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Infinitesimal Cohomology

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Introduzione

Nonostante la definizione e lo studio delle forme differenziali risalga almeno ai tempi di Eulero, è stato solamente con il teorema di de Rham che è stata messa alla luce la stretta relazione tra questi oggetti e le intrinseche proprietà topologiche delle varietà differenziabili, permettendo così un trattamento più algebrico del tema. Successivamente, nel tentativo di adattare metodi dell'analisi e della geometria complessa a diversi contesti, Kähler estese la definizione delle forme differenziali ad anelli unitari arbitrari, permettendo così di costruire un complesso di de Rham puramente algebrico. E' stato Grothendieck a provare in [15] la compatibilità tra la teoria classica di de Rham e quella definita algebricamente dimostrando che per uno schema affine non-singolare X definito sul campo complesso $\mathbb{C}$, esiste un isomorfismo canonico tra la coomologia di de Rham e la coomologia singolare del suo spazio analitico associato X^{an}:

$$H^\bullet_{dR}(X/\mathbb{C}) \xrightarrow{\cong} H^\bullet_{dR^{an}}(X^{an}) \xrightarrow{\cong} H^\bullet(X^{an}, \mathbb{C})$$

Nel caso più generale in cui X sia uno schema liscio su un campo k di caratteristica 0, la coomologia di de Rham algebrica ha tutte le buone proprietà richieste per essere definita coomologia di Weil, ma lo stesso non vale se il campo k ha caratteristica $p > 0$. Nel tentativo di definire una teoria coomologica soddisfacente nel caso di caratteristica positiva, Grothendieck introdusse tramite il linguaggio di siti e topos la coomologia ℓ-adica; quest'ultima tuttavia è una coomologia di Weil solo per varietà definite su campi di caratteristica p diversa da ℓ. Per colmare questa lacuna, nella ricerca di una buona teoria di coomologia p-adica, Grothendieck iniziò a sviluppare la coomologia cristallina e il suo analogo in caratteristica 0, la coomologia infinitesimale.

L'obiettivo di questo progetto di tesi è quello di ripercorrere il testo originale [17] con cui Grothendieck introdusse la coomologia infinitesimale e presentare una dimostrazione dettagliata del seguente teorema di confronto:

Theorem. *Sia S un $\mathbb{Q}$-schema e sia X uno schema liscio su S, allora esiste un isomorfismo:*

$$H^\bullet(X_{Inf}, \mathcal{O}_{X_{inf}}) \xrightarrow{\cong} H^\bullet_{dR}(X/S)$$

Il concetto fondamentale che permette di provare il teorema riguarda il fatto che si possa sostituire l'utilizzo delle forme differenziali con il comportamento differenziale riflesso dalle strutture di stratificazione dello spazio. Queste ultime sono delle generalizzazioni del concetto di connessione su di uno schema e uno dei passaggi chiave della dimostrazione riguarda l'equivalenza tra queste strutture di stratificazione e alcuni fasci, chiamati cristalli, sul sito infinitesimale. Come affermato da Grothendeick stesso, l'importanza di questo teorema è che fornisce una definizione di coomologia di de Rham senza utilizzare le forme differenziali, permettendo così di studiare la coomologia di de Rham anche in contesti più generali dove è noto che sia problematica, per esempio nel caso di schemi singolari o di caratteristica positiva. In effetti la coomologia cristallina, che si rivelò essere una coomologia di Weil per il caso di caratteristica $p > 0$, è solamente una generalizzazione della coomologia infinitesimale.

Il primo capitolo della tesi è dedicato allo sviluppo di nozioni tipicamente specifiche dell'analisi e della geometria diffferenziale, come operatori differenziali, connessioni e curvatura, ma con un linguaggio completamente algebrico. In particolare, seguendo [16] e [2], definiremo il concetto di stratificazione su un $\mathcal{O}_X$-modulo, che sarà di centrale importanza nel seguito, e ne daremo diverse descrizioni equivalenti.

Nel secondo capitolo introduciamo il sito infinitesimale e stratificante, in generale un fascio su questi siti può essere identificato come un particolare sistema di diversi fasci di Zariski, ma la definizione del sito infinitesimale e stratificante è motivata prevalentemente dal fatto che ogni $\mathcal{O}_X$-modulo fortificato con una stratificazione, ha una descrizione equivalente come speciale fascio, detto cristallo, su questi siti. Successivamente viene affrontato il problema del calcolo della coomologia sui topos relativi ai nuovi siti introdotti. La coomologia in un topos è definita in modo generale tramite il linguaggio dei funtori derivati, ma per una teoria di coomologia soddisfacente innanzitutto occorre ottenere le necessarie proprietà di funtorialità tra topos, mentre per il calcolo effettivo della coomologia sul topos infinitesimale e stratificante, sarà necessario esplicitare alcune relazioni dei nuovi topos con quello di Zariski.

Nel terzo capitolo, seguendo [17], viene finalmente dimostrato il teorema di confronto tra coomologia infinitesimale e coomologia di de Rham. Utilizzando i risultati del capitolo precedente è facile mostrare come la coomologia stratificante di un $\mathcal{O}_X$-modulo fortificato con una struttura di stratificazione possa essere interpretata come l'ipercoomologia di un complesso di operatori differenziali. Una buona parte del terzo capitolo è dedicata a mostrare che valga anche il viceversa, ossia l'ipercoomologia di Zariski di un complesso di operatori differenziali può essere espressa come l'ipercoomologia stratificante di un complesso di fasci stratificati. Per farlo viene costruito un appropriato funtore L, detto linearizzante, dalla categoria di $\mathcal{O}_X$-moduli e operatori differenziali alla categoria degli $\mathcal{O}_X$-moduli fortificati con una struttura di stratificazione. Applicando questa costruzione generale al complesso di de Rham $\Omega^\bullet_{X/S}$ otterremo la tesi. Nella fine del capitolo, seguendo [9] viene brevemente descritta una interpretazione più moderna della relazione tra coomologia infinitesimale e di Zariski, permettendo così di abbozzare la dimostrazione di una generalizzazione al teorema di Grothendieck.

Alla fine della tesi sono presenti tre appendici.

Nella prima appendice vengono richiamate alcune definizioni e teoremi fondamentali riguardanti la coomologia di de Rham algebrica. Viene sviluppato poi in modo particolare il concetto di connessione per un $\mathcal{O}_X$-modulo e relativa curvatura.

La seconda appendice è una breve presentazione dei principali strumenti di algebra omologica utilizzati nella tesi fin dall'inizio. In particolare, vengono affrontati i temi delle sequenze spettrali, ipercoomologia, funtori e categorie derivate. Inoltre viene descritta la stretta connessione esistente tra l'algebra omologica e quella omotopica, l'interpretazione moderna della dimostrazione di Grothendieck esposta nel finale della tesi, è basata interamente su questa relazione.

L'ultima appendice è un'esposizione rapida alla teoria dei topos di Grothendieck. Sono presenti varie definizioni necessarie per la comprensione del secondo capitolo della tesi. In particolare viene introdotto il concetto di coomologia di fasci in un topos, e di coomologia di Čech.

Introduction

Even though the study of differential forms dates back at least to the time of Euler, it was until the work of de Rham, who showed the existence of an isomorphism between the singular cohomology groups of a smooth manifold and, what we now call, its de Rham cohomology groups, that the relationship between these objects and the intrinsic topological properties of the manifold was firmly established and a modern and more algebraic treatment was allowed. Later, trying to adapt methods from calculus and complex geometry to different contexts, Kähler provided an extension of differential forms for arbitrary commutative unitary rings, thus allowing to construct a purely algebraic de Rham complex. It was Grothendieck in [15] who then showed that this algebraic de Rham theory was compatible with what was already known by proving that for an affine non-singular scheme X over the field of complex numbers $\mathbb{C}$, its de Rham cohomology is the same as the singular cohomology of its associated analytic space X^{an}:

$$H^{\bullet}_{dR}(X/\mathbb{C}) \xrightarrow{\cong} H^{\bullet}_{dR^{an}}(X^{an}) \xrightarrow{\cong} H^{\bullet}(X^{an}, \mathbb{C})$$

In the more general case where X is a scheme over a field k of characteristic 0, the de Rham cohomology is a theory which satisfies all the desirable formal properties and indeed is an example of a Weil cohomology theory, however if k is of characteristic $p > 0$, the de Rham cohomology no longer has good properties. In the attempt to construct a satisfactory cohomology theory in the case of positive characteristic $p > 0$, Grothendieck introduced, by means of the new language of sites and topos, the ℓ-adic cohomology which is a Weil cohomology theory for variety defined over a field of characteristic p different from ℓ. Thus, to fill this gap and trying to define a good p-adic cohomology that could complement the ℓ-adic cohomology, Grothendieck introduced crystalline cohomology and its characteristic zero version, infinitesimal cohomology.

The aim of this dissertation is to present a detailed proof of the following comparison theorem, following closely the original text [17] where Grothendieck introduced infinitesimal cohomology:

Theorem. *Let S be a $\mathbb{Q}$-scheme, if X is smooth over S, then there is a canonical isomorphism:*

$$H^{\bullet}(X_{Inf}, \mathcal{O}_{X_{inf}}) \xrightarrow{\cong} H^{\bullet}_{dR}(X/S)$$

The fundamental concept that allows us to prove the theorem concerns the fact that the use of differential forms can be replaced with the differential behavior reflected on the stratification structure of the space. The latter are just an extension of the concept of connection over a scheme, and one of the key step toward the proof of the comparison theorem is the fact that there is an equivalence between these stratification structures and some sheaves over the infinitesimal site called crystals. As Grothendieck stated, the significance of this theorem lies in the fact that gives a description of the de Rham cohomology without using differential forms, and it allows to study de Rham cohomology in more general context where is known to be problematic, for example non smoothness and positive characteristic. In fact crystalline cohomology, which turned out to be a Weil cohomology for the case of characteristic $p > 0$, it's only a generalization of the infinitesimal cohomology.

In the first chapter of the thesis, following [16] and [2] we will study the Grothendieck's way of geometrizing the notions of calculus and differential geometry, such as differential operators, connection and curvature. In particular we will give various presentations of the concept of a stratification on a module, which will be of central importance later.

The second chapter is dedicated to the definition and analysis of the main properties of the infinitesimal and stratifying site. In particular a sheaf over one of these sites can be identified with a system of different Zariski sheaves. The definition of the infinitesimal and stratifying sites are motivated by the fact that we can describe a stratification over an $\mathcal{O}_X$-module as a special sheaf, called crystal, over these sites. Then we have to face the problem of calculating the cohomology over the sites just introduced. Sheaf cohomology over a topos is defined by means of the language of derived functors, but to obtain a satisfactory cohomology theory first of all it is necessary to obtain some functoriality properties. Secondly, for the explicit calculation of the infinitesimal and stratifying cohomology, it will be useful to find some relations between the new topos and the Zariski topos.

In the last chapter, following [17] finally we give a proof of the comparison theorem between infinitesimal cohomology and de Rham cohomology. Using the results of the previous chapter it is easy to see that the stratifying cohomology of a stratified sheaf can be interpreted as the Zariski hypercohomology of a complex of differential operators, however it's more difficult to show that also the converse is true, namely the Zariski hypercohomology of any complex of differential operators can be expressed as the stratifying hypercohomology of a suitable complex of stratified sheaves. To do that we will construct a functor L from the category of $\mathcal{O}_X$-modules and differential operators to the category of $\mathcal{O}_X$-modules fortified with a stratification structure. Applying this construction to the de Rham complex $\Omega^\bullet_{X/S}$, we will obtain the thesis. In the final part of the chapter we just want to briefly discuss a more modern interpretation of the relation between the infinitesimal and Zariski topos which can be found in [9], together with the sketch of the proof of a slight extension of Grothendieck's comparison theorem.

At the end of the dissertation there are featured three appendixes.

The first one collects some fundamental definitions and theorems about algebraic de Rham cohomology. In particular we develop the concept of connection of an $\mathcal{O}_X$-module and relative curvature.

The second appendix is a short presentation of the main homological algebra tools used during the thesis from the very beginning. In particular are developed themes such as spectral sequences, hypercohomology, derived functors and derived categories. The last argument concerns the close connection between homological and homotopical algebra, the modern interpretation of Grothendieck's proof presented at the end of the thesis is based entirely on this relationship.

The last appendix is a brief exposition of the theory of Grothendieck's topos. There are various definitions which are necessary for the understanding of the second chapter of the thesis, in particular it is introduced the concept of cohomology of sheaves in a topos and Čech cohomology.

1 Calculus and Differential Operators

In this section, following [1],[2] and [16] we will study the Grothendieck's way of geometrizing the notions of calculus and differential geometry, in particular we will give various presentations of the concepts of connection and stratification on a module.

Let $X \to S$ be a morphism of schemes, and let F, G be $\mathcal{O}_X$-modules, then a differential operator from F to G will be an $f^{-1}(\mathcal{O}_S)$-linear map $h : F \to G$, which is "almost" $\mathcal{O}_X$-linear. To motivate the definition we can consider the adjoint map:

$$\bar{h} : \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} F \to G$$

Using the $\mathcal{O}_X$-module structure on F, if we denote $\mathcal{P}_{X/S} = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} \mathcal{O}_X$ the sheaf of principal parts, we obtain also a natural isomorphism: $\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} F \xrightarrow{\cong} \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} F$ so that we can define the map above as:

$$\bar{h} : \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} F \to G \qquad \bar{h}(a \otimes b \otimes x) \to ah(bx)$$

Notice that we can consider two canonical maps $d_0, d_1 : \mathcal{O}_X \to \mathcal{P}_{X/S}$ sending $a \to a \otimes 1$ and $a \to 1 \otimes a$, and so we obtain two different structure of $\mathcal{O}_X$-algebra over $\mathcal{P}_{X/S}$. Now writing $\mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} F$ we are using the map d_1 for the construction of the tensor product, and d_0 to obtain a $\mathcal{O}_X$-module structure on the result. Hence the original map $h : F \to G$ induces a unique $\mathcal{O}_X$-linear map $\bar{h} : \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} F \to G$ such that $\bar{h} \circ d_{1,F} = h$, where we denoted the map $(d_1 \otimes \mathrm{id}_F) = d_{1,F}$. Now if we consider $\mathcal{I} \subseteq \mathcal{P}_{X/S}$ the sheaf of ideal defined by the kernel of the multiplication map, i.e. the sheaf of ideal defined by the diagonal Δ in $X \times_S X$ (which is generated by element of the form $1 \otimes b - b \otimes 1$ with $b \in \mathcal{O}_X$), we can easily see that:

$$h(bx) = bh(x) \quad \forall b \in \mathcal{O}_X \iff \bar{h}(1 \otimes b \otimes x) = \bar{h}(b \otimes 1 \otimes x)$$

this means that h is $\mathcal{O}_X$-linear if and only if $\bar{h}$ annihilates $\mathcal{I} \otimes F$. With this argument in mind we will define a differential operator of order n as a $f^{-1}(\mathcal{O}_S)$-linear map $h : F \to G$ such that $\bar{h}$ annihilates $\mathcal{I}^{n+1} \otimes F$, hence:

$$F \xrightarrow{d_1^n \otimes id_F} \mathcal{P}_{X/S}/\mathcal{I}^{n+1} \otimes F \xrightarrow{\bar{h}^n} G, \qquad h = \bar{h}^n \circ (d_1^n \otimes id_F) : F \dashrightarrow G$$

1.1 Sheaves of Principal parts

To formalize the previous discussion it is essential to define $\mathcal{P}^n_{X/S}$ the sheaves of principal parts of order $n \geq 0$. In the standard references that we will follow these sheaves have different but equivalent definitions, the first definition has a clear geometric interpretation, but the second is more explicit.

Definition 1.1. Let $(g, g^\sharp) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ a morphism of schemes, such that the morphism:

$$g^\sharp : g^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y$$

is surjective, so that we can identify $\mathcal{O}_Y$ with a quotient sheaf of rings $g^{-1}(\mathcal{O}_X)/\mathcal{J}$. We call the n-th infinitesimal neighborhood of Y for the morphism g the scheme $Y_g^{(n)} = (Y, g^{-1}(\mathcal{O}_X)/\mathcal{J}^{n+1})$.

Note that $\{\mathcal{O}_{Y_g^{(n)}}\}_n$ form a projective system of sheaves of rings with canonical transition morphisms:

$$\varphi_{n,m} : \mathcal{O}_{Y_g^{(m)}} \to \mathcal{O}_{Y_g^{(n)}}$$

for $n \leq m$. In particular we define the augmentation ideal of $\mathcal{O}_{Y_g^{(n)}}$ as $\mathcal{J}/\mathcal{J}^{n+1}$ i.e. the kernel of $\varphi_{0,n} : \mathcal{O}_{Y_g^{(n)}} \to \mathcal{O}_Y$. Since all the $Y_g^{(n)}$ have the same underlying topological space Y, they form an inductive system of schemes.

Grothendieck defines the sheaf of principal parts of X over S of order n as the structural sheaf of the n-th neighborhood of the diagonal morphism $\Delta : X \to X \times_S X$, we will use the following notation:

$$\Delta^n_{X/S} = (X, \mathscr{P}^n_{X/S}) \quad \text{instead of} \quad X^{(n)}_\Delta = (X, \Delta^{-1}(\mathcal{O}_{X\times X})/\mathcal{J}^{n+1})$$

more explicitly we can give the following definition:

Definition 1.2. [16] Let $f : X \to S$ be a morphism of schemes and consider the diagonal morphism $(\Delta, \Delta^\sharp) : (X, \mathcal{O}_X) \to (X \times_S X, \mathcal{O}_{X\times_S X})$ which is a locally closed embedding, so that we can define a sheaf of ideal $\mathcal{J} = \text{Ker}(\Delta^\sharp)$ and we obtain a short exact sequence:

$$0 \to \mathcal{J} \to \Delta^{-1}\mathcal{O}_{X\times_S X} \xrightarrow{\Delta^\sharp} \mathcal{O}_X \to 0$$

the sheaves of principal parts of order n are the quotient sheaves:

$$\mathscr{P}^n_{X/S} = (\Delta^{-1}\mathcal{O}_{X\times_S X})/\mathcal{J}^{n+1}$$

When X/S is separated then we also have that $\mathscr{P}^n_{X/S} \cong \Delta^{-1}(\mathcal{O}_{X\times_S X}/\mathcal{K}^{n+1})$ where $\mathcal{K}$ is the kernel of $\mathcal{O}_{X\times_S X} \to \Delta_*\mathcal{O}_X$.

We can give two different $\mathcal{O}_X$-algebra structure to $\mathscr{P}^n_{X/S}$. Consider the canonical projections $p_0, p_1 : X \times_S X \to X$, since the diagonal morphism Δ is a section of $X \times_S X$ for both p_0, p_1, from the composition of morphism of schemes:

$$(X, \mathscr{P}^n_{X/S}) \xrightarrow{h_n} (X \times_S X, \mathcal{O}_{X\times_S X}) \xrightarrow{p_i} (X, \mathcal{O}_X)$$

with $h_n = (\Delta, \Delta^\sharp_n)$ and $\Delta^\sharp_n : \Delta^{-1}(\mathcal{O}_{X\times_S X}) \to \Delta^{-1}(\mathcal{O}_{X\times_S X})/\mathcal{J}^{n+1}$, we obtain maps:

$$(p_i \circ \Delta)^{-1}(\mathcal{O}_X) = \mathcal{O}_X \to \mathscr{P}^n_{X/S}$$

which are right inverse of the canonical augmentation $\mathscr{P}^n_{X/S} \to \mathcal{O}_X$ and hence ([16]16.1.5) they give to every $\mathscr{P}^n_{X/S}$ the structure of a quasi-coherent $\mathcal{O}_X$-algebra. In particular we have:

$$\begin{aligned} &p_0^{(n)} : \Delta^n_{X/S} \xrightarrow{h_n} X \times_S X \xrightarrow{p_0} X \quad && p_1^{(n)} : \Delta^n_{X/S} \xrightarrow{h_n} X \times_S X \xrightarrow{p_1} X \\ &d^n_0 : \mathcal{O}_X \to \mathscr{P}^n_{X/S} && d^n_1 : \mathcal{O}_X \to \mathscr{P}^n_{X/S} \end{aligned} \tag{1.2.1}$$

We will refer to the $\mathcal{O}_X$-module structure induced by d^n_0 as the left structure and to that one induced by d^n_1 as the right structure. If not otherwise stated we will use the left $\mathcal{O}_X$-module

structure on $\mathscr{P}^n_{X/S}$, and we will denote the other canonical morphism as $d_1^n := d^n$.
Note that we can also identify:

$$\mathscr{P}^n_{X/S} = ((p_0^{(n)})_*(p_1^{(n)})^*(\mathcal{O}_X)) \tag{1.2.2}$$

so that more in general, given F a $\mathcal{O}_X$-module, we can define:

$$\mathscr{P}^n_{X/S}(F) = \mathscr{P}^n_{X/S} \otimes_{\mathcal{O}_X} F = ((p_0^{(n)})_*(p_1^{(n)})^*(F))$$

were to form the tensor product $\otimes$ we use the right structure, and we use the left structure to give a $\mathcal{O}_X$-module structure on the result, so that for all open subset $U \subset X$ if $a \in \Gamma(U, \mathcal{O}_X)$, $b \in \Gamma(U, \mathscr{P}^n_{X/S})$, $t \in \Gamma(U, F)$ we have:

$$a(b \otimes t) = (ab) \otimes t \qquad (b \otimes t)a = b \otimes (at) = (b.d^n a) \otimes t = (d^n a).(b \otimes t) \tag{1.2.3}$$

Note that for every $\mathcal{O}_X$-module F we have also a canonical map:

$$d^n_{X/S,F} : F \to \mathscr{P}^n_{X/S}(F) \qquad t \to 1 \otimes t \tag{1.2.4}$$

such that:

$$\begin{aligned} d^n_{X/S,F}(at) =& (1 \otimes t)a = (d^n_{X/S,F}(t)).a \\ d^n_{X/S,F}(at) =& (d^n_{X/S}(a).(1 \otimes t)) = d^n_{X/S}(a)(d^n_{X/S,F}(t)) \end{aligned} \tag{1.2.5}$$

so that $d^n_{X/S,F}$ is $\mathcal{O}_X$-linear for the right structure of $\mathscr{P}^n_{X/S}(F)$ but not for the left structure.

