne's A-mixed Hodge structures [11], *i.e.*, for A a noetherian subring of $\mathbb{R}$ such that $A \otimes \mathbb{Q}$ is a field, an object H of MHS is defined as a triple $H = (H_A, W, F)$ where H_A is a finitely generated A-module, W is a finite increasing filtration on $H_A \otimes \mathbb{Q}$ and F is a finite decreasing filtration on $H_A \otimes \mathbb{C}$ such that W, F and $\overline{F}$ is a system of opposed filtrations.

Definition 3.1.1. An ∞-mixed Hodge structure H is a triple (H_A, W, F) where H_A is any A-module, W is a finite increasing filtration on $H_A \otimes \mathbb{Q}$ and F is a finite decreasing filtration on $H_A \otimes \mathbb{C}$ such that W, F and $\overline{F}$ is a system of opposed filtrations.

Denote MHS^∞ the category of ∞-mixed Hodge structures or "infinite dimensional" mixed Hodge structures, *i.e.*, where the morphisms are those which are compatible with the filtrations.

The category MHS^∞ is abelian and MHS is a fully faithful abelian subcategory of MHS^∞. Note that similar categories of infinite dimensional mixed Hodge structures already appeared in the literature, see Hain [20] and Morgan [31]. For example the category of limit mixed Hodge structures $\mathrm{MHS}^{\lim}$, *i.e.*, whose objects and morphisms are obtained by formally add to MHS (small) filtered colimits of objects in MHS with colimit morphisms.

Consider the case $A = \mathbb{Q}$. In this case, in the category MHS^∞ we have infinite products of those families of objects $\{H_i\}_{i\in I}$ such that the induced families of filtrations $\{W_i\}_{i\in I}$ and $\{F_i\}_{i\in I}$ are finite. Moreover such a (small) product of epimorphisms is an epimorphism.

For the sake of exposition we often call mixed Hodge structures the objects of MHS as well as those of MHS^∞ (or $\mathrm{MHS}^{\lim}$).

3.2. Zariski sheaves of mixed Hodge structures.

Let X denote the (big or small) Zariski site on an algebraic $\mathbb{C}$-scheme. However, most of the results in this section are available for any topological space or Grothendieck site.

Denote $X_{\bullet}$ a simplicial object of the category of algebraic $\mathbb{C}$-schemes over X: recall that (see [11, 5.1.8]) simplicial sheaves on $X_{\bullet}$ can be regarded as objects of a Grothendieck topos with enough points.

Consider presheaves of mixed Hodge structures. Note that a presheaf of usual Deligne's A-mixed Hodge structures will have its stalks in $\mathrm{MHS}^{\mathrm{lim}}$.

Consider those presheaves (resp. simplicial presheaves) of $\mathbb{Q}$-mixed Hodge structures on X (resp. on $X_{\bullet}$) such that the filtrations are finite as filtrations of sub-presheaves on X. These presheaves can be sheafified to sheaves having finite filtrations and preserving the above conditions on the stalks.

Make the following working definition of sheaves (or simplicial sheaves) of mixed Hodge structures. Let $A, \mathbb{Q}$ and $\mathbb{C}$ denote as well the constant sheaves on X (or $X_{\bullet}$) associated to the ring A, the rationals and the complex numbers.

Definition 3.2.1. A (simplicial) sheaf $\mathcal{H}$ of A-mixed Hodge structures, or "A-mixed sheaf" for short, is given by the following set of datas:

i) a (simplicial) sheaf $\mathcal{H}_A$ of A-modules,

ii) a finite (exhaustive) increasing filtration $\mathcal{W}$ by $\mathbb{Q}$-subsheaves of $\mathcal{H}_{\mathbb{Q}} \overset{\mathrm{def}}{=} \mathcal{H}_A \otimes \mathbb{Q}$,

iii) a finite (exhaustive) decreasing filtration $\mathcal{F}$ by $\mathbb{C}$-subsheaves of $\mathcal{H}_{\mathbb{C}} \overset{\mathrm{def}}{=} \mathcal{H}_A \otimes \mathbb{C}$;

satisfying the condition that $\mathcal{W}, \mathcal{F}$ and $\overline{\mathcal{F}}$ is a system of opposed filtrations, *i.e.*, we have that

$$gr^p_{\mathcal{F}} gr^q_{\overline{\mathcal{F}}} gr^{\mathcal{W}}_n(\mathcal{H}) = 0$$

for $p+q \neq n$.

There is a canonical decomposition

$$gr^{\mathcal{W}}_n(\mathcal{H}) = \bigoplus_{p+q=n} \mathcal{A}^{p,q}$$

where $\mathcal{A}^{p,q} \overset{\mathrm{def}}{=} \mathcal{F}^p \cap \overline{\mathcal{F}}^q$ and conversely.

In the case of a simplicial sheaf assume that the filtrations are given by simplicial subsheaves, *i.e.*, the simplicial structure should be compatible with the filtrations on the components. A simplicial A-mixed sheaf $\mathcal{H}_{X_{\bullet}}$ on the simplicial space $X_{\bullet}$ can be regarded (*cf.* [11, 5.1.6]) as a family of A-mixed sheaves $\mathcal{H}_{X_i}$ (on the components X_i) such that the simplicial structure is also compatible with the filtrations $\mathcal{W}_{X_i}$ and $\mathcal{F}_{X_i}$ of $\mathcal{H}_{X_i}$.

A morphism of A-mixed sheaves is a morphism of (simplicial) sheaves of A-modules which is compatible with the filtrations. Denote $\mathcal{MHS}_X$ and $\mathcal{MHS}_{X_{\bullet}}$ the corresponding categories.

In order to show the following Lemma one can just use Deligne's Theorem [11, 1.2.10].

Lemma 3.2.2. *The categories $\mathcal{MHS}_X$ and $\mathcal{MHS}_{X_\bullet}$ of $\mathbb{Q}$-mixed sheaves are abelian categories. The kernel (resp. the cokernel) of a morphism $\varphi : \mathcal{H} \to \mathcal{H}'$ has underlying $\mathbb{Q}$ and $\mathbb{C}$-vector spaces the kernels (resp. the cokernels) of $\varphi_{\mathbb{Q}}$ and $\varphi_{\mathbb{C}}$ with induced filtrations. Any morphism is strictly compatible with the filtrations. The functors $gr_{\mathcal{W}}$ and $gr_{\mathcal{F}}$ are exacts.*

Note that if $X = \{\infty\}$ is the singleton then $\mathcal{MHS}_\infty$ is equal to MHS^∞. Examples of $\mathbb{Q}$-mixed sheaves are clearly given by constant sheaves associated to $\mathbb{Q}$-mixed Hodge structures, yielding a canonical fully faithful functor

$$\mathrm{MHS}^\infty \to \mathcal{MHS}_X.$$

Stalks of a $\mathbb{Q}$-mixed sheaf $\mathcal{H}$ are in MHS^∞, the filtrations being induced stalkwise. In fact, the condition on the filtrations given with any $\mathbb{Q}$-mixed sheaf is local, at any point of X. Skyscraper sheaves $x_*(H)$ associated to an object $H \in \mathrm{MHS}^\infty$ and a point x of X are in $\mathcal{MHS}_X$. There is a natural isomorphism

$$\mathrm{Hom}_{\,\mathrm{MHS}^\infty}(\mathcal{H}_x, H) \cong \mathrm{Hom}_{\,\mathcal{MHS}_X}(\mathcal{H}, x_*(H)).$$

More generally, a presheaf in MHS^∞, with finite filtrations presheaves, can be sheafified to an A-mixed sheaf, in a canonical way, by applying the usual sheafification process to the filtrations together with the presheaf.

Definition 3.2.3. Say that an A-mixed sheaf $\mathcal{H}$ is flasque if $\mathcal{H}_A$ is a flasque sheaf.