Proposition 1.3. *As a left $\mathcal{O}_X$-module $\mathscr{P}^n_{X/S}(F)$ is generated by the image of F through $d^n_{X/S,F}$.*

Proof. First of all we prove the special case $\mathcal{O}_X = F$. In this case we want to show that $\mathscr{P}^n_{X/S}$ is generated by the image of $d^n_{X/S} : \mathcal{O}_X \to \mathscr{P}^n_{X/S}$. It is sufficient to check it on affine, so that we can consider $X = \text{Spec}(B)$ and $S = \text{Spec}(A)$. From the definition we have that $\mathscr{P}^n_{X/S}$ is the structural sheaf of the affine scheme $\text{Spec}(P^n_{B/A})$ where:

$$P^n_{B/A} = (B \otimes_A B)/I^{n+1}$$

and I is the kernel of the multiplication map $\mu : B \otimes_A B \to B$. We can always consider the canonical surjective map:

$$\pi_n : B \otimes_A B \to P^n_{B/A}$$

and from the definition we will have:

$$\pi_n(b \otimes b') = b.\pi_n(1 \otimes b') = b.d^n_{B/A}(b')$$

hence in general the image of $d^n_{X/S}$ generates $\mathscr{P}^n_{X/S}$ as a left $\mathcal{O}_X$-module. The general case follows easily from this and the characterization that we gave of the elements $d^n_{X/S,F}(at)$ in (1.2.3) and in (1.2.5). $\square$

We can also give a different and more explicit definition of the sheaves of principal parts for a morphism of schemes.

Definition 1.4. [2] Let $f : X \to S$ be a morphism of schemes, consider the tensor product sheaf $\mathcal{P}_{X/S} = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} \mathcal{O}_X$, the sheaves of principal parts of X/S of order n are the quotient sheaves:

$$\mathcal{P}^n_{X/S} = (\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} \mathcal{O}_X)/(\mathcal{I}^{n+1})$$

where $\mathcal{I}$ is the sheaf of ideal defined as the kernel of the multiplication map from the tensor product to $\mathcal{O}_X$, so that we have an exact sequence:

$$0 \to \mathcal{I} \to \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} \mathcal{O}_X \to \mathcal{O}_X \to 0$$

we can obtain two different structure of $\mathcal{O}_X$-algebra over each $\mathcal{P}^n_{X/S}$ via the two canonical maps d^n_0, d^n_1 coming from the natural map of the tensor product and passing through the quotient:

$$\mathcal{O}_X \xrightarrow{d_i} \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} \mathcal{O}_X \xrightarrow{\pi_n} \mathcal{P}^n_{X/S}$$

Remark 1.5. The definitions of sheaves of principal parts 1.2 and 1.4 are equivalent. To prove the equivalence we first need to find maps from $\mathcal{P}^n_{X/S}$ to $\mathscr{P}^n_{X/S}$, so let's consider the following canonical $f^{-1}(\mathcal{O}_S)$-linear morphism of sheaves for $i = 0, 1$:

$$\mathcal{O}_X \to p_{i*}\mathcal{O}_{X\times_S X} \to p_{i*}\Delta_*\Delta^{-1}\mathcal{O}_{X\times_S X} = (p_i\Delta)_*\Delta^{-1}\mathcal{O}_{X\times_S X} = \Delta^{-1}\mathcal{O}_{X\times_S X}$$

from the universal property of tensor product, we get a map $\phi : \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} \mathcal{O}_X \to \Delta^{-1}\mathcal{O}_{X\times_S X}$. In general ϕ need not to be an isomorphism, indeed if we compute the stalk at some point $x \in X$:

$$(\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} \mathcal{O}_X)_x = \mathcal{O}_{X,x} \otimes_{f^{-1}(\mathcal{O}_S),f(x)} \mathcal{O}_{X,x} \tag{1.5.1}$$

$$(\Delta^{-1}\mathcal{O}_{X\times_S X})_x = \mathcal{O}_{X\times_S X,\Delta(x)} = (\mathcal{O}_{X,x} \otimes_{f^{-1}(\mathcal{O}_S),f(x)} \mathcal{O}_{X,x})_{\mathfrak{q}} \tag{1.5.2}$$

where $\mathfrak{q}$ is the kernel of the canonical homomorphism:

$$\mathcal{O}_{X,x} \otimes_{f^{-1}(\mathcal{O}_S),f(x)} \mathcal{O}_{X,x} \to \kappa(x) \qquad a \otimes b \to \overline{ab}$$

we can see that the first stalk (1.5.1) need not to be a local ring whereas the second one (1.5.2) is always a local ring. However ϕ induce an isomorphism between $\mathcal{P}^n_{X/S}$ and $\mathscr{P}^n_{X/S}$, to see this again we can consider the stalks:

$$\phi_x : \mathcal{P}^n_{X/S_x} \to \mathscr{P}^n_{X/S_x}$$

$$\phi_x : (\mathcal{O}_{X,x} \otimes_{f^{-1}(\mathcal{O}_S),f(x)} \mathcal{O}_{X,x})/\mathfrak{p}^n \to ((\mathcal{O}_{X,x} \otimes_{f^{-1}(\mathcal{O}_S),f(x)} \mathcal{O}_{X,x})/\mathfrak{p}^n)_{\mathfrak{q}}$$

where $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p}$ is the kernel of the multiplication map:

$$\mathcal{O}_{X,x} \otimes_{f^{-1}(\mathcal{O}_S),f(x)} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$$

Notice that $\mathfrak{p}^n$ and $\mathfrak{p}^n_{\mathfrak{q}}$ are coming from different sheaf of ideal, namely the former is the stalk of $\mathcal{I}$, the kernel of $\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} \mathcal{O}_X \to \mathcal{O}_X$, and the latter is the stalk of $\mathcal{J}$, the kernel of $\Delta^\sharp : \Delta^{-1}\mathcal{O}_{X\times_S X} \to \mathcal{O}_X$. That being said, for every $x \in X$ we want to show that ϕ_x is an isomorphism or rather that the localization at $\mathfrak{q}$ is not needed; this means that every element in $\mathcal{O}_{X,x} \otimes_{f^{-1}(\mathcal{O}_S),f(x)} \mathcal{O}_{X,x}$ whose image in $\mathcal{O}_{X,x}$ is a unit, actually is already invertible modulo $\mathfrak{p}^n$. To do this remember that we have a canonical projection:

$$(\mathcal{O}_{X,x} \otimes_{f^{-1}(\mathcal{O}_S),f(x)} \mathcal{O}_{X,x})/\mathfrak{p}^n \to (\mathcal{O}_{X,x} \otimes_{f^{-1}(\mathcal{O}_S),f(x)} \mathcal{O}_{X,x})/\mathfrak{p} \cong \mathcal{O}_{X,x}$$

and we want that preimages of units are units, but this is true because the kernel of the projection is the nilpotent ideal $\mathfrak{p}^1/\mathfrak{p}^n$. It is indeed a general fact that a ring homomorphism $f : A \to A/I$ with I a nilpotent ideal, induce a surjective morphism on the underlying group of units $f^\times : A^\times \to (A/I)^\times$. If I is nilpotent, then of course I is contained in the Jacobson radical of A, hence maximal ideal of A/I correspond to maximal ideal of A because they always contains I, and since the units are exactly those elements not contained is some maximal ideal then of course every preimage of units is a unit.

Given the equivalence of the two definitions, from now on we will use $\mathcal{P}^n_{X/S}$ to denote the n-th sheaf of principal parts of X/S.

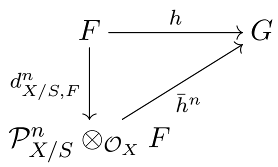
1.2 Differential Operators

Let $f : X \to S$ be a morphism of schemes, in (1) we were trying to linearize a $f^{-1}(\mathcal{O}_S)$-linear map of $\mathcal{O}_X$-modules $h : F \to G$. To do that we have defined explicitly the sheaf of principal parts of n-th order as $\mathcal{P}^n_{X/S} = (\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} \mathcal{O}_X)/\mathcal{I}^{n+1}$, and from the property of the tensor product we have also obtained two canonical $\mathcal{O}_X$-linear map:

$$d_0^n, d_1^n : \mathcal{O}_X \to \mathcal{P}^n_{X/S} \qquad a \to a \otimes 1 \qquad a \to 1 \otimes a \tag{1.5.3}$$

then we have used d_0 to give a $\mathcal{O}_X$-algebra structure over $\mathcal{P}^n_{X/S}$ and we have used d_1 to define a factorization of $h = \bar{h} \circ (d_1^n \otimes \mathrm{id})$. In the previous discussion we gave a more intrinsic definition of these maps and we are now ready to define the algebraic notion of a differential operator.

Definition 1.6. Let $f : X \to S$ be a morphism of schemes, F, G two $\mathcal{O}_X$-modules, $n \in \mathbb{N}$. Using the previous notations, an $f^{-1}(\mathcal{O}_S)$-linear map $h : F \to G$ is a differential operator of order $\leq n$ if and only if its adjoint map $\bar{h} : \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_S)} F \to G$ annihilates $\mathcal{I}^{n+1} \otimes F$ equivalently, if and only if h factors as:



with $\bar{h}^n$ an $\mathcal{O}_X$-linear map, $\mathcal{P}^n_{X/S} = \mathcal{P}_{X/S}/\mathcal{I}^{n+1}$ is the sheaf of principal parts of order n, and $d^n_{X/S,F}$ is the universal differential operator of order $\leq n$ which is induced by $d^n_{X/S,F} = (d_1^n \otimes \mathrm{id}_F)$ as in (1.2.4).

Note that, with this definition, every $\mathcal{O}_X$-linear map is a differential operator of order 0. Before proceeding, we need an explicit description for the composition of differential operators. In particular, given an $f^{-1}(\mathcal{O}_S)$-linear map $h : F \to G$, we have:

$$\bar{h} : \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} F \to G \qquad a \otimes b \otimes x \to a h(bx) \tag{1.6.1}$$

now considering two $f^{-1}(\mathcal{O}_S)$-linear map $g : F \to G$ and $h : G \to H$, we want to define $\overline{h \circ g}$ knowing $\bar{g}$ and $\bar{h}$. In order to be more clear and to obtain more readable formulas we shall often write $\otimes_{\mathcal{O}_S}$ for $\otimes_{f^{-1}(\mathcal{O}_S)}$. First from (1.6.1), we can see that:

$$\overline{h \circ g} = \bar{h}(\mathrm{id}_{\mathcal{O}_X} \otimes_{\mathcal{O}_S} g) : \mathcal{O}_X \otimes_{\mathcal{O}_S} F \to H \tag{1.6.2}$$

So we have to find a method to build in a natural way the map $\mathrm{id}_{\mathcal{O}_X} \otimes_{\mathcal{O}_S} g : \mathcal{O}_X \otimes_{\mathcal{O}_S} F \to \mathcal{O}_X \otimes_{\mathcal{O}_S} G$, starting from $\bar{g} : \mathcal{O}_X \otimes_{\mathcal{O}_S} F \cong \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} F \to G$.

Proposition 1.7. *With the identification of $\mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} = \mathcal{O}_X \otimes_{\mathcal{O}_S}$, consider the map:*

$$\delta : \mathcal{P}_{X/S} \to \mathcal{P}_{X/S} \otimes \mathcal{P}_{X/S} \qquad a \otimes b \to a \otimes_{\mathcal{O}_S} 1 \otimes_{\mathcal{O}_X} 1 \otimes_{\mathcal{O}_S} b \tag{1.7.1}$$

Given $g : F \to G$ and $h : G \to H$ two $f^{-1}(\mathcal{O}_S)$ linear map we have that:

- *δ is an $\mathcal{O}_X$-linear map (using simultaneously the left or right $\mathcal{O}_X$-module structure) and so we can form the $\delta \otimes id_F : \mathcal{P}_{X/S} \otimes F \to \mathcal{P}_{X/S} \otimes \mathcal{P}_{X/S} \otimes F$ which is $\mathcal{O}_X$-linear (using the left structure) and we can define:*

$$\bar{\delta}(g) = (id_{\mathcal{P}_{X/S}} \otimes_{\mathcal{O}_X} \bar{g}) \circ (\delta \otimes_{\mathcal{O}_X} id_F) : \mathcal{P}_{X/S} \otimes F \to \mathcal{P}_{X/S} \otimes G$$

- *$\bar{\delta}(g) = id_{\mathcal{O}_X} \otimes_{\mathcal{O}_S} g : \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} F \to \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} G$.*
- *$\overline{h \circ g} = \bar{h} \circ \bar{\delta}(g) : \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} F \to H$.*
- *The map δ induces maps δ^{n+m}:*

$$\begin{array}{ccc} \mathcal{P}_{X/S} & \xrightarrow{\delta} & \mathcal{P}_{X/S} \otimes \mathcal{P}_{X/S} \\ \downarrow & & \downarrow \\ \mathcal{P}^{n+m}_{X/S} & \xrightarrow{\delta^{n,m}} & \mathcal{P}^n_{X/S} \otimes \mathcal{P}^m_{X/S} \end{array} \tag{1.7.2}$$

Proof. First of all notice that if we identify $\mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S}$ with $\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X$, then δ corresponds to the geometric map:

$$X \times X \times X \to X \times X \qquad (x, y, z) \to (x, z)$$

The linearity of δ comes from the definition, to check the second point we can consider element of the form $a \otimes_{\mathcal{O}_S} 1 \otimes_{\mathcal{O}_X} x$ in $\mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} F$:

$$\begin{array}{ccccc} \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} F & \xrightarrow{\delta \otimes_{\mathcal{O}_X} id_F} & \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} G & \xrightarrow{id_{\mathcal{P}_{X/S}} \otimes_{\mathcal{O}_X} \bar{g}} & \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} G \\ a \otimes_{\mathcal{O}_S} 1 \otimes_{\mathcal{O}_X} x & \longrightarrow & a \otimes_{\mathcal{O}_S} 1 \otimes_{\mathcal{O}_X} 1 \otimes_{\mathcal{O}_S} 1 \otimes_{\mathcal{O}_X} x & \longrightarrow & a \otimes_{\mathcal{O}_S} 1 \otimes_{\mathcal{O}_X} g(x) \end{array}$$

The third point now is a consequence of the second and the formula (1.6.2). For the fourth point we have to show that the upper composition in the diagram annihilates $\mathcal{I}^{n+m+1}$. We know that $\mathcal{I}$ is generated by elements of the form $\xi = 1 \otimes x - x \otimes 1$. Since in $\mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S}$ we have that $1 \otimes x \otimes 1 \otimes 1 = 1 \otimes 1 \otimes x \otimes 1$, then we have:

$$\begin{aligned} \delta(\xi) =& 1 \otimes 1 \otimes 1 \otimes x - x \otimes 1 \otimes 1 \otimes 1 = (1 \otimes 1) \otimes (1 \otimes x - x \otimes 1) + (1 \otimes x - x \otimes 1) \otimes (1 \otimes 1) \\ & \mathbb{I} \otimes \xi + \xi \otimes \mathbb{I} \end{aligned}$$

With this description it is easy to see that if $\zeta = \prod_{i=0}^{n+m} \xi_i \in \mathcal{I}^{n+m+1}$ then:

$$\delta(\zeta) = \prod_{i=0}^{n+m} (\mathbb{I} \otimes \xi_i + \xi_i \otimes \mathbb{I}) \tag{1.7.3}$$

so that each element of the finite sum in $\delta(\zeta)$ must have at least $m + 1$ element in one side of the product or $n + 1$ in the other side. □

This said we can define the composition of differential operator.

Lemma 1.8. *If $g : F \to G$ and $h : G \to H$ are differential operator of orders $\leq n, m$ respectively, then the composition $h \circ g : F \to H$ is a differential operator of order $\leq m + n$ and we have a commutative diagram:*

$$\begin{array}{ccccccc} F & & \xrightarrow{\quad g \quad} & G & & \xrightarrow{\quad h \quad} & H \\ \downarrow & \searrow & \nearrow_{\bar{g}^n} & & \searrow & \nearrow_{\bar{h}^m} & \\ & & \mathcal{P}^n_{X/S} \otimes F & & \mathcal{P}^m_{X/S} \otimes G & & \\ & & & \searrow & \uparrow{\scriptstyle id \otimes \bar{g}^n} & & \\ \mathcal{P}^{n+m}_{X/S} \otimes F & & \xrightarrow{\delta^{m,n} \otimes id} & & \mathcal{P}^m_{X/S} \otimes \mathcal{P}^n_{X/S} \otimes F & & \end{array} \tag{1.8.1}$$

Proof. The nontrivial commutativity in the diagram (1.8.1) is the lower left region, which is obtained tensoring the square below with F:

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{d^n_{X/S}} & \mathcal{P}^n_{X/S} \\ {\scriptstyle d^{n+m}_{X/S}} \downarrow & & \downarrow {\scriptstyle d^m_{X/S, \mathcal{P}^n_{X/S}}} \\ \mathcal{P}^{n+m}_{X/S} & \xrightarrow{\delta^{m,n}} & \mathcal{P}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} \end{array} \tag{1.8.2}$$

We can reduce ourselves to proving the commutativity of (1.8.2) in the affine case. If S=Spec(A) and $X =$ Spec(B) we need to find a canonical homomorphism of B-module:

$$\delta^{m,n} : B \otimes_A B / I^{n+m+1} \to (B \otimes_A B / I^{m+1}) \otimes_B (B \otimes_A B / I^{n+1})$$

Equivalently we can define a homomorphism of B-module δ such that the following composition annihilates I^{n+m+1}:

$$B \otimes_A B \xrightarrow{\delta} B \otimes_A B \otimes_B B \otimes_A B \xrightarrow{\pi_m \otimes \pi_n} (B \otimes_A B / I^{m+1}) \otimes_B (B \otimes_A B / I^{n+1})$$

To do that we can repeat the same argument used for the commutativity of (1.7.2) but considering the following map:

$$\begin{array}{ccc} B \otimes_A B & \xrightarrow{\quad \delta \quad} & (B \otimes_A B) \otimes_B (B \otimes_A B) \\ \downarrow & & \downarrow \\ B \otimes_A B / I^{n+m+1} & \longrightarrow & (B \otimes_A B / I^{m+1}) \otimes_B (B \otimes_A B / I^{n+1}) \end{array} \qquad \begin{array}{ccc} b \otimes b' & \xrightarrow{\quad \delta \quad} & b \otimes 1 \otimes 1 \otimes b' \\ \downarrow & & \downarrow \\ \pi_{n+m}(b \otimes b') & \longrightarrow & \pi_m(b \otimes 1) \otimes \pi_n(1 \otimes b') \end{array}$$

Notice also that from (1.3) the map $\delta^{m,n}$ is unique. The proof of this lemma gives a motivation for the definition of δ in (1.7.1) and another explanation for the previous proposition (1.7). □

Definition 1.9. If E, F are $\mathcal{O}_X$-modules, let Diff$^n_{X/S}(E, F)$ denote the sheaf of germs of differential operators of order $\leq n$. From what we have said we have an identification:

$$\text{Diff}^n_{X/S}(E, F) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S} \otimes E, F) \tag{1.9.1}$$

Where to give a structure of $\mathcal{O}_X$-module compatible with the structure given by F, if D is a differential operator and α is a section of $\mathcal{O}_X$, we can consider αD as the composition of D with the multiplication by α which is a differential operator of order zero. Moreover we denote $\mathrm{Diff}_{X/S}(E,F) = \varinjlim_n \mathrm{Diff}^n_{X/S}(E,F)$. In particular the functor $\mathrm{Diff}(E, \_)$ is represented by $\varprojlim_n(\mathcal{P}^n_{X/S} \otimes E)$.

As an example we can give an explicit description of the sheaves of principal parts $\mathcal{P}^n_{X/S}$ and the sheaf of differential operators $\mathrm{Diff}_{X/S}(\mathcal{O}_X, \mathcal{O}_X)$ in the special case where X/S is smooth.