For a given $\mathbb{Q}$-mixed sheaf $\mathcal{H}$ we then dispose of a canonical flasque $\mathbb{Q}$-mixed sheaf

$$\prod_{x\in X} x_*(\mathcal{H}_x)$$

where the product is taken over a set of points of X.

3.3. Hodge structures and Zariski cohomology.

We show that, if $\mathcal{H}$ is a $\mathbb{Q}$-mixed sheaf then there is a unique $\mathbb{Q}$-mixed Hodge structure on the sections such that $\Gamma(-, gr(\dagger)) = \mathrm{gr}\,\Gamma(-,\dagger)$. In fact, the mixed Hodge structure on the $\mathbb{Q}$-vector space of (global) sections is such that the following

$$\Gamma(X, \mathcal{H}_{\mathbb{Q}}) \subset \prod_{x\in X} \mathcal{H}_x$$

is strictly compatible with the filtrations; in the same way, for a simplicial sheaf, the following

$$\Gamma(X_\bullet, \mathcal{H}_{\mathbb{Q},\bullet}) \subset \ker \prod_{x\in X_0} \mathcal{H}_x \to \prod_{x\in X_1} \mathcal{H}_x$$

is strictly compatible with the filtrations.

Proposition 3.3.1. *Let $\mathcal{H}_X \in \mathcal{MHS}_X$ and $\mathcal{H}_{X_\bullet} \in \mathcal{MHS}_{X_\bullet}$ as above. There are left exact functors*

$$\mathcal{H}_X \mapsto \Gamma(X, \mathcal{H}_X) \qquad \mathcal{MHS}_X \to \mathrm{MHS}^\infty$$

and

$$\mathcal{H}_{X_\bullet} \mapsto \Gamma(X_\bullet, \mathcal{H}_{X_\bullet}) \qquad \mathcal{MHS}_{X_\bullet} \to \mathrm{MHS}^\infty$$

These functors yield $\mathbb{Q}$-mixed Hodge structures on the usual cohomology, which we denote by $H^*(X, \mathcal{H}_X)$ and $H^*(X_\bullet, \mathcal{H}_{X_\bullet})$ respectively, such that if

$$0 \to \mathcal{H}' \to \mathcal{H} \to \mathcal{H}'' \to 0$$

is exact in $\mathcal{MHS}_X$, respectively in $\mathcal{MHS}_{X_\bullet}$, then

$$\cdots \to H^i(X, \mathcal{H}_X) \to H^i(X, \mathcal{H}''_X) \to H^{i+1}(X, \mathcal{H}'_X) \to \cdots$$

is exact in MHS^∞*, and respectively for the cohomology of $X_\bullet$: moreover, in the latter case we have a spectral sequence*

$$E_1^{p,q} = H^q(X_p, \mathcal{H}_{X_p}) \Rightarrow \mathbb{H}^{p+q}(X_\bullet, \mathcal{H}_{X_\bullet})$$

in the category MHS^∞.

Proof. In fact, there is an extension in $\mathcal{MHS}$

$$0 \to \mathcal{H} \to \prod_{x \in X} x_*(\mathcal{H}_x) \to \mathcal{Z}^1 \to 0$$

where $\mathcal{Z}^1$ has the quotient $\mathbb{Q}$-mixed structure; as usual, we then get another extension

$$0 \to \mathcal{Z}^1 \to \prod_{x \in X} x_*(\mathcal{Z}^1_x) \to \mathcal{Z}^2 \to 0$$

and so on. We therefore get a flasque resolution

$$\prod_{x \in X} x_*(\mathcal{H}_x) \to \prod_{x \in X} x_*(\mathcal{Z}^1_x) \to \prod_{x \in X} x_*(\mathcal{Z}^2_x) \to \cdots$$

in $\mathcal{MHS}$: the canonical $\mathbb{Q}$-mixed flasque resolution.

If $\mathcal{H}_X$ is flasque then $\Gamma(X, \mathcal{H}_X)$ has a canonical $\mathbb{Q}$-mixed Hodge structure as claimed above; note that the filtrations would be given by flasque sub-sheaves.

In general, by construction, the cohomology is the homology of the complex of sections in MHS^∞. Thus $H^*(X, \mathcal{H}_X)$ has a canonical $\mathbb{Q}$-mixed Hodge structure. The same argument applies to the total complex of the double complex of flasque $\mathbb{Q}$-mixed sheaves given by the canonical resolutions on each component of a simplicial sheaf.

Refer to [SGA4] and [26, Chapter IV] for a construction of canonical Godement resolutions available on any site and compare [11, 1.4.11] for the existence of bifiltered resolutions. □

In particular, the mixed Hodge structure $H^*(X,\mathcal{H}_X)$ is such that $H^*(X,\mathcal{H}_X)_{\mathbb{Q}} = H^*(X,\mathcal{H}_{\mathbb{Q}})$, $WH^*(X,\mathcal{H}_X)_{\mathbb{Q}} = H^*(X,\mathcal{W}\mathcal{H}_{\mathbb{Q}})$ and $FH^*(X,\mathcal{H}_X)_{\mathbb{C}} = H^*(X,\mathcal{F}\mathcal{H}_{\mathbb{C}})$. There is a decomposition

$$\mathrm{gr}_n^W H^*(X,\mathcal{H}_X) = \bigoplus_{p+q=n} H^*(X,\mathcal{A}_X^{p,q}).$$

Remark 3.3.2. Note that any (non-canonical) $\mathbb{Q}$-mixed flasque resolution in $\mathcal{MHS}$ yields a bifiltered complex and a bifiltered quasi-isomorphism with the canonical resolution. Therefore, the so obtained ∞-mixed Hodge structure on the cohomology is unique up to isomorphism.

Considering complexes in $\mathcal{MHS}_X$ and $\mathcal{MHS}_{X_\bullet}$ we construct the derived categories of $\mathbb{Q}$-mixed sheaves $\mathcal{D}^*(\mathcal{MHS}_X)$ and $\mathcal{D}^*(\mathcal{MHS}_{X_\bullet})$ as usual, as well as $\mathcal{D}^*(\mathrm{MHS}^\infty)$. We have a total derived functor

$$R\Gamma(X,-) : \mathcal{D}^*(\mathcal{MHS}_X) \to \mathcal{D}^*(\mathrm{MHS}^\infty)$$

sending a complex of $\mathbb{Q}$-mixed sheaves to the total complex of sections of its canonical resolution. Moreover, if $f : X \to Y$ is a continuous map, we have a higher direct image $\mathbb{Q}$-mixed sheaf $R^q f_*(\mathcal{H}_X)$ on Y, which corresponds as well to an exact functor

$$Rf_* : \mathcal{D}^*(\mathcal{MHS}_X) \to \mathcal{D}^*(\mathcal{MHS}_Y).$$

There is an inverse image exact functor $f^* : \mathcal{D}^*(\mathcal{MHS}_Y) \to \mathcal{D}^*(\mathcal{MHS}_X)$. Moreover, Grothendieck six standard operations can be otained in the derived category of $\mathbb{Q}$-mixed sheaves.

3.4. Local-to-global properties.

Let X be an algebraic $\mathbb{C}$-scheme and let X_{an} be the associated analytic space. For any Zariski open subset $U \subseteq X$ the corresponding integral cohomology $H^r(U_{\mathrm{an}},\mathbb{Z}(t))$ carry a mixed Hodge structure (see [11, 8.2]) such that the restriction maps $H^r(U_{\mathrm{an}},\mathbb{Z}(t)) \to H^r(V_{\mathrm{an}},\mathbb{Z}(t))$ for $V \subseteq U$ are maps of mixed Hodge structures. Thus the presheaf of mixed Hodge structures

$$U \mapsto H^r(U_{\mathrm{an}},\mathbb{Q}(t)) \tag{9}$$

can be sheafified to a Zariski $\mathbb{Q}$-mixed sheaf. In fact, for a fixed r, the resulting non-zero Hodge numbers of $H^r(U_{\mathrm{an}},\mathbb{Q})$, for any U, are in the finite set $[0,r]\times[0,r]$ (see [11, 8.2.4]).

Definition 3.4.1. Denote $\mathcal{H}^r_X(\mathbb{Q}(t))$ the $\mathbb{Q}$-mixed sheaf obtained hereabove. For $X_\bullet$ a simplicial $\mathbb{C}$-scheme denote $\mathcal{H}^r_{X_\bullet}$ the simplicial $\mathbb{Q}$-mixed sheaf given by $\mathcal{H}^r_{X_p}$ on the component X_p.

If X has algebraic dimension n then all its Zariski open affines U do have dimension $\leq n$ thus $\mathcal{H}^r_X = 0$ for $r > n$.

Scholium 3.4.2. *The Zariski cohomology groups $H^*(X, \mathcal{H}^r_X)$ carry ∞-mixed Hodge structures. Possibly non-zero Hodge numbers of $H^*(X, \mathcal{H}^r_X)$ are in the finite set $[0, r] \times [0, r]$. The Zariski cohomology $\mathbb{H}^*(X_\bullet, \mathcal{H}^r_{X_\bullet})$ carry ∞-mixed Hodge structure and the canonical spectral sequence*

$$E_1^{p,q} = H^q(X_p, \mathcal{H}^r_{X_p}) \Rightarrow \mathbb{H}^{p+q}(X_\bullet, \mathcal{H}^r_{X_\bullet})$$

is in the category MHS^∞.

Let $\omega : X_{\mathrm{an}} \to X_{\mathrm{Zar}}$ be the continuous map of sites induced by the identity mapping. We then have a Leray spectral sequence

$$L_2^{q,r} = H^q(X_{\mathrm{Zar}}, R^r\omega_*(\mathbb{Z})) \Rightarrow H^{q+r}(X_{\mathrm{an}}, \mathbb{Z})$$

of abelian groups. Since $R^r\omega_*(\mathbb{Q}) \cong \mathcal{H}^r_X$ these sheaves can be regarded as $\mathbb{Q}$-mixed sheaves and its Zariski cohomology carry ∞-mixed Hodge structures as above.

For $X_\bullet$ a simplicial scheme we thus have $\omega_\bullet : (X_\bullet)_{\mathrm{an}} \to (X_\bullet)_{\mathrm{Zar}}$ as above and a Leray spectral sequence

$$L_2^{q,r} = \mathbb{H}^q(X_\bullet, R^r(\omega_\bullet)_*(\mathbb{Z})) \Rightarrow \mathbb{H}^{q+r}(X_\bullet, \mathbb{Z})$$

where $R^r(\omega_\bullet)_*(\mathbb{Z}) \cong \mathcal{H}^r_{X_\bullet}$.

Claim 3.4.3. (Local-to-global) *There are spectral sequences*

$$L_2^{q,r} = H^q(X, \mathcal{H}^r_X) \Rightarrow H^{q+r}(X_{\mathrm{an}}, \mathbb{Q})$$

and

$$L_2^{q,r} = \mathbb{H}^q(X_\bullet, \mathcal{H}^r_{X_\bullet}) \Rightarrow \mathbb{H}^{q+r}(X_\bullet, \mathbb{Q})$$

in the category of ∞-mixed Hodge structures.

The proof of this compatibility result will appear elsewhere; however, for smooth $\mathbb{C}$-schemes and using (12) below, the compatibility follows from [35, Corollary 4.4].

4. Edge maps

Recall that the classical cycle class maps can be obtained *via* edge homomorphisms in the coniveau spectral sequence. This is a consequences of Bloch's formula

[8]. Working simplicially we then construct certain cycle class maps for singular varieties *via* edge maps in the local-to-global spectral sequence. We first show that the results of [8] hold in the category of ∞-mixed Hodge structures.

4.1. Bloch-Ogus theory.

From Deligne [11, 8.2.2 and 8.3.8] the cohomology groups $H^*_Z(X)$ ($= H^*(X \mathrm{mod} X - Z, \mathbb{Z})$ in Deligne's notation) carry a mixed Hodge structure fitting into long exact sequences

$$\cdots \to H^j_Z(X) \to H^j_T(X) \to H^j_{T-Z}(X-Z) \to H^{j+1}_Z(X) \to \cdots \tag{10}$$

for any pair $Z \subset T$ of closed subschemes of X.

Since classical Poincaré duality is compatible with the mixed Hodge structures involved, then the functors

$$Z \subseteq X \mapsto (H^*_Z(X), H_*(Z))$$

yield a Poincaré duality theory with supports (see [8] and [23]) in the abelian tensor category of mixed Hodge structures. Furthermore we have that the above theory is appropriate for algebraic cycles in the sense of [1].

Let X^p be the set of codimension p points in X. For $x \in X^p$ let

$$H^*_x(X) \stackrel{\mathrm{def}}{=} \varinjlim_{U \subset X} H^*_{\overline{\{x\}} \cap U}(U).$$

Taking direct limits of (10) over pairs $Z \subset T$ filtered by codimension and applying the exact couple method to the resulting long exact sequence we obtain the coniveau spectral sequence

$$C_1^{p,q} = \coprod_{x \in X^p} H^{q+p}_x(X) \Rightarrow H^{p+q}(X)$$

in the abelian category MHS^∞ (*cf.* [1]).

Consider X smooth over $\mathbb{C}$. By local purity, we have that $H^{q+p}_x(X, \mathbb{Z}(r)) \cong H^{q-p}(x, \mathbb{Z}(r-p))$ if x is a codimension p point in X, *i.e.*, here we have set

$$H^*(x) \stackrel{\mathrm{def}}{=} \varinjlim_{V \text{ open } \subset \overline{\{x\}}} H^*(V).$$

Sheafifying the (limit) sequences (10), we obtain the following exact sequences of $\mathbb{Q}$-mixed sheaves on X:

$$0 \to \mathcal{H}^r_{Z^p} \to \coprod_{x \in X^p} x_*(H^{r-2p}(x)) \to \mathcal{H}^{r+1}_{Z^{p+1}} \to 0 \tag{11}$$

where $\mathcal{H}^r_{Z^p}$ is the $\mathbb{Q}$-mixed sheaf associated to the presheaf

$$U \mapsto \varinjlim_{\mathrm{codim}_X Z \geq p} H^r_{Z \cap U}(U).$$

In fact, the claimed short exact sequencs (11) are obtained *via* the "locally homologically effaceable" property (see [8, Claim p. 191]), *i.e.*, the following map of sheaves on X

$$\mathcal{H}^*_{Z^{p+1}} \stackrel{\text{zero}}{\to} \mathcal{H}^*_{Z^p}$$

vanishes for all $p \geq 0$.

Proposition 4.1.1. (Arithmetic resolution) *Let $\mathcal{H}^q_X(\mathbb{Q}(t))$ be the $\mathbb{Q}$-mixed sheaf defined in (9). Assuming X smooth over $\mathbb{C}$ then*

$$0 \to \mathcal{H}^q(\mathbb{Q}(t)) \to \coprod_{x\in X^0} x_*(H^q(x)(t)) \to \coprod_{x\in X^1} x_*(H^{q-1}(x)(t-1)) \to \cdots \tag{12}$$

$$\cdots \to \coprod_{x\in X^{q-1}} x_*(H^1(x)(t-q+1)) \to \coprod_{x\in X^q} x_*(\mathbb{Q}(t-q)) \to 0$$

is a flasque resolution in the category $\mathcal{MHS}_X$. Therefore, the coniveau spectral sequence

$$C_2^{p,q} = H^p(X,\mathcal{H}^q(\mathbb{Q}(t)))\Rightarrow H^{p+q}(X,\mathbb{Q}(t)) \tag{13}$$

is in the category MHS^∞.