Example 1.10. [16] If X/S is smooth and $x_1, \ldots, x_n$ are local coordinates, i.e. sections of $\mathcal{O}_X$ defining an étale map $X \to \mathbb{A}^n_S$.

- Let $\alpha_i = 1 \otimes x_i - x_i \otimes 1$, then $\mathcal{P}^m_{X/S}$ is the free $\mathcal{O}_X$-module with basis $\{\alpha_1^{t_1} \ldots \alpha_n^{t_n} \,|\, \sum_i t_i \le m\}$. In this case $\mathcal{P}^n_{X/S} \cong \Delta^{-1}(\mathcal{O}_{X\times_S X}/\mathcal{I}^{n+1})$ where $\mathcal{I}$ is the kernel of $\mathcal{O}_{X\times_S X} \to \Delta_* \mathcal{O}_X$. From the hypothesis we obtain (A.11) that that the augmentation ideal $\Omega^1_{X/S} = \Delta^{-1}(\mathcal{I}/\mathcal{I}^2)$ is generated as an $\mathcal{O}_X$-module by $\{dx_1, \ldots dx_n\}$, where dx_i is the class of α_i in $\Delta^{-1}(\mathcal{I}/\mathcal{I}^2)$. From this we know that the monomials of degree j in the α_i are forming a free basis for the grading $\mathcal{I}^j/\mathcal{I}^{j+1}$. Now by induction and considering the exactness of the following sequence we conclude:
$$0 \to \mathcal{I}^j/\mathcal{I}^{j+1} \to \mathcal{P}^j_{X/S} \to \mathcal{P}^{j-1}_{X/S} \to 0$$
- Consider now a multi-index $t = (t_1, \ldots, t_n)$ and denote:
$$\alpha^t = \prod_{i=1}^n \alpha_i^{t_i} \qquad t! = \prod_{i=1}^n (t_i!) \qquad |t| = \sum_{i=1}^n t_i$$
With this notation $\{\alpha^t : |t| \le m\}$ is a basis for $\mathcal{P}^m_{X/S}$. Let $\overline{D_t}$ be the elements of the dual basis of $\mathcal{H}om\,(\mathcal{P}^m_{X/S}, \mathcal{O}_X)$ and let $D_t : \mathcal{O}_X \to \mathcal{O}_X$ be the differential operator from the identification (1.6). Then $\{D_t : |t| \le m\}$ is a basis for $\mathrm{Diff}^m_{X/S}(\mathcal{O}_X, \mathcal{O}_X)$ where the composition is given by:
$$D_t \circ D_{t'} = \frac{(t+t')!}{(t!)(t'!)} D_{t+t'}$$
To verify this last statement we have to show that:
$$\overline{D_t \circ D_{t'}} = \overline{D}_t \circ (\mathrm{id}_{\mathcal{P}^m} \otimes \overline{D}_{t'}) \circ \delta : \; \mathcal{P}^{m+n}_{X/S} \longrightarrow \mathcal{O}_X$$
$$\alpha^q \longrightarrow \binom{t+t'}{t'} \gamma_{t+t',q}$$
where $\gamma_{a,b} = 0$ if $a \ne b$ and $\gamma_{a,b} = 1$ if $a = b$. We can compute:
$$\delta(\alpha^q) \overset{(1.7.3)}{=} (1 \otimes \alpha + \alpha \otimes 1)^q = \sum_{i+j=q} \binom{q}{i} \alpha^i \otimes \alpha^j$$
Applying $\mathrm{id}_{\mathcal{P}^m} \otimes \overline{D}_{t'}$ we get zero unless $j = t'$ and so we have $\binom{q}{t'}\alpha^{q-t'}$, and then applying $\overline{D_t}$ we get zero unless $q - t' = t$ as we wanted.
- Over $\mathbb{Q}$ then $\mathrm{Diff}^1(\mathcal{O}_X, \mathcal{O}_X)$ generates $\mathrm{Diff}(\mathcal{O}_X, \mathcal{O}_X)$, since we can write $D_q = \frac{1}{q!} D^q$ with $D^q = \prod_{i=1}^n D_i^{q_i}$ where the multi-index is $i = (0, \ldots, 1, \ldots, 0)$ at the i-th spot. Note that the same fails over $\mathbb{Z}$ or in characteristic p because we can't take fractions or consider factorials.

1.2.1 Formal grupoid

Let $f : X \to S$ be a morphism of schemes. So far, we have built a collection of different maps, now we just want to summarize all the existing relations among them, this will be useful especially in (3.1) because some of these relations are implicitly used to build the linearizing functor L. So consider the structural sheaf $\mathcal{O}_X$ and the sheaves of principal parts $\mathcal{P}^n_{X/S}$, we have:

$$d_0^n, d_1^n : \mathcal{O}_X \to \mathcal{P}^n_{X/S} \qquad \varphi_{m,n} : \mathcal{P}^n_{X/S} \to \mathcal{P}^m_{X/S}$$
$$\pi^n : \mathcal{P}^n_{X/S} \to \mathcal{O}_X \qquad \sigma^n : \mathcal{P}^n_{X/S} \to \mathcal{P}^n_{X/S}$$
$$\delta^{m,n} : \mathcal{P}^{m+n}_{X/S} \to \mathcal{P}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S}$$

Recall that d_0^n, d_1^n are defined in (1.2.1), the morphisms $\varphi_{m,n}$ with $m \leq n$ are the canonical restriction morphisms and $\pi^n = \varphi_{0,n}$. The map σ^n is just the canonical automorphism coming from the symmetry of $\mathcal{P}^n_{X/S}$, whereas $\delta^{m,n}$ is the map defined in (1.7). These maps commute with respect to the transition maps $\varphi_{m,n}$, moreover, the following relations are also satisfied:

$$\pi^n \circ d_0^n = \mathrm{Id}_{\mathcal{O}_X} \qquad \pi^n \circ d_1^n = \mathrm{Id}_{\mathcal{O}_X} \qquad \sigma^n \circ \sigma^n = \mathrm{Id}_{\mathcal{P}^n_{X/S}}$$
$$\sigma^n \circ d_0^n = d_1^n \qquad \sigma^n \circ d_1^n = d_0^n \qquad \pi^n \circ \sigma^n = \pi^n$$

$$\begin{array}{ccc} \mathcal{P}^{m+n}_{X/S} & \xrightarrow{\varphi_{m,m+n}} & \mathcal{P}^m_{X/S} \\ \downarrow{\scriptstyle \delta^{m,n}} & \nearrow{\scriptstyle 1\otimes\pi^n} & \\ \mathcal{P}^m_{X/S} \otimes \mathcal{P}^n_{X/S} & & \end{array} \qquad \begin{array}{ccc} \mathcal{P}^{m+n}_{X/S} & \xrightarrow{\varphi_{n,m+n}} & \mathcal{P}^n_{X/S} \\ \downarrow{\scriptstyle \delta^{m,n}} & \nearrow{\scriptstyle \pi^m\circ 1} & \\ \mathcal{P}^m_{X/S} \otimes \mathcal{P}^n_{X/S} & & \end{array}$$

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{d_0^m} & \mathcal{P}^m_{X/S} \\ \downarrow{\scriptstyle d_0^{m+n}} & & \downarrow \\ \mathcal{P}^{m+n}_{X/S} & \xrightarrow{\delta^{m,n}} & \mathcal{P}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} \end{array} \qquad \begin{array}{ccc} \mathcal{O}_X & \xrightarrow{d_1^n} & \mathcal{P}^n_{X/S} \\ \downarrow{\scriptstyle d_1^{m+n}} & & \downarrow \\ \mathcal{P}^{m+n}_{X/S} & \xrightarrow{\delta^{m,n}} & \mathcal{P}^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} \end{array}$$

$$\begin{array}{ccc} \mathcal{P}^n_{X/S} \otimes \mathcal{P}^n_{X/S} & \xrightarrow{1\otimes\sigma^n} & \mathcal{P}^n_{X/S} \\ \uparrow{\scriptstyle \delta^{n,n}} & & \uparrow{\scriptstyle d_0^n} \\ \mathcal{P}^{2n}_{X/S} & \xrightarrow{\pi^{2n}} & \mathcal{O}_X \end{array} \qquad \begin{array}{ccc} \mathcal{P}^n_{X/S} \otimes \mathcal{P}^n_{X/S} & \xrightarrow{\sigma^n\otimes 1} & \mathcal{P}^n_{X/S} \\ \uparrow{\scriptstyle \delta^{n,n}} & & \uparrow{\scriptstyle d_1^n} \\ \mathcal{P}^{2n}_{X/S} & \xrightarrow{\pi^{2n}} & \mathcal{O}_X \end{array}$$

$$\begin{array}{ccc} \mathcal{P}^{n+m+p}_{X/S} & \xrightarrow{\delta^{m+n,p}} & \mathcal{P}^{m+n}_{X/S} \otimes \mathcal{P}^p_{X/S} \\ \downarrow{\scriptstyle \delta^{m,n+p}} & & \downarrow{\scriptstyle \delta^{m,n}\otimes 1} \\ \mathcal{P}^m_{X/S} \otimes \mathcal{P}^{n+p}_{X/S} & \xrightarrow{1\otimes\delta^{n+p}} & \mathcal{P}^m_{X/S} \otimes \mathcal{P}^n_{X/S} \otimes \mathcal{P}^p_{X/S} \end{array}$$

These maps and properties form the definition of a formal grupoid associated to a morphism of schemes $f : X \to S$ given in [[1],II.1.1] where the notion of formal grupoid is used to generalize in different context the theory of calculus and differential operators that we have seen.

1.3 Connection and Stratification

Now we will introduce the concept of connection on a $\mathcal{O}_X$-module. In the appendix (A) we analyze more in detail the concept of a connection for a quasi-coherent $\mathcal{O}_X$-module and we will introduce this notion in a different way trying to motivate the following construction.

Definition 1.11. Let $f : X \to S$ a morphism of schemes, a connection relative to S on an $\mathcal{O}_X$-module E is a $\mathcal{P}^1_{X/S}$-linear isomorphism:

$$\epsilon : \mathcal{P}^1_{X/S} \otimes E \to E \otimes \mathcal{P}^1_{X/S}$$

which, modulo the kernel $\Omega^1_{X/S}$ of the map $\mathcal{P}^1_{X/S} \to \mathcal{O}_X$, reduces to the identity endomorphism on E.

For our purpose we give this definition of connection because it is easier to generalize to higher order, in particular:

Definition 1.12. Let $f : X \to S$ a morphism of schemes, a n-connection relative to S on an $\mathcal{O}_X$-module E is a $\mathcal{P}^n_{X/S}$-linear isomorphism:

$$\epsilon_n : \mathcal{P}^n_{X/S} \otimes E \to E \otimes \mathcal{P}^n_{X/S}$$

such that it reduce to the identity modulo the augmentation ideal of $\mathcal{P}^n_{X/S}$.

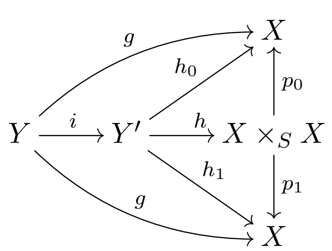
Since ϵ_n is not only $\mathcal{O}_X$-linear but also $\mathcal{P}^n_{X/S}$-linear, and since we have said (1.2.2) that we have an identification: $\mathcal{P}^n_{X/S} = ((p_0{}^{(n)})_*(p_1{}^{(n)})^*(\mathcal{O}_X))$, then this definition is equivalent to the definition given in the appendix of [17], i.e. a n-connection on E relative to S is an isomorphism:

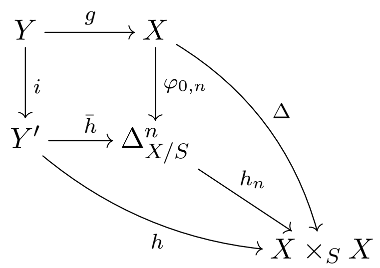
$$\epsilon_n : (p_1{}^{(n)})^*(E) \xrightarrow{\cong} (p_0{}^{(n)})^*(E)$$

Proposition 1.13. *Let X, Y, Y' be S-schemes, let $g : Y \to X$ be a S-morphism and $i : Y \to Y'$ a S closed immersion which defines a sheaf of ideal $\mathcal{I}$ on $\mathcal{O}_{Y'}$ such that $\mathcal{I}^{n+1} = 0$. If $h_0, h_1 : Y' \to X$ are two S-extensions of g over Y, and if E is a $\mathcal{O}_X$-module with a n-connection ϵ_n relative to S, then exists an isomorphism:*

$$\epsilon_{h_0,h_1} : h_1^*(M) \xrightarrow{\cong} h_0^*(M)$$

Proof. We can define a S-morphism $h = (h_0, h_1) := Y' \to X \times_S X$ such that $h_0 = p_0 \circ h$ and $h_1 = p_1 \circ h$. Since we have a factorization $\Delta \circ g = h \circ i$, and since the ideal induced by the closed immersion i is nilpotent we can factorize h through a morphism $\bar{h} : Y' \to \Delta^n_{X/S}$ so that:





Hence if $p_0^{(n)}, p_1^{(n)} : \Delta^n_{X/S} \to X$ are the projections, since we can interpret a n-connection as an isomorphism:

$$\epsilon_n : p_1^{(n)^*}(M) \xrightarrow{\cong} p_0^{(n)^*}(M)$$

taking the inverse image of $\bar{h}$ we find the required isomorphism $\epsilon_{h_0,h_1} : h_1^*(M) \xrightarrow{\cong} h_0^*(M)$. $\square$

Now adding some compatibility condition we obtain a structure that will be of central importance later.

Definition 1.14. A stratification on E is a collection of isomorphisms $\epsilon_n : \mathcal{P}^n_{X/S} \otimes E \to E \otimes \mathcal{P}^n_{X/S}$ for $n \geq 0$ such that:

- ϵ_0 is the identity map.
- ϵ_n is $\mathcal{P}^n_{X/S}$-linear.
- ϵ_n and ϵ_m are compatible, via the restriction $\varphi_{m,n} : \mathcal{P}^n_{X/S} \to \mathcal{P}^m_{X/S}$ with $m \leq n$:

$$\begin{CD} \mathcal{P}^n_{X/S} \otimes E @>{\epsilon_n}>> E \otimes \mathcal{P}^n_{X/S} \\ @V{\varphi_{mn}\otimes 1}VV @VV{1\otimes\varphi_{mn}}V \\ \mathcal{P}^m_{X/S} \otimes E @>{\epsilon_m}>> E \otimes \mathcal{P}^m_{X/S} \end{CD}$$

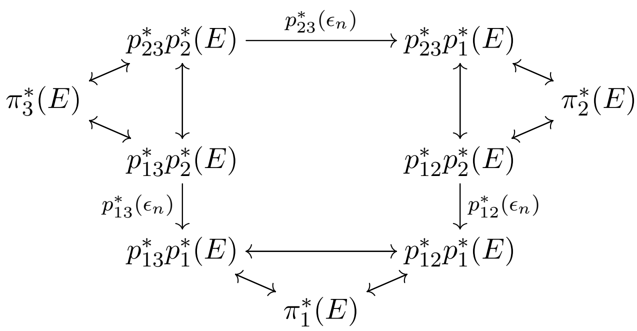
- If $\mathcal{P}^n_{X/S}(2)$ is the n^{th}-infinitesimal neighborhood of the diagonal of X in $X \times_S X \times_S X$, and $p_{ij} : \mathcal{P}^n_{X/S}(2) \to \mathcal{P}^n_{X/S}$ is the projection via the coordinates i and j, then the cocycle condition holds:

$$p_{12}^*(\epsilon_n) \circ p_{23}^*(\epsilon_n) = p_{13}^*(\epsilon_n) \qquad \text{for all } n \tag{1.14.1}$$

To understand better the last cocycle condition we can consider the canonical projections $\pi_j : \mathcal{P}^n_{X/S}(2) \to X$ with $j = 1, 2, 3$ and $p_i : \mathcal{P}^n_{X/S} \to X$ with $i = 1, 2$, then note that:

$$\pi_1 = p_1 \circ p_{13} = p_1 \circ p_{12} \qquad \pi_2 = p_2 \circ p_{12} = p_1 \circ p_{23} \qquad \pi_3 = p_2 \circ p_{23} = p_2 \circ p_{13}$$

and the cocycle condition says that if we use the stratification ϵ_n to construct the isomorphisms $\pi_3^*(E) \to \pi_2^*(E)$ and $\pi_2^*(E) \to \pi_1^*(E)$ then the composition of these two corresponds to the isomorphism $\pi_3^*(E) \to \pi_1^*(E)$ provided by ϵ_n, i.e. the following diagram commute:



Considering the situation of (1.13), if we have three S-morphisms h_0, h_1, h_2 of Y' in X, extending g on Y, in general we are not able to compare the isomorphisms $\epsilon_{h_0,h_1}, \epsilon_{h_1,h_2}$ and ϵ_{h_1,h_2}. The cocycle condition that we added for the definition of a stratification structure allows us to solve this problem and thus we obtain a more useful and manageable structure.

Proposition 1.15. *With the notation of (1.13), let X, Y, Y' be S-schemes, $g : Y \to X$ a S-morphism, $Y \to Y'$ a closed S-immersion defining a nilpotent sheaf of ideal $\mathcal{I}$ of $\mathcal{O}_{Y'}$. Given three S-morphisms $h_0, h_1, h_2 : Y' \to X$ inducing g on Y. Then if E is an $\mathcal{O}_X$-module fortified with a stratification structure relative to S, we have:*

$$\epsilon_{h_0,h_2} = \epsilon_{h_0,h_1} \circ \epsilon_{h_1,h_2} \tag{1.15.1}$$

Proof. Similarly to what we did in the proof of (1.13), we can use h_0, h_1, h_2 to define an S-morphism $h : Y' \to X \times_S X \times_S X$ such that $h_i = p_i \circ h$. Since the different morphisms h_i agree modulo a nilpotent ideal, h factorizes through an infinitesimal neighborhood of the diagonal of X^3_S defining a map:

$$\bar{h} : (Y', \mathcal{O}_{Y'}) \to (\Delta^n_{X/S}(2), \mathcal{P}^n_{X/S}(2))$$

Then the relation (1.15.1) is the inverse image through h of the cocycle relations (1.14.1). □

There are different ways of giving a stratification on a $\mathcal{O}_X$-module. To motivate the proposition below we can for example consider a morphism of schemes $f : X \to S$ and an $\mathcal{O}_X$-module $E = f^*(T)$ for some $\mathcal{O}_S$-module T. For any two $\mathcal{O}_X$-module F and G, from the isomorphism $E \otimes_{\mathcal{O}_X} F \cong f^{-1}(T) \otimes_{f^{-1}(\mathcal{O}_S)} F$, we get a natural transformation:

$$\mathrm{Hom}_{f^{-1}(\mathcal{O}_S)}(F, G) \to \mathrm{Hom}_{f^{-1}(\mathcal{O}_S)}(E \otimes_{\mathcal{O}_X} F, E \otimes_{\mathcal{O}_X} G) \qquad h \to id_{f^{-1}(T)} \otimes_{\mathcal{O}_S} h$$

We shall see that a stratification on E allows us to do the same when h is a differential operator.

Proposition 1.16. *Suppose X/S is smooth and E is an $\mathcal{O}_X$-module. Then the following data are equivalent:*

A) A stratification on E.

B) A collection of compatible, right $\mathcal{O}_X$-linear maps $\theta_n : E \to E \otimes \mathcal{P}^n_{X/S}$, with $\theta_0 = id_E$, such that the cocycle condition is satisfied, i.e. the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\theta_n} & E \otimes \mathcal{P}^n_{X/S} \\ \downarrow{\scriptstyle \theta_{m+n}} & & \downarrow{\scriptstyle \theta_m \otimes id_{\mathcal{P}^n_{X/S}}} \\ E \otimes \mathcal{P}^{m+n}_{X/S} & \xrightarrow{id \otimes \delta^{m,n}} & E \otimes \mathcal{P}^m_{X/S} \otimes \mathcal{P}^n_{X/S} \end{array} \tag{1.16.1}$$

C) An $\mathcal{O}_X$-linear ring homomorphism:

$$\mathit{Diff}_{X/S}(\mathcal{O}_X, \mathcal{O}_X) \to \mathit{Diff}_{X/S}(E, E)$$

D) A collection of $\mathcal{O}_X$-linear maps, for any two $\mathcal{O}_X$-modules F, G:

$$\nabla : \mathit{Diff}_{X/S}(F, G) \to \mathit{Diff}_{X/S}(E \otimes F, E \otimes G)$$

compatible with composition and taking identity of $\mathit{Diff}(F, F)$ to the identity.

Proof. $A \Rightarrow B$) We can define $\theta_n : \epsilon_n \circ (d_1 \otimes id) : E \to \mathcal{P}^n_{X/S} \otimes E \to E \otimes \mathcal{P}^n_{X/S}$, the commutativity of the diagram (1.16.1) is a consequence of the cocycle condition (1.14.1).