Proof. Follows from construction as sketched above. In fact, all axioms stated in [8, Section 1] are verified in MHS and the results in [8, Sections 3-4] can be obtained in MHS^∞. □

In particular, consider the presheaf of vector spaces

$$U \mapsto F^iH^*(U) \qquad \text{(resp. } U \mapsto W_iH^*(U))$$

and the associated Zariski sheaves $\mathcal{F}^i\mathcal{H}^*$ (resp. $\mathcal{W}_i\mathcal{H}^*$) on X filtering the sheaves $\mathcal{H}^*(\mathbb{C})$ (resp. $\mathcal{H}^*(\mathbb{Q})$). These filtrations are defining the sheaf of mixed Hodge structures $\mathcal{H}^*_X$ above according to (9). From Lemma 3.2.2 (*cf.* [11, Theor.1.2.10 and 2.3.5]) the functors F^i, W_i, gr_i^W and gr_F^i (any $i \in \mathbb{Z}$) from the category of $\mathbb{Q}$-mixed sheaves to that of ordinary sheaves are exact. Applying these functors to the arithmetic resolution (12) we obtain resolutions of $\mathcal{F}^i\mathcal{H}^*$ (resp. $\mathcal{W}_i\mathcal{H}^*$) as follows.

Scholium 4.1.2. *The arithmetic resolution (12) yields a bifiltered quasi-isomorphism*

$$(\mathcal{H}^*,\mathcal{F}^\dagger,\mathcal{W}_\sharp) \stackrel{\simeq}{\to} (\coprod_{x\in X^\odot} x_*H^{*-\odot}(x), \coprod_{x\in X^\odot} x_*F^{\dagger-\odot}, \coprod_{x\in X^\odot} x_*W_{\sharp-2\odot}),$$

i.e., there are flasque resolutions:

$$0 \to \mathcal{F}^i\mathcal{H}^q \to \coprod_{x\in X^0} x_*(F^iH^q(x)) \to \coprod_{x\in X^1} x_*(F^{i-1}H^{q-1}(x)) \to \cdots$$

and

$$0 \to \mathcal{W}_j\mathcal{H}^q \to \coprod_{x\in X^0} x_*(W_jH^q(x)) \to \coprod_{x\in X^1} x_*(W_{j-2}H^{q-1}(x)) \to \cdots$$

as well as

$$0 \to gr^i_{\mathcal{F}}gr^{\mathcal{W}}_j\mathcal{H}^q(\mathbb{C}) \to \coprod_{x\in X^0} x_*(gr^i_F gr^W_j H^q(x)) \to \cdots$$

$$\cdots \to \coprod_{x\in X^q} x_*(gr^{i-q}_F gr^W_{j-2q}H^0(x)) \to 0.$$

Consider the twisted Poincaré duality theory $(F^nH^*, F^{-m}H_*)$ where the integers n and m play the role of twisting, *i.e.*, we have

$$F^{d-n}H^{2d-k}_Z(X) \cong F^{-n}H_k(Z)$$

for X smooth of dimension d. From the arithmetic resolution of $\mathcal{F}^i\mathcal{H}^q$ (see the Scholium 4.1.2) we obtain the following:

Scholium 4.1.3. *Assume X smooth and let i be a fixed integer. We then have a coniveau spectral sequence*

$$F^iC^{p,q}_2 = H^p(X, \mathcal{F}^i\mathcal{H}^q) \Rightarrow F^iH^{p+q}(X) \tag{14}$$

where $H^p(X, \mathcal{F}^i\mathcal{H}^q) = 0$ if $q < min(i,p)$.

Concerning the Zariski sheaves $gr^i_{\mathcal{F}}\mathcal{H}^q$ and $\overline{\mathcal{F}}^i$ we indeed obtain corresponding coniveau spectral sequences as above.

Note that the spectral sequence F^iC_2 can be obtained applying F^i to the coniveau spectral sequence (13).

Remark 4.1.4. *i)* Remark that applying F^i (resp. W_i) to the long exact sequences (10), taking direct limits over pairs $Z \subset T$ filtered by codimension and sheafifying, we do obtain the claimed flasque resolutions of $\mathcal{F}^i\mathcal{H}^*$ and $\mathcal{W}_i\mathcal{H}^*$ without reference to the category of $\mathbb{Q}$-mixed sheaves.

ii) Note that for X (equidimensional) of dimension d, the fundamental class η_X belongs to $W_{-2d}H_{2d}(X) \cap F^{-d}H_{2d}(X)$ so that "local purity" yields the shift by two for the weight filtration and the shift by one for the Hodge filtration. Therefore one has to keep care of Tate twists when dealing with arithmetic resolutions.

iii) Note that for $x \in X^0$ we have $H^q(x) = \mathcal{H}^q_x$ and there is a natural projection in $\mathcal{MHS}_{\mathcal{X}}$

$$\coprod_{x\in X} x_*(\mathcal{H}^q_x) \to \coprod_{x\in X^0} x_*(H^q(x))$$

which is the identity on $\mathcal{H}^q$. The mixed Hodge structure induced by the arithmetic resolution on $H^*(X,\mathcal{H}^q)$ is not the canonical one (which is the one induced by the canonical $\mathbb{Q}$-mixed flasque resolution) but yields a mixed Hodge structure which is naturally isomorphic to the canonical one (being induced by a natural isomorphism in the derived category $\mathcal{D}^*(\mathcal{MHS}_X)$).

4.2. Coniveau filtration.

Let X be a smooth $\mathbb{C}$-scheme. The coniveau filtration (*cf.* [19]) $N^iH^j(X)$ is a filtration by (mixed) sub-structures of $H^j(X)$. This filtration is clearly induced from the coniveau spectral sequence (13) *via* (10). Remark that from the coniveau spectral sequence (13)

$$\mathrm{gr}_N^{i-1}H^j(X) = C_\infty^{i-1,j-i+1}$$

which is a substructure of $H^{i-1}(X,\mathcal{H}^{j-i+1})$ for $i\leq 2$. In fact, from the arithmetic resolution we have that $C_2^{p,q} = H^p(X,\mathcal{H}^q(t)) = 0$ for $p>q$.

Case $i=1$ Let X be a proper smooth $\mathbb{C}$-scheme. Note that we have $N^1H^j(X) = \ker(H^j(X)\to H^0(X,\mathcal{H}^j)) = \{$ Zariski locally trivial classes in $H^j(X)\}$. Thus

$$\frac{H^j(X,\mathbb{Q})\cap F^1H^j(X)}{N^1H^j(X)} = \mathrm{gr}_N^0H^j(X)\cap F^1 \subseteq H^0(X,\mathcal{H}^j)\cap F^1.$$

We remark that $\mathcal{H}^j/\mathcal{F}^1$ is the constant sheaf associated to $H^j(X,\mathcal{O}_X)$. Thus

$$F^1\cap H^0(X,\mathcal{H}^j)\cong \ker(H^0(X,\mathcal{H}^j)\to H^j(X,\mathcal{O}_X)).$$

If $j=1$ then $H^1(X)=H^0(X,\mathcal{H}^1)$ from (13) and (8) is trivially an equality. If $j=2$ then $F^1\cap H^0(X,\mathcal{H}^2)=0$ from the exponential sequence. But for $j=3$ and X the threefold product of an elliptic curve with itself Grothendieck's argument in [19] yields a non-trivial element in $F^1\cap H^0(X,\mathcal{H}^3)$.

Case $i=p$ **and** $j=2p$ Let X be a smooth $\mathbb{C}$-scheme. If $j=2p$ we then have $N^iH^{2p}(X)(t)=0$ for $i>p$ and $N^pH^{2p}(X)(t)=C_\infty^{p,p}$. Moreover, from (13) there is an induced edge map

$$s\ell_0^p : H^p(X,\mathcal{H}_X^p(p)) \to H^{2p}(X)(p)$$

which is a map of ∞-mixed Hodge structures and whose image is $N^pH^{2p}(X)(p)$. This equal the image of the classical cycle class map $c\ell^p : CH^p(X)\to H^{2p}(X)(p)$. In fact, by [8, 7.6], the cohomology group

$$H^p(X,\mathcal{H}_X^p(p)) \cong \mathrm{coker}(\coprod_{x\in X^{p-1}} H^1(x) \to \coprod_{x\in X^p}\mathbb{Z})$$

coincide with $NS^p(X)$, the group of algebraic cycles of codimension p in X modulo algebraic equivalence. Thus $c\ell^p$ factors through $s\ell_0^p$ and the canonical projection (see [1]).

Recall that $F^iH^p(X,\mathcal{H}^q) \stackrel{\text{def}}{=} H^p(X,\mathcal{F}^i\mathcal{H}^q) \hookrightarrow H^p(X,\mathcal{H}^q(\mathbb{C}))$ is injective and

$$H^p(X,\mathcal{F}^p\mathcal{H}^p) \cong \operatorname{coker}(\coprod_{x\in X^{p-1}} F^1H^1(x) \to \coprod_{x\in X^p} \mathbb{C})$$

whence the canonical map $H^p(X,\mathcal{F}^p\mathcal{H}^p) \to NS^p(X)\otimes\mathbb{C}$ is also surjective. As an immediate consequence of this fact, *e.g.*, from the coniveau spectral sequence (14), we get the following.

Scholium 4.2.1. *Let X be a proper smooth $\mathbb{C}$-scheme. Then*

$$F^0H^p(X,\mathcal{H}^p(p)) \stackrel{\text{def}}{=} H^p(X,\mathcal{F}^0\mathcal{H}^p(p)) \cong NS^p(X)\otimes\mathbb{C}$$

and the image of the cycle map is in $H^{2p}(X,\mathbb{Q}(p))\cap F^0H^{2p}(X,\mathbb{C}(p))$.

Now $\operatorname{im} c\ell^p_{\mathbb{Q}} = N^pH^{2p}(X,\mathbb{Q}(p))$ and $H^{2p}(X,\mathbb{Q}(p))\cap F^0H^{2p}(X,\mathbb{C}(p))$ is equal to $H^{p,p}_{\mathbb{Q}}$, *i.e.*, the sub-structure of rational (p,p)-classes in $H^{2p}(X)$. Note that $H^{p,p}_{\mathbb{Q}}$ corresponds to the 1-motivic part of $H^{2p}(X)(p)$. For X a smooth proper $\mathbb{C}$-scheme, the Hodge conjecture then claims that $\operatorname{im} c\ell^p_{\mathbb{Q}} = H^{p,p}_{\mathbb{Q}}$.

In this case $\operatorname{gr}^{p-1}_N H^{2p}(X)(p) = C^{p-1,p+1}_{\infty}$ is a quotient of $H^{p-1}(X,\mathcal{H}^{p+1}_X(p))$. For example: $\operatorname{gr}^1_N H^4(X)(2) = H^1(X,\mathcal{H}^3_X(2))$.

Problem 4.2.2. *Is $F^2\cap H^1(X,\mathcal{H}^3_X) = 0$?*

Case $i=p$ and $j=2p+1$ Let X be a smooth $\mathbb{C}$-scheme. If $j=2p+1$ then $N^iH^{2p+1}(X)=0$ for $i>p$ and $N^pH^{2p+1}(X)(t) = C^{p,p+1}_{\infty}$ which is a quotient of $H^p(X,\mathcal{H}^{p+1}_X)$, *i.e.*, there is an edge map

$$s\ell^{p+1}_{-1} : H^p(X,\mathcal{H}^{p+1}_X(p+1)) \to H^{2p+1}(X)(p+1)$$

with image $N^pH^{2p+1}(X)(p+1)$. In this case the Grothendieck-Hodge conjecture characterize $N^pH^{2p+1}(X)$ as the largest sub-Hodge structure of type $\{(p,p+1),(p+1,p)\}$. This is the same as the 1-motivic part of $H^{2p+1}(X)(p+1)$.

This 1-motivic part yields an abelian variety which is the maximal abelian subvariety of the intermediate jacobian $J^{p+1}(X)$. On the other hand, it is easy to see that $N^pH^{2p+1}(X)(p+1)$ yields the algebraic part of $J^{p+1}(X)$, *i.e.*, defined by the images of codimension $p+1$ cycles on X which are algebraically equivalent to zero modulo rational equivalence (*cf.* [19] and [33]).

4.3. Exotic $(1,1)$-classes.

Consider X singular. We briefly explain the Conjecture 2.3.4 for $p=1$. Moreover we show that there are edge maps generalizing the cycle class maps constructed in the previous section.

For X a proper irreducible $\mathbb{C}$-scheme, consider the mixed Hodge structure on $H^{2+i}(X,\mathbb{Z})$ modulo torsion. The extension (5) is the following

$$0\to H^{1+i}((H^1)^\bullet) \to W_2H^{2+i}(X)/W_0 \to H^i((H^2)^\bullet)\to 0. \qquad (15)$$

Since the complex $(H^1)^\bullet$ is made of level 1 mixed Hodge structures then

$$H^{2+i}(X)^h = H^{2+i}(X)^e$$

in our notation.

If X is nonsingular then $H^{2+i}(X)$ is pure and there are only two cases where this extension is non-trivial. In the case $i=-1$ the above conjecture corresponds to the well known fact that $H_1(\mathrm{Pic}^0(X)) = H^1(X,\mathbb{Z})$. The case $i=0$ corresponds to the celebrated theorem by Lefschetz showing that the subgroup $H^{1,1}_{\mathbb{Z}}$ of $H^2(X,\mathbb{Z})$ of cohomology classes of type $(1,1)$ is generated by c_1 of line bundles on X. Since homological and algebraic equivalences coincide for divisors, the Néron-Severi group $\mathrm{NS}^1(X)$ coincide with $H^{1,1}_{\mathbb{Z}}$. For such a nonsingular variety X we then have

$$\mathrm{NS}^1(X) = F^1\cap H^2(X,\mathbb{Z}) = H^{1,1}_{\mathbb{Z}} = H^1(X,\mathcal{H}^1_X) = N^1H^2(X,\mathbb{Z}).$$

For $i=-1$ and X possibly singular, the conjecture corresponds to the fact (proved in [4]) that the abelian variety corresponding to $\mathrm{gr}^W_1 H^1$ is

$$\ker^0(\mathrm{Pic}^0(X_0)\to \mathrm{Pic}^0(X_1)).$$

For $i=0$ the Conjecture 2.3.4 is quite easily verified by checking the claimed compatibility of the extension class map. Such a statement then corresponds to a Lefschetz $(1,1)$-theorem for complete varieties with arbitrary singularities.

For $i\geq 1$ we may get *exotic* $(1,1)$*-classes* in the higher cohomology groups $H^{2+i}(X)$ of an higher dimensional singular variety X. We ignore the geometrical meaning of these exotic $(1,1)$-classes. It will be interesting to produce concrete examples. The conjectural picture is as follows.

Let $\pi : X_\bullet \to X$ be an hypercovering. Let $(H^q(\mathcal{H}^1))^\bullet$ be the complex of $E_1^{\bullet,q}$-terms of the spectral sequence in Corollary 3.4.2 for $r=1$. Now $E_1^{i,q} = H^q(X_i,\mathcal{H}^1_{X_i}) = 0$ for $q\geq 2$ (where X_i are the smooth components of the hypercovering $X_\bullet$ of X) and all non-zero terms are pure Hodge structures: therefore the spectral sequence degenerates at E_2. Thus, from Corollary 3.4.2, we get an extension

$$0\to H^{1+i}((H^0(\mathcal{H}^1))^\bullet)\to \mathbb{H}^{1+i}(X_\bullet,\mathcal{H}^1_{X_\bullet})\to H^i((H^1(\mathcal{H}^1))^\bullet)\to 0 \qquad (16)$$

in the category of mixed $\mathbb{Q}$-Hodge structures. We have $H^{i+1}((H^0(\mathcal{H}^1))^\bullet) = H^{i+1}((H^1)^\bullet) = \mathrm{gr}^W_1 H^{2+i}$ and $H^i((H^1(\mathcal{H}^1))^\bullet) = H^i((NS)^\bullet) = H^i((N^1H^2)^\bullet)$.