$B \Rightarrow D \Rightarrow C$) If we show that the θ_n can induce the data of (D) then (C) will follow. If $h : F \to G$ is a differential operator of order $\leq n$ and $\bar{h} : \mathcal{P}^n_{X/S} \otimes F \to G$ is its $\mathcal{O}_X$-linearization, we can consider the differential operator of order $\leq n$ given by the following composition:

$$\nabla(h) : \; E \otimes F \xrightarrow{\theta_n \otimes id_F} E \otimes \mathcal{P}^n_{X/S} \otimes F \xrightarrow{id_E \otimes \bar{h}} E \otimes G$$

First of all the fact that $\nabla(id_F) = id_{E\otimes F}$ is a direct consequence of the condition $\theta_0 = id_E$. Indeed consider the natural projections $\mu : \mathcal{P}_{X/S} \to \mathcal{P}^0_{X/S} \cong \mathcal{O}_X$ and $\mu_n : E \otimes \mathcal{P}^n_{X/S} \to E \otimes \mathcal{P}^0_{X/S} \cong E$, then from the definition we obtain:

$$\begin{array}{rl} \overline{id}_F : \mathcal{P}^n_{X/S} \otimes F \to F & \qquad \xi \otimes x \to \mu(\xi) \otimes x \cong \mu(\xi)x \\ id_E \otimes \overline{id}_F : E \otimes \mathcal{P}^n_{X/S} \otimes F \to E \otimes F & \qquad (e \otimes \xi \otimes x) \to \mu(\xi)e \otimes x = (\mu_n \otimes id_F)(e \otimes \xi \otimes x) \end{array}$$

From this and the compatibility of the θ_n follows easily that:

$$\nabla(id_F) = (\mu_n \otimes id_F)(\theta_n \otimes id_F) = (\mu_n \circ \theta_n) \otimes id_F = \theta_0 \otimes id_F = id_{E\otimes F}$$

We have now to check that ∇ preserves composition, and this is a consequence of the cocycle condition. Let $f : F \to G$ and $g : G \to H$ be differential operator of order $\leq n$ and $\leq m$ respectively, and consider the diagram:

$$\begin{array}{ccccc} E \otimes G & \xrightarrow{\theta^m \otimes id_G} & E \otimes \mathcal{P}^m_{X/S} \otimes G & \xrightarrow{id_E \otimes \bar{g}^m} & E \otimes H \\ \uparrow{\scriptstyle \nabla(f)} & & \uparrow{\scriptstyle id_E \otimes \overline{\delta^{n,m}}(f)} & \nearrow{\scriptstyle id_E \otimes \overline{(g\circ f)}^{n+m}} & \\ E \otimes F & \xrightarrow[\theta^{n+m} \otimes id_F]{} & E \otimes \mathcal{P}^{n+m}_{X/S} \otimes F & & \end{array}$$

Now we can see that the composition in the top of the diagram is $\nabla(g) \circ \nabla(f)$ and the composition of the bottom part of the diagram is $\nabla(g \circ f)$. So we have to prove the commutativity of the diagram, in particular for the right triangle the commutativity is a consequence of the propositions (1.7) and (1.8). The commutativity of the square in the left is an easy consequence of the cocycle condition (1.16.1) and the definitions of ∇ and $\bar{\delta}$.
$C \Rightarrow A$) This implication is more difficult, here we will simply sketch how to define the map ϵ_n and we will not prove the cocycle condition. In this case from (1.9.1) we can consider the maps $\nabla_n : \mathrm{Diff}^n(\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{Diff}^n(E, E)$ as maps between the corresponding $\mathcal{O}_X$-linearization: $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}, \mathcal{O}_X) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S} \otimes E, E)$ for every $n \geq 0$. These maps ∇_n are $\mathcal{O}_X$-linear where we are considering $\mathcal{P}^n_{X/S}$ and $\mathcal{P}^n_{X/S} \otimes E$ with the left $\mathcal{O}_X$-module structure, moreover it is possible to prove that in general they are even $\mathcal{P}_{X/S}$-linear i.e. $\nabla_n(\xi\partial) = \xi\nabla_n(\partial)$ for all $\xi \in \mathcal{P}_{X/S}$ and $\partial \in \mathrm{Diff}^n(F, G)$. To see this observe that the $\mathcal{P}_{X/S}$-module structure of $\mathrm{Diff}^n(F, G) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S} \otimes F, G)$ is defined by $\xi\partial = \partial \circ (\mu_{\bar{\xi}} \otimes id_F)$ where $\mu_{\bar{\xi}} : \mathcal{P}^n_{X/S} \to \mathcal{P}^n_{X/S}$ is multiplication by the class of ξ in $\mathcal{P}^n_{X/S}$. If $\partial \in \mathrm{Diff}^n(F, G)$ and $x \in \mathcal{O}_X$, then recalling that $d_0, d_1 : \mathcal{O}_X \to \mathcal{P}_{X/S}$ are the maps defined in (1.5.3), we obtain:

$$d_1(x)\partial = \partial \circ (\mu_{\overline{1\otimes x}} \otimes id_F) = \partial \circ \mu_x$$

where $\mu_x \in \text{Diff}^0(F, F)$ is the multiplication by x. Whereas since ∂ is $\mathcal{O}_X$-linear:

$$d_0(x)\partial = \partial \circ (\mu_{\overline{x \otimes 1}} \otimes id_F) = \mu_{\bar{x}} \circ \partial$$

with $\mu_{\bar{x}} \in \text{Diff}^0(G, G)$ the multiplication by x. Now we know that the maps $\{\nabla_n\}$ preserve composition:

$$\nabla_n(d_1(x)\partial) = \nabla_n(\partial \circ \mu_x) = \nabla_n(\partial) \circ \nabla_0(\mu_x) = \nabla_n(\partial) \circ \mu_x = d_1(x)\nabla_n(x)$$

It follows that $\nabla_n(\xi\partial) = \xi\nabla_n(\partial)$ for all $\xi \in \mathcal{P}_{X/S}$ because the maps ∇_n are $\mathcal{O}_X$-linear, and $\{d_1(x) : x \in \mathcal{O}_X\}$ generates $\mathcal{P}_{X/S}$. From this we can conclude the construction of the ϵ_n by considering the map ∇^* obtained from ∇_n applying the functor $\text{Hom}_{\mathcal{O}_X}(\_, E)$, so that we have the following diagram:

$$\begin{array}{ccc} \mathcal{P}^n_{X/S} \otimes E & \xrightarrow{\eta} & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S} \otimes E, E), E) \\ \downarrow{\scriptstyle \epsilon_n} & & \downarrow{\scriptstyle \nabla^*} \\ E \otimes \mathcal{P}^n_{X/S} & \overset{\xi}{\longleftrightarrow} & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}, \mathcal{O}_X), E) \end{array}$$

where η is the evaluation and ξ is the following composition of canonical map:

$$E \otimes \mathcal{P}^n_{X/S} \to E \otimes (\mathcal{P}^{n\vee}_{X/S})^\vee \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}, \mathcal{O}_X), E)$$

As X/S is smooth then $\mathcal{P}^n_{X/S}$ is locally free and in the previous composition we obtain isomorphisms. Indeed if $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module so is its dual $\mathcal{F}^\vee$ and we have well known canonical isomorphism:

$$\mathcal{F} \xrightarrow{\cong} (\mathcal{F}^\vee)^\vee \qquad \mathcal{F}^\vee \otimes \mathcal{G} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

and putting $\mathcal{F} = \mathcal{P}^n_{X/S}$, and $\mathcal{G} = E$ we obtain that ξ is indeed an isomorphism. Note that ξ induce an isomorphism with $E \otimes \mathcal{P}^n_{X/S}$ and not with $\mathcal{P}^n_{X/S} \otimes E$ since we have computed $\mathcal{P}^{n\vee}_{X/S} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}, \mathcal{O}_X)$ with the left $\mathcal{O}_X$-module structure. Finally we obtain the map ϵ_n as the composition in the diagram using the inverse of ξ. □

1.3.1 Curvature

We close this chapter analyzing the relation between connection and stratification, in particular we will see that the curvature of a connection can be interpreted as the obstruction to extend the connection to a stratification. So first recall that in the appendix (A) we have characterized a connection as an additive map $\nabla : E \to E \otimes \Omega^1_{X/S}$, and given the canonical exterior derivative $d_k : \Omega^k_{X/S} \to \Omega^{k+1}_{X/S}$, which is a differential operator of order ≤ 1, using the construction of (1.16) we can extend ∇ to get an homomorphism of abelian sheaves $\nabla_k : E \otimes \Omega^k_{X/S} \to E \otimes \Omega^{k+1}_{X/S}$ which satisfies the identity:

$$\nabla_k(x \otimes \omega) = x \otimes d_k\omega + (-1)^k \nabla(x) \wedge \omega$$

We have then defined the curvature of a connection as the $\mathcal{O}_X$-linear map:

$$K = \nabla_1 \circ \nabla : E \to E \otimes \Omega^2_{X/S} \qquad \nabla_{k+1} \circ \nabla_k(x \otimes \omega) = K(x) \wedge \omega$$

Proposition 1.17. *Suppose that X/S is smooth and $x_1, \ldots, x_n$ are local coordinates. Let E be an $\mathcal{O}_X$-module and ϵ a connection on E. Suppose that the connection maps $D_i \in Der^1(\mathcal{O}_X, \mathcal{O}_X)$ to $\theta_i \in Diff^1(E, E)$ then the curvature K of the connection over some $\mathcal{O}_X$-module E is given by:*

$$K = \sum_{i<j} [\theta_i, \theta_j] \otimes dx_i \wedge dx_j$$

In particular $K = 0$ if and ony if the θ_i commute.

Proof. From the example (1.10) and from the smoothness hypothesis we get that the set $\{dx_1, dx_2, \ldots, dx_n\}$ is a free basis for $\Omega^1_{X/S}$ and $\{D_1, \ldots, D_n\}$ is a free basis for $\mathrm{Der}^1(\mathcal{O}_X, \mathcal{O}_X)$. From the previous general construction of (1.16) we obtain the θ_i from D_i thanks to the connection, more explicitly we have:

$$\theta_i : E \cong \mathcal{O}_X \otimes E \xrightarrow{d_1 \otimes id} \mathcal{P}^1_{X/S} \otimes E \xrightarrow{\epsilon} E \otimes \mathcal{P}^1_{X/S} \xrightarrow{id \otimes \bar{D}_i} E \otimes \mathcal{O}_X \cong E$$

Recall that in the proposition (A.29) we have shown how to obtain a connection $\nabla : E \to E \otimes \Omega^1_{X/S}$ starting with ϵ, from this follows that for every section $x \in E$ then:

$$\begin{aligned} \nabla(x) &= \sum_i \theta_i(x) \otimes dx_i \\ \nabla_1 \circ \nabla(x) &= \nabla_1(\sum_i \theta_i(x) \otimes dx_i) = \sum_i [\nabla(\theta_i(x)) \wedge dx_i - \theta_i(x) \otimes d^2 x_i] \\ &= \sum_i \sum_j \theta_j \circ \theta_i(x) \otimes dx_j \wedge dx_i = \sum_{i<j} [\theta_i, \theta_j] \otimes dx_i \wedge dx_j \end{aligned}$$

□

Theorem 1.18. *Suppose X/S is smooth and S is a $\mathbb{Q}$-scheme. Then a connection ∇ on E extends to a stratification if and only if the connection is integrable i.e. $K = 0$.*

Proof. If the connection ∇ extends to a stratification then of course the θ_i commute and the curvature is $K = 0$.
From (1.16) we know that a connection is equivalent to a map $\rho_1 : \mathrm{Diff}^1(\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{Diff}^1(E, E)$ and to extend the connection to a stratification means that we want an extension of ρ_1 to a $\mathcal{O}_X$-linear homomorphism $\rho : \mathrm{Diff}(\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{Diff}(E, E)$. We can do this locally, from the example (1.10) we know that we have a basis $\{D_q = \frac{1}{q!} D_1^{q_1} \ldots D_n^{q_n}\}$ for $\mathrm{Diff}(\mathcal{O}_X, \mathcal{O}_X)$. Using the notation of the previous proposition, we can define $\theta_i \in \mathrm{Diff}^1(E, E)$ which are the element corresponding to the D_i by the connection ∇. So we can define:

$$\rho(D_q) = \frac{1}{q!} \theta_1^{q_1} \ldots \theta_n^{q_n}$$

Since S is a $\mathbb{Q}$-scheme this elements exist, and ρ is well defined because $K = 0$ and from the characterization of (1.17) the θ_i need to commute so that the ordering is not important. □

2 The Infinitesimal and Stratifying site

In this chapter, following [17] and [2] we will introduce the infinitesimal site and the stratifying site, and we will show that we can describe a stratification over an $\mathcal{O}_X$-module in terms of an object over these sites. Grothendieck's philosophy relies in the fact that the notion of a site is too rigid and a better structure is the topos to which it gives rise, namely the category of sheaves of sets on the site. Different sites can give rise to equivalent topos, but the "geometry" (e.g. cohomology) of all such sites should be the same. It is possible to give an intrinsic description of what makes a category a topos. In the appendix (C) we can find some background material about Grothendieck topos and sheaf cohomology in a topos.

Definition 2.1. Let $X \xrightarrow{f} S$ be an S-scheme, the category Inf(X/S) consists as objects, morphisms $g : U \to T$ where $U \subset X$ is a Zariski open subset of X and g is a closed S-immersion such that its associated ideal on T is nilpotent. The morphisms between $U \to T$ and $U' \to T'$ are S-morphisms where $u : U \to U'$ is an open immersion and $t : T \to T'$ makes the following diagram commute:

$$\begin{array}{ccc} U & \xrightarrow{u} & U' \\ \downarrow & & \downarrow \\ T & \xrightarrow{t} & T' \end{array}$$

Now we have to define a topology on this category to obtain the infinitesimal site.

Proposition 2.2. *We can induce the structure of a site in the category Inf(X/S) considering the Grothendieck topology generated by the pretopology defined by taking as covering family of an object $(U \to T)$ a set of morphisms $\{(U_i \to T_i) \to (U \to T)\}$ such that $T = \bigcup T_i$ and $U_i = U \times_T T_i$.*

Proof.

- (Identity axiom) Every isomorphism $(U' \to T') \to (U \to T)$ is a covering family indeed $U' \to U$ is obviously an open map and $T \cong T'$ and $U' \cong U \times_T T'$.

- (Stability Axiom) The stability under pullback is easy. If $U_i \to T_i$ is an element of a covering family of $U \to T$ and $U' \to T'$ is any morphism. Then the pullback in the category is given by $(U_i) \times_U U' \to T_i \times_T T'$. The only fact to check is that $U_i \times_U U' \cong U' \times_{T'} (T_i \times_T T')$, but this is true because using the commutativity of the diagram is possible to see that the left hand side satisfies the same universal property of the right hand side.

- (Transitivity axiom) If we have covering family $\{U_i \to T_i\}_i$ of $(U \to T)$ and $\{U_{ij} \to T_{ij}\}$ is a covering family for $(U_i \to T_i)$ then $\{U_{ij} \to T_{ij}\}_{ij}$ is also a covering for $(U \to T)$, indeed we know that $U_i = U \times_T T_i$ and $U_{ij} = U_i \times_{T_i} T_{ij}$ but this means also that $U_{ij} = U \times_T T_{ij}$.

□

Definition 2.3. The infinitesimal topos is defined as the topos of abelian groups associated to the infinitesimal site: $(X/S)_{Inf} =$ Sh(Inf(X/S)).

Proposition 2.4. *There is a fully faithful functor from the infinitesimal site into its associated topos:*

$$Y : \textit{Inf}(X/S) \to (X/S)_{Inf}$$

Proof. We just have to remember that we always have a natural embedding of the site into its associated topos if the topology defining the site is subcanonical, i.e. all the representable presheaves of the site are in fact sheaves. This follows essentially from the Yoneda embedding, let the category $\mathcal{C}$ be a site and take an object $A \in \mathrm{Obj}(\mathcal{C})$ we will denote $\tilde{A}$ the functor $\mathrm{Hom}_{\mathcal{C}}(\_, A)$ which is an object in the category of presheaves $\tilde{\mathcal{C}} = \mathrm{Psh}(\mathcal{C})$. The assignment of an object A of $\mathcal{C}$ to its representable presheaf $A \to \tilde{A}$ extends to a functor $Y : \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$. Now for any other presheaf $F \in \tilde{C}$ we have a canonical Yoneda identification:

$$\mathrm{Hom}_{\tilde{C}}(\tilde{A}, F) \cong F(A)$$

The Yoneda lemma implies that the functor Y is full and faithful and hence realizes $\mathcal{C}$ as a full subcategory inside its category of presheaves. Therefore to obtain a functor from a site to its related topos we have just to check that the topology is subcanonical. In our case, for the infinitesimal site $\mathrm{Inf}(X/S)$ for every element $(U \to T)$ then the presheaves $(\widetilde{U \to T})$ are indeed sheaf, so that by definition the topology defining the infinitesimal site is subcanonical thus we get the thesis. To check the sheaf condition for $(\widetilde{U \to T})$=$\mathrm{Hom}_{Inf(X/S)}(\_, T)$ consider a covering family $\{T'_i \to T'\}$ in $\mathrm{Inf}(X/S)$ and a matching family of element $\{h_i \in \mathrm{Hom}_{Inf(X/S)}(T'_i, T)\}$. We have to define an element $h \in \mathrm{Hom}_{Inf(X/S)}(T', T)$ inducing all the $\{h_i\}$. To do that since $\bigcup_i T'_i$=T' and since the h_i are compatible with each other i.e:

$$h_{i_0} \circ p_0 = h_{i_1} \circ p_1 \in \mathrm{Hom}_{Inf(X/S)}(T'_{i_0} \times_{T'} T'_{i_1}, T)$$

we can just define h by gluing all the local definitions $h_{|T'_i}$=h_i. □

From now on we will denote the element $(U \to T)$ of $\mathrm{Inf}(X/S)$ as pair (U, T). In [17] we can find a more explicit description of the sheaves on $\mathrm{Inf}(X/S)$.

Proposition 2.5. *A sheaf of sets on Inf(X/S) can be identified with a system of Zariski sheaves of sets $F_{(U,T)}$ on T, one for each object (U,T) of Inf(X/S), together with, for each morphism (u,t) : $(U,T) \to (U',T')$, a homomorphism: $\rho_t : t^{-1}F_{(U'T')} \to F_{(UT)}$, such that this homomorphism is an isomorphism when $T \xrightarrow{t} T'$ is an open immersion and that the resulting system of homomorphisms is transitive, i.e. if we have morphism: $(U'',T'') \xrightarrow{v} (U',T') \xrightarrow{u} (U,T)$ then the following diagram commutes:*

$$\begin{array}{ccccc} v^{-1}u^{-1}F_{(U,T)} & \xrightarrow{v^{-1}(\rho_u)} & v^{-1}F_{(U',T')} & \xrightarrow{\rho_v} & F_{(U'',T'')} \\ \| & & & & \| \\ (u \circ v)^{-1}F_{(U,T)} & & \xrightarrow{\rho_{u \circ v}} & & F_{(U''T'')} \end{array}$$

The same description holds for sheaves of groups, rings etc.

Proof. Given a sheaf $F \in (X/S)_{Inf}$, an object $(U,T) \in \mathrm{Inf}(X/S)$ and an open $V \subset T$ one can associate a sheaf $F_{(U,T)}$ on T, as $F_{(U,T)}(V) = F((U \cap V) \to V)$. Then $F_{(U.T)}$ is indeed a sheaf because inherits all the required properties from $F \in (X/S)_{Inf}$. We now have to show that this correspondence is well behaved under morphisms.
Let (u,t) : $(U,T) \to (U',T')$ be a morphism and consider two open subsets $V' \subseteq T'$ and

$t^{-1}(V')=V \subseteq T$, so that we get commutative diagram:

$$
\begin{array}{ccc}
((U \cap V) \to V) & \longrightarrow & ((U' \cap V') \to V') \\
\downarrow & & \downarrow \\
(U \to T) & \longrightarrow & (U' \to T')
\end{array}
\qquad
\begin{array}{ccc}
F(U \cap V \to V) & \longleftarrow & F(U' \cap V' \to V')) \\
\uparrow & & \uparrow \\
F(U \to T) & \longleftarrow & F(U' \to T')
\end{array}
$$

Now rewriting the same diagram in term of the associated sheaves on T and T', we can see that we have obtained a morphism $F_{(U' \to T')} \to t_* F_{(U \to T)}$, therefore from the well known adjunction $()^{-1} \dashv ()_*$ we obtain a morphism $\rho_{T,T'} : t^{-1}F_{(U' \to T')} \to F_{(U \to T)}$. Now if $T \xrightarrow{t} T'$ is an open immersion then $t^{-1}(V') = V' \cap T$ for every $V' \subset T'$, and then $\rho_{T,T'}$ must be an isomorphism. □

Remark 2.6. With this characterization a sheaf $F \in (X/S)_{Inf}$ can be thought of as a system of Zariski sheaves, one for each thickening T of an open U of X but such that in some sense they are relative to U, that is, they reflect behavior around the closed U in T. One of the advantage of the Zariski interpretation of $(X/S)_{inf}$ is the fact that it has enough points, namely we can show that a morphism of sheaves $u : F \to G$ is an isomorphism looking at the stalks. It is enough to check that for each point $x \in X$ and each thickening T of a Zariski open neighborhood of x, the map of stalks: $(F_{(U,T)})_x \xrightarrow{u} (G_{(U,T)})_x$ is an isomorphism.