Moreover, from the local-to-global spectral sequence in Claim 3.4.3 and cohomological descent we get the following edge map

$$s\ell^{1+i} : \mathbb{H}^{1+i}(X_\bullet,\mathcal{H}^1_{X_\bullet})\to W_2H^{2+i}(X)/W_0.$$

In fact, first observe that $W_0H^{2+i}(X) = \mathbb{H}^{2+i}(X_\bullet,\mathcal{H}^0_{X_\bullet})$. From (16) above we then see that $W_0\mathbb{H}^{1+i}(X_\bullet,\mathcal{H}^1_{X_\bullet}) = 0$. The map $s\ell^{1+i}$ is then easily obtained as an edge homomorphism of the cited local-to-global spectral sequence and weight arguments. This cycle map will fit in a diagram

$$\begin{array}{ccccccccc} 0\to & H^{1+i}((H^1)^\bullet) & \to & W_2H^{2+i}(X)/W_0 & \to & H^i((H^2)^\bullet) & \to 0\\ & \uparrow \| & & \uparrow s\ell^{1+i} & & \uparrow & \\ 0\to & H^{1+i}((H^0(\mathcal{H}^1))^\bullet) & \to & \mathbb{H}^{1+i}(X_\bullet, \mathcal{H}^1_{X_\bullet}) & \to & H^i((H^1(\mathcal{H}^1))^\bullet) & \to 0 \end{array}$$

mapping the extension (16) to (15).

Scholium 4.3.1. *The image of the map* $s\ell^{1+i}$ *is* $H^{2+i}(X)^e$.

Following [4] consider the simplicial sheaf $\mathcal{O}^*_{X_\bullet}$ and the corresponding Zariski cohomology groups $\mathbb{H}^{1+i}(X_\bullet, \mathcal{O}^*_{X_\bullet})$. Since the components of $X_\bullet$ are smooth, the canonical spectral sequence

$$E_1^{p,q} = H^q(X_p, \mathcal{O}^*_{X_p}) \Rightarrow \mathbb{H}^{p+q}(X_\bullet, \mathcal{O}^*_{X_\bullet})$$

yields a long exact sequence

$$H^{1+i}((H^0(\mathcal{O}^*))^\bullet) \to \mathbb{H}^{1+i}(X_\bullet, \mathcal{O}^*_{X_\bullet}) \to H^i((\mathrm{Pic}\,))^\bullet) \xrightarrow{d^i} H^{2+i}((H^0(\mathcal{O}^*))^\bullet) \to \cdots$$

According to [4] (see the construction in [2]) we may regard $\mathbb{H}^{1+i}(X_\bullet, \mathcal{O}^*_{X_\bullet})$ as the group of k-points of a group scheme whose connected component of the identity yields a semi-abelian variety

$$0 \to H^{1+i}((H^0(\mathcal{O}^*))^\bullet)/\sigma \to \mathbb{H}^{1+i}(X_\bullet, \mathcal{O}^*_{X_\bullet})^0 \to H^i((\mathrm{Pic}^{\,0}))^\bullet)^0 \to 0$$

where σ is a finite group. The Hodge realization of the so obtained isogeny 1-motive is

$$T_{\mathrm{Hodge}}([0 \to \mathbb{H}^{1+i}(X_\bullet, \mathcal{O}^*_{X_\bullet})^0]_{\mathbb{Q}}) = W_1H^{1+i}(X, \mathbb{Q})(1).$$

This last claim is clearly related to Deligne's conjecture [11, 10.4.1]. For $i = -1, 0$ this is actually proven in [4] and for all i in [2].

Recall the existence of a canonical map of sheaves $c_1 : \mathcal{O}^*_{X_\bullet} \to \mathcal{H}^1_{X_\bullet}$ yielding a map

$$c_1 : \mathbb{H}^{1+i}(X_\bullet, \mathcal{O}^*_{X_\bullet}) \to \mathbb{H}^{1+i}(X_\bullet, \mathcal{H}^1_{X_\bullet}).$$

By composing $s\ell^{1+i}$ and c_1 we then obtain a cycle map

$$\mathbb{H}^{1+i}(X_\bullet, \mathcal{O}^*_{X_\bullet}) \to W_2H^{2+i}(X)/W_0.$$

We may regard the image of this cycle map as the discrete part of $\mathbb{H}^{1+i}(X_\bullet, \mathcal{O}^*_{X_\bullet})$. Over $\mathbb{Q}$, it is clearly equal to $F^1 \cap H^{2+i}(X, \mathbb{Q})$. The reader can easily check that this is the case, *e.g.*, $\mathbb{H}^{2+i}(X_\bullet, \mathcal{O}^*_{X_\bullet}[-1])$ coincides with Deligne-Beilinson cohomology (see [5, 5.4]). In general, we may expect the following picture for cycle maps.

4.4. $\mathcal{K}$-cohomology and motivic cohomology.

Let $X_\bullet$ be a smooth simplicial scheme. Consider the local-to-global spectral sequence in Claim 3.4.3. For a fixed p we then obtain a spectral sequence

$$W_{2p}\mathbb{H}^q(X_\bullet,\mathcal{H}^r_{X_\bullet})/W_{2p-2}\Rightarrow W_{2p}\mathbb{H}^{q+r}(X_\bullet,\mathbb{Q})/W_{2p-2}.$$

The sheaf $\mathcal{H}^r_{X_\bullet}$ has weights $\leq 2r$ and so the mixed Hodge structure on its cohomology has weights $\leq 2r$. Thus $W_{2p}\mathbb{H}^q(X_\bullet,\mathcal{H}^p_{X_\bullet}) = \mathbb{H}^q(X_\bullet,\mathcal{H}^p_{X_\bullet})$ and $W_{2p}\mathbb{H}^q(X_\bullet,\mathcal{H}^r_{X_\bullet})/W_{2p-2} = 0$ if $r < p$. Thus, there is an edge map

$$s\ell^{p+i} : \mathbb{H}^{p+i}(X_\bullet,\mathcal{H}^p_{X_\bullet})/W_{2p-2} \to W_{2p}\mathbb{H}^{2p+i}(X_\bullet)/W_{2p-2}. \tag{17}$$

Note that if $X_\bullet = X$ is constant then $s\ell^{p+0} = s\ell^p_0$ and $s\ell^{p-1} = s\ell^p_{-1}$ in the notation of Section 4.2, modulo W_{2p-2}.

Scholium 4.4.1. *The image of the edge map $s\ell^{p+i}$ is $H^{2p+i}(X)^h$.*

Consider Quillen's higher $K-$theory. Consider Zariski sheaves associated to Quillen's K-functors. The $\mathcal{K}$-cohomology groups are $\mathbb{H}^*(X_\bullet,\mathcal{K}_p)$ (as usual we consider Zariski simplicial sheaves $\mathcal{K}_p$). Local higher Chern classes give us maps of simplicial sheaves $c_p : \mathcal{K}_p \to \mathcal{H}^p_{X_\bullet}(p)$ for each $p \geq 0$ (*cf.* [1]). We thus obtain a map

$$c_p : \mathbb{H}^{p+i}(X_\bullet,\mathcal{K}_p) \to \mathbb{H}^{p+i}(X_\bullet,\mathcal{H}^p_{X_\bullet}(p)). \tag{18}$$

Note that in the canonical spectral sequence

$$E_1^{s,t} = H^t(X_s,\mathcal{K}_p)\Rightarrow\mathbb{H}^{s+t}(X_\bullet,\mathcal{K}_p)$$

we have $H^t(X_s,\mathcal{K}_p) = 0$ if $t > p$. The same hold for the sheaf $\mathcal{H}^p_{X_\bullet}$. We then have a commutative square

$$\begin{array}{ccc}
\mathbb{H}^{p+i}(X_\bullet,\mathcal{K}_p) & \stackrel{c_p}{\to} & \mathbb{H}^{p+i}(X_\bullet,\mathcal{H}^p_{X_\bullet}(p)) \\
\downarrow & & \downarrow \\
H^i((CH^p)^\bullet) & \to & H^i((NS^p)^\bullet).
\end{array}$$

Thus the image of c_p in $H^i((NS^p)^\bullet)$ is clearly contained in the kernel of the map λ^i_a defined in (4). We also have the following commutative square

$$\begin{array}{ccc}
\mathbb{H}^{p+i}(X_\bullet,\mathcal{H}^p_{X_\bullet})/W_{2p-2} & \stackrel{s\ell^{p+i}}{\to} & W_{2p}\mathbb{H}^{2p+i}(X_\bullet)/W_{2p-2} \\
\downarrow & & \downarrow \\
H^i((NS^p)^\bullet) & \to & \mathrm{gr}_{2p}\mathbb{H}^{2p+i}(X_\bullet).
\end{array}$$

Composing $s\ell^{p+i}$ and c_p above we then obtain a simplicial cycle map

$$c\ell^{p+i} : \mathbb{H}^{p+i}(X_\bullet,\mathcal{K}_p) \to W_{2p}H^{2p+i}(X_\bullet)/W_{2p-2}. \tag{19}$$

Let X be a proper $\mathbb{C}$-scheme and let $X_\bullet \to X$ be a universal cohomological descent morphism. By descent, $\mathbb{H}^*(X_\bullet) \cong H^*(X)$ as mixed Hodge structures. Let H denote the mixed Hodge structure on $H^{2p+i}(X,\mathbb{Z})/(\text{torsion})$. Let F^p denote the Hodge filtration. Note that

$$F^p \cap H_{\mathbb{Z}} = \operatorname{Hom}_{\mathrm{MHS}}(\mathbb{Z}(-p), H) = \operatorname{Hom}_{\mathrm{MHS}}(\mathbb{Z}(-p), W_{2p}H).$$

Moreover

$$\operatorname{Hom}_{\mathrm{MHS}}(\mathbb{Z}(-p), W_{2p}H) \subseteq \operatorname{Hom}_{\mathrm{MHS}}(\mathbb{Z}(-p), W_{2p}H/W_{2p-2}H) = F^p \cap H^e_{\mathbb{Z}}.$$

Thus

$$F^p \cap H_{\mathbb{Z}} \subseteq F^p \cap H^e_{\mathbb{Z}} = F^p \cap H^h_{\mathbb{Z}} = \ker(H^{p,p}_{\mathbb{Z}} \xrightarrow{e^p} J^p(H)).$$

Therefore we have the following natural question.

Problem 4.4.2. *Let X be a proper $\mathbb{C}$-scheme and let $\pi : X_\bullet \to X$ be a proper smooth hypercovering. Is $F^p \cap H^{2p+i}(X,\mathbb{Q})$ the image of the cycle class map $c\ell^{p+i}$ in (19) ?*

Bloch's counterexample answer this question in the negative (see Section 5.1 below). Note that here we actually deal with the simplicial scheme $X_\bullet$ (not just X) as, *e.g.*, in [3] it is shown that $F^1 \cap H^2(X,\mathbb{Z})$ can be larger than the image of $\operatorname{Pic}(X)$, if X is singular. However, $F^1 \cap H^2(X,\mathbb{Z})$ is the image of the Pic of any hypercovering of X. However, $F^2 \cap H^4(X,\mathbb{Q})$ is larger than the image of $\mathbb{H}^2(X_\bullet, \mathcal{K}_2)$ if X is the singular 3-fold in Section 5.1 below.

Let's then consider the case $p = 2$ in the above. In this case we have that $H^q(X_i, \mathcal{H}^2_{X_i})$ is purely of weight $q+2$ by the coniveau spectral sequence (14). Thus the canonical spectral sequence in Corollary 3.4.2 degenerates yielding the following extension

$$0 \to H^{1+i}((H^1(\mathcal{H}^2))^\bullet) \to \mathbb{H}^{2+i}(X_\bullet, \mathcal{H}^2_{X_\bullet})/W_2 \to H^i((NS^2)^\bullet) \to 0. \tag{20}$$

Note that $H^1(X_i, \mathcal{H}^2_{X_i}) = N^1H^3(X_i)$ and $W_2\mathbb{H}^j(X_\bullet, \mathcal{H}^2_{X_\bullet}) = H^j((H^0(\mathcal{H}^2))^\bullet)$. The map in (17) is mapping the extension (20) to the following canonical extension

$$0 \to H^{1+i}((H^3)^\bullet) \to W_4H^{4+i}(X)/W_2 \to H^i((H^4)^\bullet) \to 0.$$

According to Conjectures 2.3.2–2.3.4 we may expect that the image of the mixed Hodge structure $\mathbb{H}^{2+i}(X_\bullet, \mathcal{H}^2_{X_\bullet})/W_2$ under this map is $H^{4+i}(X)^h$.

Finally, making use of the triangulated category of motives (see [39] and [26]) let $H^*_m(X,\mathbb{Q}(\cdot))$ denote the motivic cohomology of the proper $\mathbb{C}$-scheme X. Since motivic cohomology is universal we may get a canonical map $H^{2p+i}_m(X,\mathbb{Q}(p)) \to H^{2p+i}(X,\mathbb{Q}(p))$ compatibly with the weight filtrations. This map will factors trhough Beilinson's absolute Hodge cohomology [5]. However, in general, its image will not be larger than $c\ell^{p+i}$ in (19), *i.e.*, smaller than the rational part of $F^pH^{2p+i}(X,\mathbb{C})$. In fact, we can see that only Beilinson's absolute Hodge cohomology (or Deligne-Beilinson cohomology) would have image equal to $F^2 \cap H^4(X, \mathbb{Q})$ if X is the singular 3-fold in Bloch's counterexample below.

5. Examples

We finally discuss a couple of examples where one can test the conjectures.

5.1. Bloch's example.

We now consider Bloch's example explained in a letter to U. Jannsen, reproduced in the Appendix A of [23] (see also Appendix A.I in [27]). This example, originally requested by Mumford, is a counterexample to a naive extension of the cohomological Hodge conjecture to the singular case. Moreover (as indicated by Bloch's Remark 1 in [23, Appendix A]) it shows that no cohomological invariants of algebraic varieties, that agree with Chow groups of non-singular varieties, can provide all Hodge cycles for singular varieties.

Let P be the blow-up of $\mathbb{P}^3$ at a point x in $S_0 \subset \mathbb{P}^3$ a smooth hypersurface of degree ≥ 4 over $\overline{\mathbb{Q}}$. The point x is assumed $\overline{\mathbb{Q}}$-generic. Let S be the blow-up of S_0 at x over $\mathbb{C}$. Thus $S \subset P$ and $H^3(S, \mathbb{Q}) = 0$.

Let X be the gluing of two copies of P along S, *i.e.*, the singular projective variety defined as the pushout

$$\begin{array}{ccc} S \coprod S & \stackrel{i \coprod i}{\to} & P \coprod P \\ c \downarrow & & \downarrow f \\ S & \stackrel{j}{\to} & X \end{array}$$

Such a Mayer-Vietoris diagram always defines a cohomological descent morphism $X_{\bullet} \to X$ (*e.g.*, a distinguished (semi)simplicial resolution in the sense of Carlson [9, §3 and §13]).