Example 2.7. If for each object $U \to T$ of Inf(X/S) we take the structural sheaf of rings $\mathcal{O}_T$, we can obtain a canonical sheaf of rings $\mathcal{O}_{X_{inf}}$ on Inf(X/S) (which will also be denoted as $\mathcal{O}_{X/S}$), so that we can also consider $(X/S)_{inf}$ as a ringed topos. We can define in the same way also two other sheaves of rings over Inf(X/S) as the sheaves $\mathscr{O}_X$ and $\mathscr{K}_{X/S}$ induced by the assignments $(U,T) \to \mathcal{O}_U$ and $(U,T) = \mathrm{Ker}(\mathcal{O}_T \to \mathcal{O}_U)$. We then have an exact sequence:

$$0 \to \mathscr{K}_{X/S} \to \mathcal{O}_{X_{inf}} \to \mathscr{O}_X \to 0 \tag{2.7.1}$$

Now we will introduce the stratifying topos which is important because we can interpret a $\mathcal{O}_X$-module on X, fortified with a stratification relative to S, as a special sheaf on this new site.

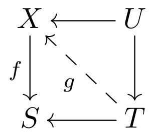
Definition 2.8. Define the category Strat(X/S) as the full sub-category of Inf(X/S) consisting of those objects $U \to T$ such that there exists locally a retraction $T \to X$. This category inherits the induced topology of Inf(X/S), hence Strat(X/S) is a site and we will denote the related topos as $(X/S)_{Strat}$.

Sheaves on Strat(X/S) can be described exactly in the same way as sheaf on Inf(X/S), so we can consider again a canonical sheaf of rings $\mathcal{O}_{X_{strat}}$ obtained by taking for each object (U,T) of Strat(X/S), the structural sheaf of rings $\mathcal{O}_T$ of T. Thus the category of all sheaves on Strat(X/S) can be considered as a ringed topos. From now on, given a morphism u we shall write u^{-1} for pull-back of a sheaf of sets, and u^* for module pull-back, that is u^{-1} followed by the tensor product with the structure sheaf.

Remark 2.9. Note that if X is smooth over S, then Strat(X/S) and Inf(X/S) are the same. This is true because in general given a morphism of schemes $f : X \to S$ it is equivalent to ask that f is smooth or that f is locally of finite presentation and formally smooth (in fact in [16] this is exactly the definition of smooth morphism). Recall that f is formally smooth if for all affine scheme T and all closed subscheme U of T defined by a nilpotent ideal I of $\mathcal{O}_T$, then for every morphism $T \to S$, the natural map:

$$\mathrm{Hom}_S(T, X) \xrightarrow{\Phi} \mathrm{Hom}_S(U, X)$$

induced by the canonical injection $U \to T$ is an epimorphism, equivalently it exists always the map g making commutative the following diagram of solid arrow:

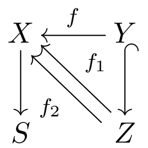


The importance of the stratifying topos relies in the following proposition.

Definition 2.10. A crystal of $\mathcal{O}_{X_{strat}}$-module is a sheaf F of $\mathcal{O}_{X_{strat}}$-module such that for any morphism $(U,T) \xrightarrow{t} (U', T')$ the homomorphism $t^* F_{(U',T')} \to F_{(U,T)}$ is an isomorphism.

Proposition 2.11. *The category of $\mathcal{O}_X$-modules on X fortified with a stratification structure relative to S is equivalent to the category of crystals of $\mathcal{O}_{X_{strat}}$-module.*

Proof. If F is an $\mathcal{O}_X$-module with a stratification relative to S, then for each diagram of S-morphisms:



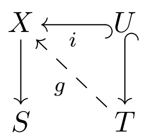
where $Y \to Z$ is a nilpotent immersion, and f_1, f_2 are any two extension of f, there is a canonical isomorphism:

$$f_1^*(F) \xrightarrow{\cong} f_2^*(F) \tag{2.11.1}$$

to obtain the isomorphisms above we can define a map:

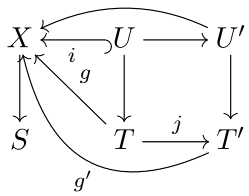
$$g = (f_1, f_2) : Z \to X \times_S X$$

as f_1, f_2 agree locally modulo a nilpotent ideal, then g factors through some infinitesimal neighborhood of the diagonal $\Delta^n_{X/S} = (X, \mathcal{P}^n_{X/S})$, so that we obtain the desired isomorphism from the proposition (1.13). Recall that the data of a stratification are a collection of $\mathcal{P}^n_{X/S}$-linear isomorphisms $\epsilon_n : \mathcal{P}^n_{X/S} \otimes F \to F \otimes \mathcal{P}^n_{X/S}$ which satisfy the cocycle condition. The cocycle condition gives us the compatibility between the maps ϵ_n that we need to obtain a transitive system of isomorphisms between the pull-back of F by the various extensions of f as in (2.11.1).
This said we can associate to F a crystal of $\mathcal{O}_{X_{strat}}$-modules $\mathcal{F}$. For each object of Strat(X/S):



(2.11.2)

where g is some retraction which exists locally, consider $\mathcal{F}_{(U \to T)} = g^*(F)$, since on F is defined a stratification structure, we have seen that this definition does not depend, up to canonical isomorphism, on the choice of the particular extension of i, hence we can glue together the local construction to build a $\mathcal{O}_{X_{strat}}$-module. We have only to prove that this $\mathcal{O}_{X_{strat}}$-module $\mathcal{F}$ on Strat(X/S) is a crystal. So let $(U,T) \xrightarrow{j} (U',T')$ be a morphism in Strat(X/S) and consider the following diagram:



(2.11.3)

where $g, g' \circ j$ are extension of i hence there are canonical isomorphism:

$$j^*g'^*(F) \xrightarrow{\cong} g^*(F) \quad \text{hence} \quad \rho_j : j^*\mathcal{F}_{(U',T')} \xrightarrow{\cong} \mathcal{F}_{(U,T)}$$

The cocycle condition satisfied by the stratification give us compatibility of these isomorphism with composition, in particular if we have morphisms:

$$(U,T) \xrightarrow{v} (U',T') \xrightarrow{u} (U'',T'')$$

then the following diagram commute:

$$\begin{array}{ccc} v^*u^*\mathcal{F}_{(U'',T'')} & \xrightarrow{v^*(\rho_u)} & v^*\mathcal{F}_{(U',T')} \\ \downarrow & & \downarrow{\scriptstyle \rho_v} \\ (u\circ v)^*\mathcal{F}_{(U''.T'')} & \xrightarrow{\rho_{u\circ v}} & \mathcal{F}_{(U,T)} \end{array}$$

indeed writing down a diagram analogue to (2.11.3), it is easy to see that the commutativity of the previous diagram is a consequence of (1.15) and the cocycle condition:

$$p_{12}^*(\epsilon_n) \circ p_{23}^*(\epsilon_n) = p_{13}^*(\epsilon_n) \tag{2.11.4}$$

where $p_{ij} : X_S^3 \to X_S^2$ are the projection via the coordinates i,j. Conversely if a $\mathcal{O}_{X_{strat}}$-module $\mathcal{F}$ on Strat(X/S) is a crystal we can obtain a $\mathcal{O}_X$-module with a stratification on S considering $F = \mathcal{F}_{(X \xrightarrow{id} X)}$. Indeed from the definition of $\mathcal{O}_{X_{strat}}$-module we know that $\mathcal{F}_{(X \xrightarrow{id} X)}$ is a $\mathcal{O}_X$-module on the Zariski site, to see that it is also fortified with a stratification structure we will use the fact that $\mathcal{F}$ is a crystal of $\mathcal{O}_{X_{strat}}$-module on Strat(X/S). We know that we always have a map from X to the n-th neighbourhood of the diagonal $X \to \Delta^n_{X/S}$ which is a closed S-immersion associated to a nilpotent ideal, and we have a morphism on Strat(X/S):

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ \downarrow & & \downarrow{\scriptstyle id} \\ \Delta^n_{X/S} & \xrightarrow{p_i^n} & X \end{array}$$

from the definition of special $\mathcal{O}_{X_{strat}}$-module we have an isomorphism:

$$\rho_i : (p_i^n)^*(\mathcal{F}_{(X\to X)}) \to (\mathcal{F}_{(X\to\Delta^n_{X/S})})$$

since this is an isomorphism we can consider the following composition:

$$\epsilon_n : (p_0^n)^*(F) \to \mathcal{F}_{(X \to \Delta^n_{X/S})} \to (p_1^n)^*(F)$$

these isomorphisms for every $n \geq 0$ can be interpreted as n-connection. In order to form a stratification structure we have to prove the cocycle condition, but this follows immediately from the transitivity condition of the special $\mathcal{O}_{X_{strat}}$-module $\mathcal{F}$ for the different inverse images of $\mathcal{F}$ over the projections of $X^3_S \to X^2_S$, in particular we have:

$$\begin{array}{ccccc} X & \longrightarrow & X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^n_{X/S}(2) & \xrightarrow{p_{ij}} & \Delta^n_{X/S} & \xrightarrow{p_k} & X \end{array}$$

with the usual identity:

$$\pi_1 = p_1 \circ p_{13} = p_1 \circ p_{12} \qquad \pi_2 = p_2 \circ p_{12} = p_1 \circ p_{23} \qquad \pi_3 = p_2 \circ p_{23} = p_2 \circ p_{13}$$

so that:

$$\begin{array}{ll} p_{12}^*(\epsilon_n) \circ p_{23}^*(\epsilon_n) : & (p_{23}^* \circ p_2^*)(F) \xrightarrow{p_{23}^*(\epsilon_n)} (p_{23}^* \circ p_1^*)(F) = (p_{12}^* \circ p_2^*)(F) \xrightarrow{p_{12}^*(\epsilon_n)} (p_{12}^* \circ p_1^*)(F) \\ p_{13}^*(\epsilon_n) : & (p_{23}^* \circ p_2^*)(F) = (p_{13}^* \circ p_2^*)(F) \xrightarrow{p_{13}^*(\epsilon_n)} (p_{13}^* \circ p_1^*)(F) = (p_{12}^* \circ p_1^*)(F) \end{array}$$

and from the transitivity of the isomorphism due to the structure of a special $\mathcal{O}_{X_{strat}}$-module we obtain the cocycle condition (2.11.4). □

2.1 Functoriality

It is a general fact that working with a topos over a suitable site has its advantage because usually the category of functors inherits some properties of the target category. As an example in our case one of the advantage of the infinitesimal topos over the infinitesimal site is the functoriality. If $g : X' \to X$ is an S-morphism, there isn't a canonical way to pull back thickenings in X to thickenings in X', but we will be able to pull back the sheaves they represent and hence obtain a morphism of topos $(X'/S)_{inf} \xrightarrow{g} (X/S)_{inf}$.
Functoriality is also the first step to obtain an interesting theory of cohomology over this new site, so now our interest is to prove the following lemma:

Lemma 2.12. *([1],III,2.2) Given two morphism of schemes $f : X' \to X$ and $S' \to S$ such that the following diagram commute:*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

Then we obtain a morphism of topos $f_{inf} : (X'/S')_{inf} \to (X/S)_{inf}$.

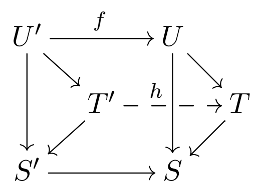
The proof of the lemma is divided in multiple part. First we define a functor for representable:

$$f^{-1} : \text{Inf}(X/S) \xrightarrow{Y} (X/S)_{inf} \to (X'/S')_{inf}$$

Where given $(U,T) \in \text{Inf}(X/S)$ then $f^{-1}(U,T)$ is the presheaf on $\text{Inf}(X'/S')$ defined by:

$$(U',T') \to \text{Hom}_f((U',T'),(U,T))$$

this means the following: $(f^{-1}(U,T))(U',T') = \emptyset$ if $f(U') \subsetneq U$ while if $f(U') \subset U$, we have $(f^{-1}(U,T))(U',T') = \{h : T' \to T\}$, where h is such that the following diagram commute:



(2.12.1)

It is easy to see that this presheaf is in fact a sheaf but it may not be representable again. Now this functor is important because we will show that there is a unique way to construct a morphism of topos $f_{inf} : (X'/S')_{inf} \to (X/S)_{inf}$ such that $f_{inf}^{-1}(\widetilde{U,T}) = f^{-1}(U,T)$ for every $(U,T) \in \text{Inf}(X/S)$. To do this we want to apply the following proposition on the infinitesimal site.

Proposition 2.13. *Let $\mathcal{C},\mathcal{D}$ be categories and call $\tilde{\mathcal{C}},\tilde{\mathcal{D}}$ the corresponding presheaves categories. Suppose that we have a functor $\eta : \mathcal{C} \to \tilde{\mathcal{D}}$. Then we have a unique pair of functor:*

$$\eta_* : \tilde{\mathcal{D}} \to \tilde{\mathcal{C}} \quad \eta^* : \tilde{\mathcal{C}} \to \tilde{\mathcal{D}}$$

such that they are adjoint $\eta^* \dashv \eta_*$ and considering the Yoneda embedding $Y : \mathcal{C} \to \tilde{\mathcal{C}}$ given by the assignment $X \to Hom_{\mathcal{C}}(\_, X)$, then $\eta^*_{|C} = \eta$.

Proof. First we have to find the definitions of the two claimed functors $\eta^* \dashv \eta_*$. If $G \in \tilde{\mathcal{D}}$, we want to find $\eta_*(G)$ a presheaf on $\mathcal{C}$, so for all $C \in \mathcal{C}$ we can set:

$$\eta_*(G)(C) \overset{\mathbb{Y}}{=} \text{Hom}_{\tilde{\mathcal{C}}}(\tilde{C}, \eta_*(G)) = \text{Hom}_{\tilde{\mathcal{D}}}(\eta^*(\tilde{C}), G) = \text{Hom}_{\tilde{\mathcal{D}}}(\eta(C), G)$$

This proves the existence and uniqueness of η_*. Since η_* is unique, so is its adjoint but it remains to check that it exists. Again to prove the existence of η^* we will give a formula for it. Let $D \in \mathcal{D}$, and consider the category $\eta\{D\}$ of all morphisms $\{\tilde{D} \to \eta(C) : C \in \mathcal{C}\}$. Then we can define:

$$\eta^*(F)(D) = \varinjlim_{\eta\{D\}} F(C) \qquad \text{with } F \in \tilde{\mathcal{C}} \text{ and } D \in \mathcal{D}$$

We can find in ([2],5.8) the checking for the adjunction, i.e:

$$\text{Hom}_{\tilde{\mathcal{D}}}(\eta^*(F), G) \cong \text{Hom}_{\tilde{\mathcal{C}}}(F, \eta_*(G))$$

□

Proof of the lemma (2.12). We have defined a functor $f^{-1} := \eta : \text{Inf}(X/S) \to \text{Pshv}(\text{Inf}(X'/S'))$, so that we can apply this last proposition in our situation. To obtain a morphism of topos $f_{inf} : (X'/S')_{inf} \to (X/S)_{inf}$ what it remains to prove is that:

(a) For any $T \in \text{Inf}(X/S)$, then $f^{-1}(T) = \eta(T)$ is a sheaf on $\text{Inf}(X'/S')$.

(b) For any sheaf F on $\text{Inf}(X'/S')$, then $f_{inf*}(F) = \eta_*(F)$ is a sheaf on $\text{Inf}(X/S)$.

(c) For any sheaf G on $\text{Inf}(X/S)$ let $f_{inf}^{-1}(G)$ be the sheafification of the presheaf $\eta^*(G)$ then we have an adjunction $f_{inf}^{-1} \dashv f_{inf*}$.

(d) f_{inf}^{-1} commute with finite inverse limit.

(a) To check the sheaf condition for $\eta(T) = f^{-1}(T)$ consider a covering family $\{T_i' \to T'\}$ in $\text{Inf}(X'/S')$ and a matching family of element $\{h_i \in \eta(T)(T_i')\}$. We have to define an element $h \in \eta(T)(T')$ such that induces all the $\{h_i\}$. To do that since $\bigcup_i T_i' = T'$ and since the h_i are compatible with each other, we can just define h locally as $h_{|T_i'} = h_i$.

(b) We have to check the sheaf condition for $f_{inf*}(F) = \eta_*(F)$. So consider a covering family $\{T_i \to T\}$ in $\text{Inf}(X/S)$ and a matching family of element $\{s_i \in \eta_*(F)(T_i)\}$. We have to find a unique section $s \in \eta_*(F)(T)$ inducing $\{s_i\}$. By definition s is a morphism of sheaves $s : \eta(T) \to F$ and is given by maps of sets $s_{T'} : \eta(T)(T') \to F(T')$ for every $T' \in \text{Inf}(X'/S')$ compatible with change of T' i.e. given any map $T'' \to T'$ then the following diagram must commute:

$$\begin{array}{ccc} \eta(T)(T') & \xrightarrow{s_{T'}} & F(T') \\ \downarrow & & \downarrow \\ \eta(T)(T'') & \xrightarrow{s_{T''}} & F(T'') \end{array} \tag{2.13.1}$$

Obviously the same description is valid for the element of the matching family $\{s_i\}$, what we have to do is to define each $s_{T'}$ in such a way that induce the relative map $s_{i|T'}$. So now suppose $h \in \eta(T)(T')$ so by definition $h : T' \to T$ is a morphism that make commutative the diagram (2.12.1). Let $T_i' = h^{-1}(T_i)$ so that we obtain a covering $\{T_i' \to T'\}$ of T' and maps $T_i' \xrightarrow{h_i} T_i \in \eta(T_i)(T_i')$ induced by h. We have a matching family of element which are sheaf maps $s_i : \eta(T_i) \to F$ on $\text{Inf}(X'/S')$, this means that every s_i has some component $s_{i|T_i'} : \eta(T_i)(T_i') \to F(T_i')$, and we can evaluate our h_i over this maps to obtain a matching family of elements $\{s_{i|T_i'}(h_i)\}$ of $F(T_i')$ and since F is indeed a sheaf there must be an unique element $s_{T'}(h)$ of $F(T')$ inducing $\{s_{i|T_i'}(h_i)\}$. Now we have defined a set of maps $s_{T'}$, but we have still to check the compatibility with T' i.e. the commutativity of the diagram (2.13.1), but this is easy since we have to remember that if we have a morphism $T'' \xrightarrow{g} T'$ in $\text{Inf}(X'/S')$ and a section $h \in \eta(T)(T')$ then the following diagram commute:

$$\begin{array}{ccccccccc} U'' & & \longrightarrow & & U' & & \xrightarrow{f} & & U & \\ \downarrow & \searrow & & & \downarrow & \searrow & & & \downarrow & \searrow \\ & & T'' & \xrightarrow{g} & & & T' & \overset{h}{\dashrightarrow} & & T \\ \downarrow & \swarrow & & & \downarrow & \swarrow & & & \downarrow & \swarrow \\ S' & & = & & S' & & \longrightarrow & & S & \end{array}$$

So that $h \circ g$ is a section of $\eta(T)(T'')$ and with the above construction is not difficult to see that $F(g)(s_{T'}(h)) = s_{T''}(h \circ g)$. This means that we have indeed defined a morphism of sheaves $s : \eta(T) \to F$ which is a section of $s \in \eta_*(F)(T)$ inducing $\{s_i\}$, hence $\eta_*(F)(T)$ is a sheaf.

(c) If G is a sheaf on Inf(X/S), and if F is a sheaf on Inf(X'/S'), we have by the universal property of sheafification:

$$\begin{aligned}\mathrm{Hom}_{(X'/S')_{inf}}(f_{inf}^{-1}(G), F) &\cong \mathrm{Hom}_{\mathrm{Psh}(Inf(X'/S'))}(\eta^*(G), F) \\ &\cong \mathrm{Hom}_{\mathrm{Psh}(Inf(X/S))}(G, \eta_*(F)) \cong \mathrm{Hom}_{(X/S)_{inf}}(G, f_{inf\,*}(F))\end{aligned}$$

(d) ([1],III,2.2.7) Given J a finite category and an inverse system of sheaves $\{G_j : j \in J\}$ in $(X/S)_{inf}$, as usual there is a natural map of sheaves $f_{inf}^*(\varprojlim G_j) \to \varprojlim f_{inf}^*(G_j)$, we want that this is an isomorphism. As we have said in (2.6) it is enough to check the stalk at every point of X'. In particular if G is a sheaf on Inf(X/S), and if (U', T') is an object of Inf(X'/S') with $x' \in T'$, then we can describe the stalk of $f_{inf}^*(G)$ at (U', T', x') as the direct limit $\varinjlim\{G(T) | h : V' \to T\}$, where the limit is taken over the small category I which has as object map $h : V' \to T$ with V' an open neighborhood of x' in T' and T a thickening of an open set U of X, in this context we can think of a morphism from h_1 to h_2 in I as commutative diagrams:

$$\begin{array}{ccc} V'_2 & \xrightarrow{h_2} & T_2 \\ \downarrow & & \downarrow \\ V'_1 & \xrightarrow{h_1} & T_1 \end{array}$$

Thus we have to show that for all (U', T', x') in Inf(X'/S') we have isomorphisms:

$$\varinjlim_I \varprojlim_J G_j(T) \xrightarrow{\cong} \varprojlim_J \varinjlim_I G_j(T)$$

this is true since the category of indices I is filtering [[2],5.10] and in general finite inverse limits commute with filtering direct limits in every Grothendieck topos [28].