Thus we obtain a short exact sequence

$$0 \to H^4(X, \mathbb{Q}(2)) \to H^4(P, \mathbb{Q}(2))^{\oplus 2} \to H^4(S, \mathbb{Q}(2)) \to 0$$

where $CH^2(P)_{\mathbb{Q}} \cong H^4(P, \mathbb{Q}(2))$. Thus $H^4(X, \mathbb{Z})$ has rank 3, is purely of type $(2,2)$ and the Hodge 1-motive is $[H^4(X, \mathbb{Q}(2)) \to 0]$. From (20) we obtain

$$\mathbb{H}^2(X_{\bullet}, \mathcal{H}^2_{X_{\bullet}})/W_2 = H^0((\mathrm{NS}^2)^{\bullet})) = H^4(X, \mathbb{Q}(2)).$$

Since the Albanese of S vanishes, the algebraically defined Hodge 1-motive is given by $H^0((\mathrm{NS}^2)^{\bullet})) = \ker(NS^2(P)^{\oplus 2} \to NS^2(S))$ and we clearly have that

$$[H^0((\mathrm{NS}^2)^{\bullet})) \to 0] \cong [H^4(X, \mathbb{Q}(2)) \to 0]$$

as predicted by Conjecture 2.3.4. However $H^0((CH^2)^{\bullet})) = \ker(CH^2(P)^{\oplus 2} \to CH^2(S))$ has rank 2, and it is strictly smaller than $H^4(X, \mathbb{Q}(2))$, as Bloch's observed. Moreover, from the above we may regard $\mathbb{H}^2(X_\bullet, \mathcal{K}_2)_\mathbb{Q}$ mapping to both $H^0((CH^2)^\bullet))$ and $H^4(X, \mathbb{Q}(2))$. Then $c\ell^2$ in (19) is not surjective because $H^0((CH^2)^\bullet)) \neq H^4(X, \mathbb{Q}(2))$. The same argument applies to motivic cohomology $H^4_m(X, \mathbb{Q}(2))$. In fact, if Y is a smooth variety $H^4_m(Y, \mathbb{Q}(2)) \cong CH^2(Y)_\mathbb{Q}$. However, Beilinson's absolute Hodge cohomology is $H^4_\mathcal{D}(X, \mathbb{Z}(2)) \otimes \mathbb{Q} \cong H^4(X, \mathbb{Q}(2))$. Thus, Beilinson's formulation of the Hodge conjecture in [5, §6] doesn't hold in the singular case.

Note that in this example, all Hodge classes are involved, as the Hodge structure is pure.

5.2. Srinivas example.

The following example has been produced by Srinivas upon author's request. It is similar to Bloch's example however, in this example, the space of "Hodge cycles" is strictly smaller than $H^{2,2}_\mathbb{Q}$.

Let Y be a smooth projective complex 4-fold with $H^1(Y, \mathbb{Z}) = H^3(Y, \mathbb{Z}) = 0$, and with an algebraic cycle $\alpha \in CH^2(Y)$ whose singular cohomology class $\overline{\alpha} \in H^4(Y, \mathbb{Q})$ is a non-zero primitive class. For example, Y could be a smooth quadric hypersurface in $\mathbb{P}^5$, and $\alpha \in CH^2(Y)$ the difference of the classes of two planes, taken from the two distinct connected families of planes in Y. Let Z be a general hypersurface section of Y of any fixed degree d such that $H^{3,0}(Z) \neq 0$ (this holds for any large enough degree d; for example, if Y is a quadric then we may take $Z = Y \cap H$ to be the intersection with a general hypersurface H of any degree ≥ 3).

Then Z is a smooth projective 3-fold, and by the theorem of Griffiths, if $i : Z \to Y$ is the inclusion, then $i^*\alpha \in CH^2(Z)$ is homologically trivial, but no non-zero multiple of $i^*\alpha$ is algebraically equivalent to 0. In fact, $H^3(Z, \mathbb{Q})$ has no proper Hodge substructures, and so (because $H^{3,0}(Z) \neq 0$) the Abel-Jacobi map vanishes on the group $CH^2(Z)_{\rm alg}$ of cycle classes algebraically equivalent to 0; on the other hand, the Abel-Jacobi image of $i^*\alpha$ is non-torsion.

Now let X be the singular projective variety defined as a push-out

$$\begin{array}{ccc} Z \coprod Z & \stackrel{i \coprod i}{\to} & Y \coprod Y \\ c \downarrow & & \downarrow f \\ Z & \stackrel{j}{\to} & X \end{array}$$

so that X is obtained by gluing two copies of Y along Z.

Consider the simplicial scheme $X_\bullet$ obtained as above (*e.g.*, the Čech hypercovering of X, with $X_0 \to X$ taken to be the quotient map $f : Y \coprod Y \to X$). Then $H^*(X_\bullet, \mathbb{Q}) \cong H^*(X, \mathbb{Q})$ as mixed Hodge structures, and we have an exact sequence of mixed Hodge structures (of which all terms except $H^4(X, \mathbb{Z})$ are in fact pure)

$$0 \to H^3(Z, \mathbb{Z}) \to H^4(X, \mathbb{Z}) \to H^4(Y, \mathbb{Z})^{\oplus 2} \stackrel{s}{\to} H^4(Z, \mathbb{Z})$$

where $s(a,b) = i^*a - i^*b$. Then $(\overline{\alpha}, 0)$ and $(0, \overline{\alpha})$ are linearly independent elements of $\ker s$, since $i^*\overline{\alpha} = 0$ in $H^4(Z, \mathbb{Q})$ (this is essentially the definition of $\overline{\alpha}$ being a primitive cohomology class).

In this situation, the group of Hodge classes in $H^4(X, \mathbb{Q})/W_3$ is non-trivial, but since $H^3(Z, \mathbb{Q})$ has no non-trivial sub-Hodge structures, the intermediate Jacobian $J^2(Z)$ has no non-trivial abelian subvariety. The extension of Hodge structures determined by the Hodge classes is not split; for example the extension class of the pullback of

$$0 \to H^3(Z, \mathbb{Z}) \to H^4(X, \mathbb{Z}) \to \ker s \to 0$$

under $\mathbb{Z}(-2) \to \ker s$ determined by $(\overline{\alpha}, 0)$ is (up to sign) the Abel-Jacobi image of $i^*\alpha$, which is non-torsion. Here $\operatorname{rank}(\ker s) = 3$, so we get an extension class map $\mathbb{Z}^3 \to J^2(Z)$; one checks that the image has rank 1, generated by the image of $(\overline{\alpha}, 0)$ (or equivalently by the image of $(0, \overline{\alpha})$).

So the lattice for the corresponding Hodge 1-motive is, by definition

$$\ker(\mathbb{Z}^3 \to J^2(Z)) = F^2 \cap H^4(X, \mathbb{Z}),$$

which is strictly smaller than the lattice of all Hodge classes in $H^4(X, \mathbb{Z})$. Moreover, since $H^1(Z, \mathcal{H}^2) = N^1H^3(Z) = 0$, from the extension (20) we obtain

$$\mathbb{H}^2(X_\bullet, \mathcal{H}^2_{X_\bullet})/W_2 \cong H^0((NS^2)^\bullet).$$

Finally, the cycle $\alpha \in CH^2(Y)$ projects to a cycle in $\mathrm{NS}^2(Y)$ which restricts to a non-zero class $i^*\alpha \in NS^2(Z)_\mathbb{Q}$ by construction. Since $CH^2(Z)_{\mathrm{ab}} = CH^2(Z)_{\mathrm{alg}}$ then the algebraically defined 1-motive is given by the image of $H^0((NS^2)^\bullet) = \ker(NS^2(Y)^{\oplus 2} \to NS^2(Z))$ in $H^4(X, \mathbb{Z})$, providing generators for $\ker(\mathbb{Z}^3 \to J^2(Z))$ as claimed in Conjecture 2.3.4.

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