□

Proposition 2.14. *The morphism f_{inf} defined in (2.12) is canonically a morphism of ringed topos.*

Proof. Recall that we have defined a structure of ringed topos $((X/S)_{inf}, \mathcal{O}_{X_{inf}})$ where for every element (U,T) then $\mathcal{O}_{X_{inf}}(U,T) = \mathcal{O}_T$. Now given morphisms of schemes $f : X' \to X$ and $S' \to S$ such that the following diagram commute:

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

we managed to define a morphism of topos f_{inf}, it remains to show that actually f_{inf} is a morphism of ringed topos:

$$(f_{inf}, f_{inf}^\sharp) : ((X'/S')_{inf}, \mathcal{O}_{X'_{inf}}) \to ((X/S)_{inf}, \mathcal{O}_{X_{inf}})$$

In fact the only thing left to define is a natural map $f^{\sharp}_{inf} : \mathcal{O}_{X_{inf}} \to f_{inf*}(\mathcal{O}_{X'_{inf}})$, so let $T \in \mathrm{Inf}(X/S)$ then we need a map:

$$f^{\sharp}_{inf}(T) : \mathcal{O}_{X_{inf}}(T) \to f_{inf*}(\mathcal{O}_{X'_{inf}})(T) = \mathrm{Hom}_{(X'/S')_{inf}}(f^{-1}(T), \mathcal{O}_{X'_{inf}})$$

for every object $T' \in \mathrm{Inf}(X'/S')$ then if $h \in f^{-1}(T)(T')$, h is a map $T' \to T$ such that the diagram (2.12.1) commute, and hence provides us a map of rings:

$$\mathcal{O}_{X_{inf}}(T) = \mathcal{O}_T(T) \to \mathcal{O}_{T'}(T') = \mathcal{O}_{X'_{inf}}(T')$$

so that $f^{\sharp}_{inf}$ is essentially the evaluation map. Note also that since for each $h \in f^{-1}(T)(T')$ we have obtained a map $\mathcal{O}_T(T) \to \mathcal{O}_{T'}(T')$ then using the notation of (2.13) we have a natural map:

$$\varinjlim_{\eta\{T'\}} \mathcal{O}_T(T) \to \mathcal{O}_{T'}(T')$$

which gives us the adjoint morphism of sheaves of rings of $f^{\sharp}_{inf}$, namely:

$$f^{\sharp}_{inf} : f^{-1}_{inf}(\mathcal{O}_{X_{inf}}) \to \mathcal{O}_{X'_{inf}}$$

□

Proposition 2.15 ([1],III.2.2.6). *Consider the following commutative diagram:*

$$\begin{array}{ccccc} X'' & \xrightarrow{g'} & X' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S'' & \longrightarrow & S' & \longrightarrow & S \end{array}$$

Then it exists a natural isomorphism of morphism of ringed topos:

$$(g \circ g')_{inf} \cong g_{inf} \circ g'_{inf}$$

Finally we can use the general theory of ringed topos to define cohomology as in the appendix (C) and all the functoriality properties of cohomology will follow.

Proposition 2.16. *Given two morphism of schemes $f : X' \to X$ and $S' \to S$ such that the following diagram commute:*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

so that f induce a morphism of topos $f_{inf} : (X'/S')_{inf} \to (X/S)_{inf}$. If F is an abelian sheaf in $(X'/S')_{inf}$ there is a Leray spectral sequence (C.33.1):

$$E_2^{pq} = H^p(X/S_{inf}, R^q f_{inf*}F) \implies H^{p+q}(X'/S'_{inf}, F)$$

2.2 Relation with the Zariski Topos

As it will become apparent later, it is useful for some calculation of cohomology on the infinitesimal site to have a comparison with the Zariski site. In particular now we describe a projection from the infinitesimal topos to the Zariski topos.

Proposition 2.17. *There is a natural morphism of topos:*

$$u_{X/S} : (X/S)_{inf} \to X_{Zar}$$

given by:

- *For $F \in (X/S)_{inf}$ and $j : U \to X$ an open immersion: $(u_{X/S_*}(F))(U) = \Gamma(U/S_{inf}, j_{inf}^{-1}F)$*
- *For $E \in X_{Zar}$ and $(U,T) \in$ Inf(X/S) then: $(u^*_{X/S}(E))(U,T) = E(U)$.*

for every abelian sheaf F in $(X/S)_{inf}$ we have the related Leray spectral sequence (C.33.1):

$$E_2^{pq} = H^p(X_{Zar}, R^q u_{X/S_*}(F)) \implies H^{p+q}(X/S_{inf}, F)$$

Proof. It is not difficult to see that we have an adjunction:

$$\text{Hom}(u_{X/S}{}^*(E), F) \xrightarrow{\cong} \text{Hom}(E, u_{X/S_*}(F))$$

the proof of the adjunction can be found in ([1],III,3.2.3). In particular we should think to u_{X/S_*} as the set of global section of F over $(U/S)_{inf}$, moreover we have that $(u_{X/S}{}^*(E)_{(U,T)}) \cong E_{|U}$, for any (U,T). This proves the fact that the functor $u_{X/S}{}^*$ commute with arbitrary inverse limit and since it is also left exact because has a right adjoint, then it is exact. Thus we do have obtained a morphism of topos. □

The previous morphism is not a morphism of ringed topos, since in general there isn't a canonical map $\mathcal{O}_X \to u_{X/S*}\mathcal{O}_{X_{inf}}$. We call $u_{X/S}$ a projection because we can define also a section $i_{X/S} : X_{Zar} \to (X/S)_{inf}$. In particular since we have seen that $u_{X/S}{}^*$ is exact, we can try to show that it has a left adjoint.

Proposition 2.18. *There is a natural morphism of ringed topos:*

$$(i_{X/S}, i_{X/S}{}^\sharp) : X_{Zar} \to (X/S)_{inf}$$

given by the pair of adjoint functor $(i_{X/S}{}^*, i_{X/S_*})$ where $i_{X/S_*} = u_{X/S}{}^*$ and, for $F \in X/S_{inf}$ consider the identity $(X,X) \in$ Inf(X/S), and define $i_{X/S}{}^*(F) = F_{(X,X)}$. The morphism $i_{X/S}$ is such that:

$$u_{X/S} \circ i_{X/S} = Id_{X_{Zar}}$$

Since from the definition i_{X/S_*} is exact the Leray spectral sequence (C.33.1) degenerates and for every sheaf E on X_{Zar} we have:

$$H^\bullet(X_{Zar}, E) \cong H^\bullet(X_{inf}, i_{X/S_*}E)$$

Proof. From the definition we get that $i^*_{X/S}$ commutes with inverse limits and that $i_{X/S}$ is indeed a section for $u_{X/S}$, the proof of the following adjunction is not difficult and it can be found in ([1],III,3.3.2):

$$\mathrm{Hom}_{Zar}(i^*_{X/S}(F), E) \cong \mathrm{Hom}_{Inf}(F, i_{X/S_*}(E))$$

Unlike $u_{X/S}$, in this case $i_{X/S}$ it is a morphism of ringed topos because there is always a map:

$$\mathcal{O}_{X_{inf}} \to i_{X/S_*}\mathcal{O}_X \cong \mathscr{O}_X$$

note that this map and the sheaf $\mathscr{O}_X$ on the infinitesimal site are exactly the same as in the remark (2.7.1). □

Remark 2.19. It is possible to show ([1],III.3.4) the compatibility between these new morphisms of topos $u_{X/S}$ and $i_{X/S}$ and the functoriality morphism of (2.12).

What we really need in the next section is a relation between X/S_{inf} and the Zariski topos of every thickening T coming from an object $(U,T) \in \mathrm{Inf}(X/S)$. In [2] using the general notion of localisation in a Grothendieck topos it is proven the following important property:

Proposition 2.20. *([2],5.26) Let $(U,T) \in Inf(X/S)$, then there is a commutative diagram of morphism of topos:*

$$\begin{array}{ccccc} (X/S)_{inf\,|(U,T)} & \xrightarrow{\varphi_T} & T_{Zar} & \xleftarrow[\cong]{} & U_{Zar} \\ \downarrow{\scriptstyle j_T} & & \downarrow{\scriptstyle i_T} & \swarrow{\scriptstyle i_U} & \\ (X/S)_{inf} & \xrightarrow{u_{X/S}} & X_{Zar} & & \end{array}$$

Where:

- *If E is a sheaf in $(X/S)_{inf}$, then $\varphi_{T*}(j_T{}^*(E)) = E_{(U,T)}$.*
- *φ_{T*} is an exact functor.*
- *If F is an abelian sheaf in $(X/S)_{inf}$ then we have a canonical isomorphism:*

$$H^\bullet_{inf}((U,T), F) \xrightarrow{\cong} H^\bullet_{zar}(T, F_{(U,T)}) \tag{2.20.1}$$

Proof. For our purpose we just check the last point. In particular if A is a sheaf of ring on $(X/S)_{Inf}$ we already know that the functor $\pi := \varphi_{T*}(j_T{}^*(\_))$ which for every A-module F on $(X/S)_{Inf}$ associates the $A_{(U,T)}$-module $\pi(F) = F_{(U,T)}$ over T_{Zar} is an exact functor (2.6) for every choice of $(U,T) \in \mathrm{Inf}(X/S)$, so that it "is" its derived functor $R\pi = \pi$:

$$D((X/S)_{Inf}, A) \xrightarrow{R\pi} D(T_{Zar}, A_{(U,T)}) \qquad F^\bullet \to \pi(F^\bullet) = F^\bullet_{(U,T)}$$

since we have also the identity:

$$\Gamma_{inf}((U,T), F) = \Gamma_{Zar}(T, F_{(U,T)}) = \Gamma_{Zar}((T, \pi(F))$$

we have also a natural isomorphism of the derived functors in $D^+(\Gamma(T, A_{(U,T)}))$:

$$R\Gamma((U,T), F^\bullet) \xrightarrow{\cong} R\Gamma(T_{Zar}, R\pi(F^\bullet))$$

which in particular since we have said that $R\pi = \pi$ tells us that:

$$H^\bullet_{inf}((U,T), F) \xrightarrow{\cong} H^\bullet_{zar}(T, F_{(U,T)}) \tag{2.20.2}$$

□

2.3 Computation of infinitesimal cohomology

So now following [17] we will give two methods to compute cohomology on the topos that we have defined. For our purpose we will take X a scheme over the base S and given F a module on Strat(X/S), we will suppose that F satisfies the following condition:

- For each object $U \to T$, then $F_{(U\to T)}$ is quasi-coherent.
- For each morphism $(U,T) \xrightarrow{j} (U',T')$ of Strat(X/S) such that $j : T \to T'$ is an immersion, we have that $F_{(U\to T)} = j^*(F_{(U'\to T')})$.

Remark 2.21. In general can be easier to define and compute cohomology if the site does have a final object, unfortunately this is not the case for the infinitesimal and stratifying topos. To prove that in general the infinitesimal site Inf(X/S) does not have a final object, essentially we will prove that given $(U,T) \in$ Inf(X/S) we can always get a morphism $(U,T) \to (U',T')$ where (U',T') is not isomorphic to (U,T). We can restrict ourselves to the situation where U is affine and let $i : U \to U'$ an open immersion such that $i(U)$ is contained in an affine subset of U'. Then we can prove that it exists a thickening T' of U' and a morphism $T \to T'$ making the following diagram commute:

$$\begin{array}{ccc} U & \xrightarrow{i} & U' \\ \downarrow & & \downarrow \\ T & \xrightarrow{i'} & T' \end{array}$$

where $(U',T') \in$ Inf(X/S). Notice that this is a push-out problem in the category of schemes, and we can construct explicitly T'. As a topological space T' must be equal to U', so that the topological map $i' : T \to T'$ equal i. We define the structure sheaf of T' as:

$$\mathcal{O}_{T'} = \mathcal{O}_{U'} \times_{i_*\mathcal{O}_U} i'_*\mathcal{O}_T$$

Now to see that $(T',\mathcal{O}_{T'})$ is a scheme we need to show that every point has an affine neighbourhood. Since the formation of fiber product of sheaves commutes with restricting to opens, we may assume that U' is affine so that also U is affine because i is an affine map. In this situation also T is affine because as we will see any thickening of an affine scheme is affine, so that $U' \leftarrow U \to T$ correspond to morphism of ring $A' \to A \leftarrow B$. From our definition then the global section of $\mathcal{O}_{T'}$ is the ring $B' = A' \times_A B$. Since $B \to A$ is surjective with locally nilpotent kernel the same is true for the canonical map $B' \to A'$, so that Spec$(A') =$ Spec(B'). Obviously we want to show that $T' =$ Spec(B'), but this is true since they are the same on principal open subset, if $b' \in B'$ with image $a' \in A'$ then $\mathcal{O}_{T'}(D(a')) = B'_{b'}$, in particular if $a \in A$ and $b \in B$ are the corresponding image of $b' \in B'$ then we have:

$$B'_{b'} = A'_{a'} \times_{A_a} B_b \cong \mathcal{O}_{U'}(D(a')) \times_{i_*\mathcal{O}_U(D(a'))} i'_*\mathcal{O}_T(D(a')) = \mathcal{O}_{T'}(D(a'))$$

With this construction T' is also a push-out because we can construct an affine open covering of $T' = \bigcup_i T'_i$ such that the corresponding opens in U, U', T are affine and in the affine situation the diagram is a push-out.

It remains to prove that any finite order thickening of an affine scheme is again affine. Therefore let X be an affine scheme, and consider $i : X \to X'$ a thickening, recall that the Serre vanishing theorem states that a scheme X' is affine if for all quasi-coherent sheaves of $\mathcal{O}_{X'}$-modules $\mathcal{F}$ we have $H^1(X', \mathcal{F}) = 0$. Now consider $\mathcal{I} = \mathrm{Ker}(i^\sharp : \mathcal{O}_{X'} \to i_*\mathcal{O}_X)$, the sheaf of ideal defined by our thickening, and let $\mathcal{F}_n = \mathcal{I}^n\mathcal{F}$ so that we have obtained a filtration:

$$0 = \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \dots \mathcal{F}_0 = \mathcal{F}$$

Then each quotient $\mathcal{F}_a/\mathcal{F}_{a+1}$ is annihilated by $\mathcal{I}$, so that can be considered as inverse image of some $\mathcal{O}_X$-modules $\mathcal{G}_a$, namely $\mathcal{F}_a/\mathcal{F}_{a+1} = i_*\mathcal{G}_a$ and we obtain:

$$H^1(X', \mathcal{F}_a/\mathcal{F}_{a+1}) \cong H^1(X', i_*\mathcal{G}_a) \cong H^1(X, \mathcal{G}_a) = 0 \tag{2.21.1}$$

where since X is affine for hypothesis the last equality is exactly the Serre vanishing theorem. The second isomorphism is again a consequence of Serre's criterion of affineness, indeed it can be proved ([16],1.7.17) that under suitable condition it is possible to characterize affine morphisms of schemes in terms of exactness properties of the related direct image functor, so that in our case since i is affine, then i_* is an exact functor and the second isomorphism of (2.21.1) is obtained from the degeneration of the usual Leray spectral sequence (B.11). Thus $\mathcal{F}$ has a finite filtration whose successive quotients have vanishing first cohomology groups and it follows by a simple induction argument that also $H^1(X', \mathcal{F}) = 0$.
Finally, the last small detail that needs to be checked is the existence of some $\mathcal{O}_X$-module $\mathcal{G}_a$ such that $\mathcal{F}_a/\mathcal{F}_{a+1} = i_*\mathcal{G}_a$. As $i : X \to X'$ is a closed immersion of schemes, recall that if $\mathcal{I} \subseteq \mathcal{O}_X$ is the quasi-coherent sheaf of ideals cutting out X, then there is a functor:

$$i_* : \mathcal{O}_X\text{−Mod} \to \mathcal{O}_{X'}\text{−Mod}$$

which is exact, fully faithful, with essential image those quasi-coherent $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

2.3.1 The stratifying topos

Despite the fact that it may happen that a site doesn't have a final object, in a general topos $\mathcal{T}$ we have a canonical way to obtain a final object e, namely it is the sheafification of the constant presheaf whose value at any element is always the singleton $\{*\}$. In particular, as the site Inf(X/S) does not have a final object, then the final object of $(X/S)_{Inf}$ is not representable, but we claim that often it can be covered by representable sheaves. For this reason we will use a Čech technique to compute cohomology in the topos that we have defined, thus first of all we are interested in coverings of the final object e. Suppose that there is a closed immersion $i : X \to Y$, with Y/S smooth, then we can consider the sheaf:

$$\tilde{Y} := i^*_{inf}(Y)$$

where remember that $i^*_{inf}(Y)$ is the sheaf which value at (U, T) is the set of S-morphisms of T in Y such that extends the restriction of i to U as in the diagram (2.12.1). The map $\tilde{Y} \to e$ is a covering of the final object since it is an epimorphism ([28],II,4.3), i.e. for every element $(U, T) \in$ Inf(X/S) then $\tilde{Y}(T) \neq \emptyset$, this in turn is true as for sufficiently small open subset T' of T, the set $\tilde{Y}(T')$ is not empty because the map $T' \cap X \xrightarrow{i} Y$ can be lifted to T' since Y/S is smooth and $X \cap T' \to T'$ is a nilpotent immersion. From the definition of $\tilde{Y}$ it follows that if $(X, Y) \in$ Inf(X/S) then $\tilde{Y}$ is representable, but in general could happen that $\tilde{Y}$ is not itself representable since the map $X \xrightarrow{i} Y$ might not be an element of Inf(X/S), in this case we can describe the sheaf as limit of representable sheaves:

$$\tilde{Y} = \varinjlim_i \widetilde{Y(i)}$$

where $\widetilde{Y(i)}$ is the sheaf represented by $(X, Y(i))$, the i-th infinitesimal neighborhood of X in Y. Note also that if X/S is smooth or equivalently is we restrict ourselves to the stratifying topos, then the covering of the final object is simply given by $\tilde{X} \to e$ where $\tilde{X}$ is the object represented by the identity $X \xrightarrow{id} X$.

Proposition 2.22. *Let X be a scheme above the base S. Let F be any module on Strat(X/S), then it exists a cosimplicial sheaf $\mathscr{F}^\bullet$ on X_{zar} such that there is a canonical isomorphism:*

$$H^\bullet(X_{strat}, F) \xrightarrow{\cong} \mathbb{H}^\bullet(X_{zar}, \mathscr{F}^\bullet)$$

Proof. If F is any module on Strat(X/S), since our covering is simply $\{\tilde{X} \to e\}$, with e the final object of the stratifying topos, we have a spectral sequence (C.41):

$$H^\bullet(X_{strat}, F) \Leftarrow E_2^{pq} = H^p(\nu \to H^q(\tilde{X}^{\nu+1}, F)) \tag{2.22.1}$$

The sheaf $\tilde{X}^{\nu+1}$ is usually not representable for $\nu \geq 1$, but we can describe it as a colimit of representable sheaf:

$$\tilde{X}^{\nu+1} = \prod^{\nu+1} \tilde{X} = \varinjlim_i (\widetilde{\Delta^i_{X/S}(\nu)})$$

where $\Delta^i_{X/S}(\nu)$ denotes the i^{th}-infinitesimal neighborhood of the diagonal of $X \times_S \cdots \times_S X$ ($\nu + 1$ times). So now we want to calculate more explicitly $H^q(\tilde{X}^{\nu+1}, F)$. If X is affine, $\Delta^i_{X/S}(\nu)$ are also affine, thus from Serre's vanishing theorem this means that:

$$H^q_{strat}((X, \Delta^i_{X/S}(\nu)), F)) \overset{(2.20.2)}{=} H^q_{Zar}(\Delta^i_{X/S}(\nu), F_{(X, \Delta^i_{X/S}(\nu))}) = 0 \quad \text{if } q > 0 \tag{2.22.2}$$

If we want to use this vanishing of cohomology to show that the spectral sequence (2.22.1) degenerates, we have to show that the cohomology functors $H^q(\_, F)$ commute with direct limit. To do this it is useful to recall the following theorem from general homological algebra.

Theorem 2.23. *([33]3.5.8) Let $\mathcal{A}$ be any abelian category satisfying Grothendieck's axiom AB4^* ($\prod_{i=1}^\infty$ is exact), and let $\ldots \to C_1 \to C_0$ be a tower of cochain complexes in $\mathcal{A}$ satisfying the Mittag-Leffler condition, namely for all k exists j such that for all $i \geq j$ the image of the C_i in C_k satisfy the descending chain condition, then there is an exact sequence for each q:*

$$0 \to \varprojlim_i{}^1 H^{q-1}(C_i) \to H^q(\varprojlim_i(C_i)) \to \varprojlim_i H^q(C_i) \to 0$$

In general the category of quasi-coherent sheaves on a scheme, even if is a nice abelian category, do not satisfies $AB4^*$. If X is affine though this condition is satisfied, and we can rephrase the theorem in our situation saying that we have a tower of cochain complexes Hom$(\widetilde{\Delta^i_{X/S}(\nu)}, I^\bullet)$

with $I^\bullet$ a injective resolution for F so that the cohomology groups of the cochain complex are exactly the Ext groups, hence for every q we have an exact sequence:

$$0 \to \varprojlim_i{}^1 \mathrm{Ext}^{q-1}(\widetilde{\Delta^i_{X/S}(\nu)}, F) \to \mathrm{Ext}^q(\varinjlim_i(\widetilde{\Delta^i_{X/S}(\nu)}), F) \to \varprojlim_i \mathrm{Ext}^q(\widetilde{\Delta^i_{X/S}(\nu)}, F) \to 0$$

The groups of the cochain complex $\{\mathrm{Ext}^{q-1}(\widetilde{\Delta^i_{X/S}(\nu)}, F)\}$ are zero for $q \geq 2$ as we have seen in (2.22.2), hence the complex satisfies the Mittag-Leffler condition and in this situation is clear that $\varprojlim_i{}^1 \mathrm{Ext}^{q-1}(\widetilde{\Delta^i_{X/S}(\nu)}, F)$=0, so that finally we obtain the isomorphism:

$$H^q(\tilde{X}^{\nu+1}, F) = H^q(\widetilde{\varinjlim_i \Delta^i_{X/S}(\nu)}, F) = \varprojlim_i H^q(\widetilde{\Delta^i_{X/S}(\nu)}, F)$$

Hence we have obtained the degeneration of the spectral sequence (2.22.1) which gives us canonical isomorphisms:

$$\left.\begin{array}{l} H^q(\tilde{X}^{\nu+1}.F) = 0 \text{ if } q > 0 \\ H^0(\tilde{X}^{\nu+1}, F) = \varprojlim_i F(\Delta^i_{X/S}(\nu)) \end{array}\right\} \longrightarrow H^\bullet(X_{strat}, F) \xrightarrow{\cong} H^\bullet(\nu \to F(X^\nu/X)) \qquad (2.23.1)$$

where X^ν/X denotes the formal scheme $\varinjlim_i \Delta^i_{X/S}(\nu)$.

If X is no longer affine, obviously we do not obtain the same degeneration of the spectral sequence (2.22.1). To deal with this situation we will denote U^ν/U the formal scheme $\varinjlim_i \Delta^i_{U/S}(\nu)$, and we introduce the sheaf $\mathscr{F}^{(\nu)}$ on X_{zar} defined by:

$$\mathscr{F}^{(\nu)} : U \to F(U^\nu/U) = \varprojlim_i F(\Delta^i_{U/S}(\nu)) \qquad (2.23.2)$$

Then, for variable ν, the $\mathscr{F}^{(\nu)}$ form a cosimplicial sheaf $\mathscr{F}^\bullet$ on X_{zar}. If we take now an affine open covering of X, we can use the previous isomorphism (2.23.1) to deduce from a Čech calculation a canonical isomorphism:

$$H^\bullet(X_{strat}, F) \xrightarrow{\cong} \mathbb{H}^\bullet(X_{zar}, \mathscr{F}^\bullet) \Leftarrow E_2^{pq} = H^p(\nu \to H^q(X_{zar}, \mathscr{F}^{(\nu)}))$$

□

2.3.2 The infinitesimal topos

In the case of the infinitesimal site the situation is similar but not identical, indeed in general is not true that the sheaf $\tilde{X}$ over Inf(X/S) covers the final object of the infinitesimal topos, however as we have said before if we can find an S-immersion $X \to Y$ with Y formally smooth in S (for example this is always possible with X and S affine), then we can consider:

$$\tilde{Y} := \varinjlim_i \widetilde{Y(i)}$$

with $Y(i)$ the i^{th}-infinitesimal neighborhood of X in Y.

Proposition 2.24. *Let X be a scheme above the base S. Let G be any module on Inf(X/S), then it exists a cosimplicial sheaf $\mathscr{G}^\bullet$ on X_{zar} such that there is a canonical isomorphism:*

$$H^\bullet(X_{inf}, G) \xrightarrow{\cong} \mathbb{H}^\bullet(X_{zar}, \mathscr{G}^\bullet)$$

Proof. We previously showed that if Y is formally smooth over S, then $\tilde{Y}$ covers the final object of Inf(X/S), so that we could repeat all the construction of (2.22). In particular given a sheaf G on Inf(X/S), we have again a spectral sequence from (C.41):

$$H^\bullet(X_{inf}, G) \Leftarrow E_2^{pq} = H^p(\nu \to H^q(\tilde{Y}^{\nu+1}, G)) \tag{2.24.1}$$

with the same argument as before, when X is affine the sequence degenerates, yielding a canonical isomorphism:

$$H^\bullet(X_{inf}, G) \xrightarrow{\cong} H^\bullet(\nu \to G(Y^\nu/X))$$

with Y^ν/X the formal completion of $Y_S^\nu = Y \times_S \cdots \times_S Y$ ($\nu+1$ times) along X. If we no longer assume X affine, we introduce again for $\nu \geq 0$ the sheaf $\mathscr{G}^{(\nu)}$ on X_{Zar}:

$$\mathscr{G}^{(\nu)} : U \to G(Y^\nu/U) = \varprojlim_i G(U \to U_Y^\nu(i)) \tag{2.24.2}$$

with Y^ν/U the formal completion of Y_S^ν along U. For variable ν, we have thus obtained a cosimplicial sheaf $\mathscr{G}^\bullet$ on X_{Zar}. If we take an an affine open covering of X, using the isomorphism obtained in the affine case, we obtain a canonical isomorphism:

$$H^\bullet(X_{inf}, G) \xrightarrow{\cong} \mathbb{H}^\bullet(X_{zar}, \mathscr{G}^\bullet) \Leftarrow E_2^{pq} = H^p(\nu \to H^q(X_{zar}, \mathscr{G}^{(\nu)})$$

□

2.3.3 Differential Operators

Since we have stated the existence of an equivalence of categories between crystals of $\mathcal{O}_{X_{strat}}$-modules and $\mathcal{O}_X$-modules fortified with a stratification structure (2.11), in view of the proposition (1.16) we have to expect a relation between differential operators and stratifying and infinitesimal sites. In fact differential operators arise naturally in the context of the infinitesimal and stratifying sites, via the cosimplicial sheaves $\mathscr{F}^\bullet$ (2.23.2) and $\mathscr{G}^\bullet$ (2.24.2).

For example let $\mathscr{F}^\bullet$ be the cosimplicial sheaf on X_{zar} associated to $\mathcal{O}_{X_{strat}}$. Recall that we can define the sheaves of principal parts $\mathcal{P}^n_{X/S}$ as the structural sheaves of the n-th infinitesimal neighbourhood of the diagonal of X, namely $(\Delta^n_{X/S}, \mathcal{P}^n_{X/S})$. Now consider as before the schemes $(\Delta^n_{X/S}(\nu), \mathcal{P}^n_{X/S}(\nu))$, in analogy with what was stated in (1.2.1), we know that for each positive integer ν the canonical projections $p_i : X^{\nu+1} \to X$ yield the family of canonical homomorphisms of sheaves of rings:

$$p_i^n(\nu) : \mathcal{O}_X \to \mathcal{P}^n_{X/S}(\nu) \qquad n \in \mathbb{N} \qquad i = 0, \ldots, \nu \tag{2.24.3}$$

which induce $(\nu+1)$ canonical homomorphisms of sheaves of rings $p_i(\nu) : \mathcal{O}_X \to \mathscr{F}^{(\nu)}$ giving to $\mathscr{F}^{(\nu)}$ different $\mathcal{O}_X$-algebras structures. Similarly to what we have seen in (1.2) if we choose one of these canonical homomorphisms, for example $p_0(\nu)$ to give the structure of $\mathcal{O}_X$-algebra, then the remaining morphisms will not be $\mathcal{O}_X$-linear and their truncations at the order $n \in \mathbb{N}$ can be interpreted as differential operators of order $\leq n$:

$$p_i^n(\nu) : \mathcal{O}_X \to \mathcal{P}^n_{X/S}(\nu)$$

note that if ν=1 we obtain simply the canonical differential operators of order $\leq n$ on $\mathcal{O}_X$ (1.6).

3 The comparison theorem

Finally we can state and prove the main theorem.

Theorem 3.1. *If S is a $\mathbb{Q}$-scheme, and X is smooth on S, there is a canonical isomorphism:*

$$H^\bullet(X_{Inf}, \mathcal{O}_{X_{inf}}) \xrightarrow{\cong} H^\bullet_{dR}(X/S)$$

This theorem is particularly significant as it gives a description of the de Rham cohomology without using differential forms and, at the same time, it also gives a sufficiently general context in which to study the de Rham cohomology. In his paper [17] Grothendieck prove this theorem but also gives some motivation that led him to a conjecture that generalize this theorem for singular schemes:

Conjecture 3.2. *If X is locally of finite type (not necessarily smooth) over $\mathbb{C}$, there are canonical isomorphisms:*

$$H^\bullet(X_{strat}, \mathcal{O}_{X_{strat}})) \xrightarrow{\cong} H^\bullet(X_{inf}, \mathcal{O}_{X_{inf}}) \xrightarrow{\cong} H^\bullet_{dR}(X/\mathbb{C}) \xrightarrow{\cong} H^\bullet(X^{an}, \mathbb{C})$$

The importance of this conjecture lies in the fact that its truth would imply that the infinitesimal or stratifying topos provides, for schemes of finite type of characteristic zero, a pure algebraic definition of the classical transcendentally defined complex cohomology. If that wasn't enough, at the end of his notes, Grothendieck managed to modify the definition of the infinitesimal site to obtain the so called crystalline site Cris(X/S), and he conjectured that the same theory given for Inf(X/S) would hold for Cris(X/S) without the hypothesis of S being of characteristic 0. Eventually he was right and nowadays we know that the crystalline site yields a good definition for a p-adic cohomology.

The proof in the smooth case requires a special construction, so first of all we want to motivate the path that will lead us to the proof. Let X be a smooth scheme over the base S, and take F a quasi-coherent module on X fortified with a stratification relative to S so that F defines a module F_{st} on Strat(X/S) (2.11). Recall that following (2.22) we can associate to the crystal F_{st} a complex of differential operators of infinite order (2.3.3) over X_{Zar}. In particular, let $\pi^i_{X/S}(\nu) : \Delta^i_{X/S}(\nu) \to X$ be the canonical projections, we can denote:

$$\mathcal{C}^\bullet(F_{st}) = \mathscr{F}^\bullet \qquad \mathcal{C}^{(\nu)}(F_{st}) = \mathscr{F}^{(\nu)} \overset{(2.23.2)}{=} \varprojlim_i F_{st(X, \Delta^i_{X/S}(\nu))} \overset{(2.11.2)}{=} \varprojlim_i \pi^i_{X/S}(\nu)^*(F)$$

where the $\pi^i_{X/S}(\nu)^*(F)$ are all canonical isomorphic thanks to the stratification structure on F. From this we showed (2.22) that there is a canonical isomorphism:

$$H^\bullet(X_{strat}, F_{st}) \xrightarrow{\cong} \mathbb{H}^\bullet(X_{zar}, \mathcal{C}^\bullet(F_{st}))$$

This means that the stratifying cohomology of a stratified sheaf can be interpreted as the Zariski hypercohomology of a complex of differential operators of infinite order. In the following we want to show that the converse is also true, namely the Zariski hypercohomology of any complex of differential operators can be expressed as the stratifying hypercohomology of a suitable complex of stratified sheaves. In fact, to prove the theorem, we want to apply this relation to the de Rham complex $\Omega^\bullet_{X/S}$.

3.1 The linearizing functor

We will construct a functor L from the category Dif(X/S) of modules on X, with as morphisms differential operators relative to S, to the category of pro-modules on X fortified with a stratification relative to S. The functor L will be called the linearizing functor, as it can be viewed intuitively as a linearization of differential operators.
To motivate this construction we can consider as usual $f : X \to S$ a morphism of schemes and E a sheaf of $f^{-1}(\mathcal{O}_S)$-modules on X, then we can induce a constant connection on $L(E)=\mathcal{O}_X \otimes_{\mathcal{O}_S} E$. Consider the following maps:

- $\tau : \mathcal{O}_X \otimes_{\mathcal{O}_S} E \to E \otimes_{\mathcal{O}_S} \mathcal{O}_X$ defined as $c \otimes x \to x \otimes c$.
- $i : E \otimes_{\mathcal{O}_S} \mathcal{O}_X \to \mathcal{O}_X \otimes_{\mathcal{O}_S} E \otimes_{\mathcal{O}_S} \mathcal{O}_X$ the canonical "inclusion" $x \otimes c \to 1 \otimes x \otimes c$.
- The natural identification: $\mathcal{O}_X \otimes_{\mathcal{O}_S} E \otimes_{\mathcal{O}_S} \mathcal{O}_X \xrightarrow{\cong} (\mathcal{O}_X \otimes_{\mathcal{O}_S} E) \otimes_{\mathcal{O}_X} \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X$ defined by $b \otimes x \otimes c \to b \otimes x \otimes 1 \otimes c = 1 \otimes x \otimes b \otimes c$.

If we compose the map τ with the inclusion i and we make these identifications, we get the map:

$$\theta : L(E) \to L(E) \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S} \qquad c \otimes x \to 1 \otimes x \otimes 1 \otimes c$$

so that in turn θ induces the following map:

$$\epsilon : \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} L(E) \to L(E) \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S} \quad (a \otimes b) \otimes (c \otimes x) \to 1 \otimes x \otimes a \otimes bc$$

Now if the $f^{-1}(\mathcal{O}_S)$-structure of E comes from an $\mathcal{O}_X$-structure, then we have the further identification $L(E)=\mathcal{O}_X \otimes_{\mathcal{O}_S} E \cong \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} E$ and we know that any $f^{-1}(\mathcal{O}_S)$-linear map of E can be linearized to $L(E)$ (1). The point of the above construction is the fact that the reductions modulo $\mathcal{I}^{n+1}$ of the map ϵ (or for (1.16) of θ) define a canonical stratification on $L(E)$.

Now if we take $E=\mathcal{O}_X$ then $L(E)=L(\mathcal{O}_X)=\mathcal{P}_{X/S}$ and the stratification obtained considering $\mathcal{P}_{X/S}$ with his left $\mathcal{O}_X$-module structure, it is obtained from the map $\theta : \mathcal{P}_{X/S} \to \mathcal{P}_{X/S} \otimes \mathcal{P}_{X/S}$ defined as $(c \otimes d) \to 1 \otimes d \otimes 1 \otimes c$. Note that using the right structure for $\mathcal{P}_{X/S}$ (i.e. repeating the construction using $L(E) = E \otimes_{\mathcal{O}_S} \mathcal{O}_X$) we would obtain a stratification from the map:

$$\delta : \mathcal{P}_{X/S} \to \mathcal{P}_{X/S} \otimes \mathcal{P}_{X/S} \qquad c \otimes d \to c \otimes 1 \otimes 1 \otimes d$$

Moreover, consider now a differential operator $h : F \to G$, reminding the equivalence and the notation given in (1.16) and considering the stratification δ for $\mathcal{P}_{X/S}$ then we can compute the map $\nabla(h) : \mathcal{P}_{X/S} \otimes F \to \mathcal{P}_{X/S} \otimes G$ which turn out to be nothing else then $\bar{\delta}(h) = 1 \otimes h$ (1.7). So note now that $\nabla(h)$ is $\mathcal{O}_X$-linear for the left structure whereas is a differential operator using the right structure, in fact even more is true since the map $\nabla(h)$ is compatible with the constant stratification that we have put on F and G because the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{P}_{X/S} \otimes F & \xrightarrow{\theta_F} & \mathcal{P}_{X/S} \otimes F \otimes \mathcal{P}_{X/S} \\ \downarrow{\scriptstyle \nabla(h)} & & \downarrow{\scriptstyle \nabla(h) \otimes id} \\ \mathcal{P}_{X/S} \otimes G & \xrightarrow{\theta_G} & \mathcal{P}_{X/S} \otimes G \otimes \mathcal{P}_{X/S} \end{array} \qquad \begin{array}{ccc} L(F) & \xrightarrow{\theta_F} & L(E) \otimes \mathcal{P}_{X/S} \\ \downarrow{\scriptstyle \nabla(h)} & & \downarrow{\scriptstyle \nabla(h) \otimes id} \\ L(G) & \xrightarrow{\theta_G} & L(G) \otimes \mathcal{P}_{X/S} \end{array}$$

This method provides us with a way of going from the category of $\mathcal{O}_X$-module and differential operators to the category of stratified $\mathcal{O}_X$-modules and horizontal maps if we were able to replace $\mathcal{P}_{X/S}$ with the inverse system $\{\mathcal{P}^n_{X/S}\}$.

Now we want to formalize the previous construction to build the true functor L. The first step is to define the target category for the linearizing functor.

Definition 3.3. Given a category $\mathcal{C}$ and an index category I, the category of pro-object of $\mathcal{C}$ is the category with as object functors $A : I \to \mathcal{C}$ (we will indicate $A(i) = A_i$). For our purpose the pro-objects are indexed by the integers $I = \mathbb{Z}$, in this situation the morphisms between two pro-objects (A_i) and (B_i) are defined to be:

$$\varinjlim_k \mathrm{Hom}((A_i)_k, (B_i))$$

where $(A_i)_k$ denote the pro-object obtained from (A_i) by shifting k places to the right, so that a morphism is given by a diagram for suitable $k \in \mathbb{Z}$:

$$\begin{array}{ccccccccc} \ldots \to & A_{i+k+1} & \to & A_{i+k} & \to & A_{i+k-1} & \to \ldots \\ & \downarrow & & \downarrow & & \downarrow & \\ \ldots \longrightarrow & B_{i+1} & \longrightarrow & B_i & \longrightarrow & B_{i-1} & \longrightarrow \ldots \end{array}$$

In our specific case we have to consider the category of pro-objects of sheaves on the underlying space X. Recall that $\Delta^i_{X/S}(\nu)$ is the i-th infinitesimal neighborhood of the diagonal of $X \times_S \cdots \times_S X$ ($\nu + 1$ times) and call $\mathcal{P}^i_{X/S}(\nu)$ its structural sheaf. With this notation we obtain pro-object $(\mathcal{P}^i_{X/S}(\nu))$ for variable i, so that for variable ν they form a cosimplicial pro-object $\mathscr{P}^{(\bullet)}$. In particular $\mathscr{P}^{(0)}=\mathcal{O}_X$ and for any ν there are $\nu + 1$ distinct canonical homomorphisms of pro-sheaves of rings as in (2.24.3):

$$p_j(\nu) : \mathcal{O}_X \to \mathscr{P}^{(\nu)} \qquad (j = 0, \ldots, \nu)$$

which define distinct structures of an $\mathcal{O}_X$-algebra on $\mathscr{P}^{(\nu)}$. In particular we want to consider the left structure given by $p_0(\nu)$ and the right structure given by $p_\nu(\nu)$.

Finally, we define another cosimplicial pro-object $L^\bullet$ as $L^\nu = \mathscr{P}^{(\nu+1)}$, so that we have a canonical homomorphism of cosimplicial pro-sheaves of rings defined as:

$$\alpha^\bullet : \mathscr{P}^{(\bullet)} \to L^\bullet \qquad \alpha^\nu : \mathscr{P}^{(\nu)} \to L^\nu \qquad \begin{array}{rcl} \alpha : \{0, \ldots, \nu\} & \longrightarrow & \{0, \ldots, \nu, \nu + 1\} \\ i & & \alpha(i) = i \end{array} \tag{3.3.1}$$

Now let M be an $\mathcal{O}_X$-module, we can consider the pro-module obtained as:

$$L^\nu(M) = L^\nu \otimes_{\mathcal{O}_X} M \qquad \nu \geq 0$$

where the tensor product is taken with respect to the right structure of $\mathcal{O}_X$-module on L^ν. From the definition we know that $L^\nu(M)$ is a L^ν-module, hence an $\mathcal{O}_X$-bimodule where the left structure is given by the left structure on L^ν, but also is a $\mathscr{P}^{(\nu)}$ module by restrictions of scalar since we have a canonical homomorphism $\alpha^\nu : \mathscr{P}^{(\nu)} \to L^\nu$, also from the definition of the map α^ν we can easily see that the structure of $\mathscr{P}^\nu$ module commute with the right structure of $\mathcal{O}_X$-module on $L^\nu(M)$.

Now we can consider a second $\mathcal{O}_X$-module N, and $D : M \to N$ a differential operator of order $\leq k$, for any $k \in \mathbb{Z}$, we know that we can factorize D as:

$$\begin{array}{ccc} M & \xrightarrow{\quad D \quad} & N \\ {\scriptstyle d^k_{1,M}}\downarrow & \nearrow_{\bar{D}} & \\ \mathcal{P}^k_{X/S}(1) \otimes_{\mathcal{O}_X} M & & \end{array}$$

where $d^k_{1,M}$ is the k-th canonical differential operator (1.6), and $\bar{D}$ is an $\mathcal{O}_X$-linear morphism with respect to the left $\mathcal{O}_X$-structure on $\mathcal{P}^k_{X/S}(1) \otimes_{\mathcal{O}_X} M$. Similarly to what we have done for the system $\{\mathcal{P}^n_{X/S}(1)\}$ in (1.7)and (1.8), we can define a unique $\mathcal{O}_X$-linear map for the right structure:

$$\delta^{i+k}(\nu) : \mathcal{P}^{i+k}_{X/S}(\nu) \to \mathcal{P}^i_{X/S}(\nu) \otimes_{\mathcal{O}_X} \mathcal{P}^k_{X/S}(\nu)$$

such that the following diagram commute:

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\quad p^{i+k}_{\nu+1}(\nu+1) \quad} & \mathcal{P}^{i+k}_{X/S}(\nu+1) \\ {\scriptstyle p^k_{\nu+1}(\nu+1)}\downarrow & & \downarrow{\scriptstyle \delta^{i+k}(\nu+1)} \\ \mathcal{P}^k_{X/S}(\nu+1) & \longrightarrow & \mathcal{P}^i_{X/S}(\nu+1) \otimes_{\mathcal{O}_X} \mathcal{P}^k_{X/S}(\nu+1) \end{array}$$

This maps allows us to associate to a differential operator $D : M \to N$ of order $\leq k$, for each non negative i, an $\mathcal{O}_X$-linear morphism for the left structure:

$$\mathcal{P}^{i+k}_{X/S}(\nu+1) \otimes_{\mathcal{O}_X} M \xrightarrow{\delta^{i+k}(\nu+1)} \mathcal{P}^i_{X/S}(\nu+1) \otimes_{\mathcal{O}_X} \mathcal{P}^k_{X/S}(\nu+1) \otimes_{\mathcal{O}_X} M \xrightarrow{id \otimes \bar{D}} \mathcal{P}^i_{X/S}(\nu+1) \otimes_{\mathcal{O}_X} N$$

hence a homomorphism between the relative pro-objects:

$$L^\nu(D) : L^\nu(M) \to L^\nu(N) \qquad \nu \geq 0$$

in particular $L^0(D)$ is $\mathcal{O}_X$-linear with respect to the left structure. That being said, it is clear that we can state the following proposition:

Proposition 3.4. *Let M be an $\mathcal{O}_X$-module, the association $M \to L^0(M)$ extends to a functor from the category of $\mathcal{O}_X$-modules with differential operators as morphism to the category of $\mathcal{O}_X$-pro-modules.*

With this construction we can replace in the discussion at the beginning of this section the sheaf $\mathcal{P}_{X/S}$ with the inverse system $\{\mathcal{P}^n_{X/S}(1)\}$.

Proposition 3.5. *Given an $\mathcal{O}_X$-module M, define its linearization as:*

$$L(M) = \varprojlim L^0(M) = \varprojlim_i \mathcal{P}^i_{X/S} \otimes_{\mathcal{O}_X} M$$

where $L(M)$ is an $\mathcal{O}_X$-module fortified with a canonical stratification structure. Furthermore L extends to a functor called linearizing functor from the category of $\mathcal{O}_X$-modules with differential operators as morphisms to the category of stratified $\mathcal{O}_X$-module and horizontal morphisms.

Proof.

- $L(M)$ is fortified with a canonical stratification relative to S, and we will call $L(M)$ the linearization of M. The stratification is obtained from the composition of morphism:

  $$\epsilon_{n_m} : \mathcal{P}^n_{X/S} \otimes \mathcal{P}^{n+m}_{X/S} \otimes M \xrightarrow{1\otimes\delta^{n,m}\otimes 1} \mathcal{P}^n_{X/S} \otimes \mathcal{P}^n_{X/S} \otimes \mathcal{P}^m_{X/S} \otimes M \xrightarrow{(1.\sigma^n)\otimes 1\otimes 1} \mathcal{P}^m_{X/S} \otimes M \otimes \mathcal{P}^n_{X/S}$$

  where the canonical map σ^n and his properties are defined in (1.2.1). Considering the limit over the m-index we obtain:

  $$\epsilon_n : \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} L(M) \to L(M) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S}$$

  The map ϵ_n is $\mathcal{P}^n_{X/S}$-linear, and it is an isomorphism since we can write his inverse as an analogue composition of maps. It is clear that ϵ_0 is the identity and the cocycle condition holds.

- Given a differential operator $D : M \to N$ of order $\leq k$, then taking the limit of the pro-morphism $L^0(D)$ we obtain $L(D)$ that is $\mathcal{O}_X$-linear and it is horizontal with respect to the canonical stratification defined over $L(M)$ and $L(N)$. This properties follows from the commutativity of the diagram:

  $$\begin{array}{ccccc}
  \mathcal{P}^n_{X/S} \otimes \mathcal{P}^{n+m+k}_{X/S} \otimes M & \xrightarrow{1\otimes\delta\otimes 1} & \mathcal{P}^n_{X/S} \otimes \mathcal{P}^{n+m}_{X/S} \otimes \mathcal{P}^k_{X/S} \otimes M & \xrightarrow{1\otimes 1\otimes \bar{D}} & \mathcal{P}^n_{X/S} \otimes \mathcal{P}^{n+m}_{X/S} \otimes N \\
  \downarrow{\scriptstyle 1\otimes\delta\otimes 1} & & \downarrow{\scriptstyle 1\otimes\delta\otimes 1\otimes 1} & & \downarrow{\scriptstyle 1\otimes\delta\otimes 1} \\
  \mathcal{P}^n_{X/S} \otimes \mathcal{P}^n_{X/S} \otimes \mathcal{P}^{m+k}_{X/S} \otimes M & \xrightarrow{1\otimes 1\otimes\delta\otimes 1} & \mathcal{P}^n_{X/S} \otimes \mathcal{P}^n_{X/S} \otimes \mathcal{P}^m_{X/S} \otimes \mathcal{P}^k_{X/S} \otimes M & \xrightarrow{1\otimes 1\otimes 1\otimes \bar{D}} & \mathcal{P}^n_{X/S} \otimes \mathcal{P}^n_{X/S} \otimes \mathcal{P}^m_{X/S} \otimes N \\
  \downarrow{\scriptstyle (1.\sigma)\otimes 1\otimes 1} & & \downarrow{\scriptstyle (1.\sigma)\otimes 1\otimes 1\otimes 1} & & \downarrow{\scriptstyle (1.\sigma)\otimes 1\otimes 1} \\
  \mathcal{P}^{m+k}_{X/S} \otimes M \otimes \mathcal{P}^n_{X/S} & \xrightarrow{\delta\otimes 1\otimes 1} & \mathcal{P}^m_{X/S} \otimes \mathcal{P}^k_{X/S} \otimes M \otimes \mathcal{P}^n_{X/S} & \xrightarrow{1\otimes \bar{D}\otimes 1} & \mathcal{P}^m_{X/S} \otimes N \otimes \mathcal{P}^n_{X/S}
  \end{array}$$

  considering the limits over the right indices we get the following diagram:

  $$\begin{array}{ccc}
  \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} L(M) & \xrightarrow{1\otimes L(D)} & \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} L(N) \\
  \downarrow{\scriptstyle \epsilon^M_n} & & \downarrow{\scriptstyle \epsilon^N_n} \\
  L(M) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S} & \xrightarrow{L(D)\otimes 1} & L(N) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S}
  \end{array}$$

- Given differential operators $D : M \to N$ and $F : N \to P$, then from a diagram similar to the previous one we obtain that $L(F \circ D) = L(F) \circ L(D)$.

- We can recover M as the subsheaf of horizontal sections of $L(M)$, and from $L(D)$ we can recover D as the induced morphism on the subsheaf of horizontal section. □

Proposition 3.6. *Composing the linearizing functor L with the functor which realizes the equivalence of categories of (1.16), we obtain a functor L_{strat} from the category of $\mathcal{O}_X$-module and differential operators as morphisms to the category of crystals of $\mathcal{O}_{X_{strat}}$-modules.*

Remark 3.7. Instead of taking the limit we can keep the entire inverse system in mind to obtain the same properties of the linearizing functor L for the functor L^0 defined in (3.4). Notice also that the cosimplicial pro-module $L^\bullet(M)$ on $\mathscr{P}^\bullet$ is just the cosimplicial pro-module induced by the stratified pro-module $L^0(M)$, because $L^k(M)$ is obtained from $L^i(M)$ with $i \leq k$ by base change with respect to any canonical morphisms $\mathscr{P}^{(i)}_{X/S} \to \mathscr{P}^{(k)}_{X/S}$.

3.2 Proof of the comparison theorem

The last step toward the proof of the main theorem (3.1) is to manage to interpret the Zariski hypercohomology of a complex of differential operators as a stratifying hypercohomology. To do this we will need the following general proposition.

Proposition 3.8. *Let $\mathcal{A}$ be an abelian category, $C^\bullet$ a cosimplicial object of $\mathcal{A}$; we will denote $d^i_j : C^i \to C^{i+1}$ the morphism corresponding to the unique strictly increasing map from the subset of the integers $(0,\ldots,i)$ in $(0,\ldots,i{+}1)$ without taking the value j. Define the complex $D^\bullet$ as:*

$$D^i = C^{i+1} \qquad d^i = \sum_{j=0}^{i+1}(-1)^j d^{i+1}_j : D^i \to D^{i+1}$$

Then $D^\bullet$ is chain homotopical to zero.

Proof. We can define a chain homotopy as follows $h^i = (-1)^{i+1}s^{i+1} : D^{i+1} \to D^i$ where the map $s^{i+1} : C^{i+2} \to C^{i+1}$ is the morphism corresponding to the increasing map from the subset of the integers $(0,\ldots,i+2)$ to $(0,\ldots,i+1)$ constant over $(0,\ldots,i+1)$. We just have to show that:

$$h^i \circ d^i + d^{i-1} \circ h^{i-1} = \mathrm{Id}$$

expanding the left hand side we obtain:

$$h^i \circ d^i + d^{i-1} \circ h^{i-1} = (-1)^{i+1}s^{i+1} \circ \sum_{j=0}^{i+1}(-1)^j d^{i+1}_j + \sum_{j=0}^{i}(-1)^j d^i_j \circ (-1)^i s^i$$

note that if $j \leq i$ we have $s^{i+1} \circ d^{i+1}_j = d^i_j \circ s^i$, so that:

$$h^i \circ d^i + d^{i-1} \circ h^{i-1} = s^{i+1} \circ d^{i+1}_{i+1} = \mathrm{Id}$$

□

In our case remember that $\mathscr{P}^{(\bullet)}$ and $L^\bullet$ as defined in (3.3.1) are cosimplicial pro-objects, with $L^\nu = \mathscr{P}^{(\nu+1)}$. From $L^\bullet$ we can obtain a pro-complex L^* with differential defined as:

$$\partial^\nu = \sum_{i=0}^{\nu+1}(-1)^i d^\nu_i : L^\nu \to L^{\nu+1}$$

with $d^\nu_i : \mathscr{P}^{(\nu+1)} \to \mathscr{P}^{(\nu+2)}$ the canonical maps. Note that there are $\nu + 2$ such maps, but the last one is not used, so that from the proposition above (3.8) we obtain that the complex $\varprojlim L^*$ is exact and, more in particular, it is a resolution for $\mathscr{P}^{(0)} = \mathcal{O}_X$. Now take a complex of differential operators $M^\bullet$ on X bounded from below, as just stated we have resolutions:

$$M^n \to \varprojlim L^0(M^n) \to \varprojlim L^1(M^n) \to \ldots$$

so that we have a resolution of the entire complex:

$$M^\bullet \to \varprojlim L^\bullet(M^\bullet)$$

and thus there is a canonical isomorphism:

$$\mathbb{H}^\bullet(X_{zar}, M^\bullet) \xrightarrow{\cong} \mathbb{H}^\bullet(X_{zar}, \varprojlim L^\bullet(M^\bullet)) \tag{3.8.1}$$

Since $M^\bullet$ is a complex of differential operators, we can apply to $M^\bullet$ the functor L_{strat} defined in (3.6), i.e. take its linearization $L(M^\bullet)$ and apply the equivalence of categories of (2.11), to obtain a complex of sheaves $L(M^\bullet)_{strat}$ on Strat(X/S). Now from the proposition (2.22) we know that there is a canonical isomorphism:

$$\mathbb{H}^\bullet(X_{strat}, L(M^\bullet)_{strat}) \xrightarrow{\cong} \mathbb{H}^\bullet(X_{zar}, \mathcal{C}^\bullet(L(M^\bullet)_{strat})) \tag{3.8.2}$$

where we remind that $\mathcal{C}^\bullet(L(M^\bullet)_{strat})$ is the Zariski complex of differential operators associated with the complex of crystals $L(M^\bullet)_{strat}$, but by construction:

$$\begin{aligned} C^\nu(L(M^\bullet)_{strat}) &= \varprojlim (L(M^\bullet)_{strat})_{(X \to \Delta^i_{X/S}(\nu))} \\ &= \varprojlim \pi^i_{X/S}(\nu)^* L^0(M^\bullet) \\ &= \varprojlim L^\nu(M^\bullet) \\ C^\bullet(L(M^\bullet)_{strat}) &= \varprojlim L^\bullet(M^\bullet) \end{aligned}$$

so that combining (3.8.1) and (3.8.2) we have in fact obtained a canonical isomorphism:

$$\mathbb{H}^\bullet(X_{zar}, M^\bullet) \xrightarrow{\cong} \mathbb{H}^\bullet(X_{strat}, L(M^\bullet)_{strat}) \tag{3.8.3}$$

and thus in particular there is a spectral sequence:

$$E_2^{pq} = H^p(X_{strat}, H^q(L(M^\bullet)_{strat})) \Rightarrow \mathbb{H}^\bullet(X_{zar}, M^\bullet) \tag{3.8.4}$$

The last fact that we need to complete the proof of (3.1) is a formal variant of the Poincaré lemma which heavily depends on the assumption that our base scheme S is of characteristic zero and that X is smooth over S.

Lemma 3.9 (Poincaré Lemma). *If S is a $\mathbb{Q}$-scheme, then there is a quasi-isomorphism of complexes:*

$$\mathcal{O}_{X_{strat}} \to L(\Omega^\bullet_{X/S})_{strat}$$

Proof. First of all considering the canonical maps $d_0 : \mathcal{O}_X \to \mathcal{P}^i_{X/S},\ x \to x \otimes 1$, defined in (1.2.1) for every $i \geq 0$, we obtain a canonical morphism $\mathcal{O}_X \to \varprojlim_i \mathcal{P}^i_{X/S} = L(\mathcal{O}_X)$ from the universal properties of the limit. Now we want to see if this morphism extends to a map of complexes, i.e. that the composition $\mathcal{O}_X \to L(\mathcal{O}_X) \to L(\Omega^1_{X/S})$ is zero. To do this we have to use the local description of $\mathcal{P}^i_{X/S}$ that we have obtained in (1.10). If $x_1, \ldots, x_n$ are local coordinates for X/S and if ξ_i denotes the image of $1 \otimes x_i - x_i \otimes 1$ in $\mathcal{P}^m_{X/S}$, then $\mathcal{P}^m_{X/S}$ is the free $\mathcal{O}_X$-module with basis $\{\xi_1^{t_1} \ldots \xi_n^{t_n} \,|\, \sum_i t_i \leq m\}$, from this description it is easy to see that locally:

$$L(\mathcal{O}_X) = \varprojlim_i \mathcal{P}^i_{X/S} = \mathcal{O}_X[\![\xi_1, \ldots, \xi_n]\!]$$

consequently $L(\Omega^p_{X/S}) = \oplus\mathcal{O}_X[\![\xi_1,\dots,\xi_n]\!]dx_{i_1}\wedge\cdots\wedge dx_{i_p}$, where the differential $L(d)$ is given by:

$$\xi_1^{i_1}\dots\xi_n^{i_n}\omega \to \sum_{j=1}^{n} i_j\xi_1^{i_1}\dots\xi_j^{i_j-1}\dots\xi_n^{i_n}dx_j\wedge\omega + \xi_1^{i_1}\dots\xi_n^{i_n}d\omega$$

This said is easy to see that the composition $\mathcal{O}_X \to L(\mathcal{O}_X) \to L(\Omega^1_{X/S})$ is zero, thus we have indeed a morphism of complexes.

To prove the lemma we may work locally on Strat(X/S), so consider $(U \to T) \in$ Strat(X/S), with T small enough so that it exists a lifting $h : T \to X$ with $h(T)$ contained in an open in X on which are defined $x_1,\dots,x_n$ local coordinates. In this case $h^*L(\Omega^\bullet_{X/S}) = (L(\Omega^\bullet_{X/S})_{strat})_{(U,T)}$, so that from the previous local description we just need to check that the complex given by $(\oplus\mathcal{O}_T[\![\xi_1\dots\xi_n]\!]dx_{i_1}\wedge\cdots\wedge dx_{i_m}, d)_m$ is a resolution of $\mathcal{O}_T$, which is true since S is a $\mathbb{Q}$-scheme and X is smooth over S and follows from the variant of the Poincaré Lemma (A.13). ☐

Finally we are ready to prove the comparison theorem (3.1), basically we just need to apply the linearizing functor to the algebraic de Rham complex $\Omega^\bullet_{X/S}$, the isomorphism (3.8.3) and the Poincaré Lemma (3.9) will end the proof.

Theorem. *If S is a $\mathbb{Q}$-scheme, and X is smooth on S, there is a canonical isomorphism:*

$$H^\bullet(X_{Inf},\mathcal{O}_{X_{inf}}) \xrightarrow{\cong} H^\bullet_{dR}(X/S)$$

Proof. Let F be a module on X fortified with a stratification structure relative to S, and F_{strat} the associated crystal on Strat(X/S). Consider the de Rham complex $\Omega^\bullet_{X/S}\otimes_{\mathcal{O}_X} F$, this is a complex of differential operators so that we can apply the linearizing functor to obtain a canonical morphism:

$$i : F_{strat} \to L(\Omega^\bullet_{X/S}\otimes_{\mathcal{O}_X} F)_{strat}$$

which is essentially induced by $\mathcal{O}_X \to \varprojlim_i \mathcal{P}^i_{X/S} = L(\mathcal{O}_X),\ x \to (x\otimes 1)_i$. Since S is a $\mathbb{Q}$-scheme and X/S is smooth, we can apply the previous lemma (3.9) to obtain that i is a quasi-isomorphism, in particular:

$$H^\bullet(X_{strat},F_{strat}) \cong \mathbb{H}^\bullet(X_{strat}, L(\Omega^\bullet_{X/S}\otimes_{\mathcal{O}_X} F)_{strat})$$

Now using the isomorphism (3.8.3) we can relate the stratified hypercohomology of the linearization of the differential complex $\Omega^\bullet_{X/S}\otimes_{\mathcal{O}_X} F$ with its Zariski hypercohomology:

$$\mathbb{H}^\bullet(X_{strat}, L(\Omega^\bullet_{X/S}\otimes_{\mathcal{O}_X} F)_{strat}) \xrightarrow{\cong} \mathbb{H}^\bullet(X_{zar},\Omega^\bullet_{X/S}\otimes_{\mathcal{O}_X} F)$$

Finally by composition we obtain that the following canonical morphism, which always exists, is in fact an isomorphism in our case:

$$H^\bullet(X_{strat},F_{strat}) \xrightarrow{\cong} \mathbb{H}^\bullet(X_{zar},\Omega^\bullet_{X/S}\otimes_{\mathcal{O}_X} F)$$

Now remember that in the smooth case the infinitesimal and stratified topos are the same, and taking $F = \mathcal{O}_X$ with the canonical constant stratification, we obtain the proof of the theorem:

$$H^p(X_{Inf},\mathcal{O}_{X_{Inf}}) \xrightarrow{\cong} \mathbb{H}^p((X/S)_{strat}, L(\Omega^\bullet_{X/S})_{strat}) \xrightarrow{\cong} \mathbb{H}^p(X_{Zar},\Omega^\bullet_{X/S}) = H^p_{dR}(X)$$

☐

Remark 3.10. As Grothendieck stated, the significance of this theorem lies in the fact that gives a description of the de Rham cohomology without using differential forms, and it allows to study de Rham cohomology in more general context where is known to be problematic, for example non smoothness and positive characteristic. In fact P. Berthelot and A. Ogus adapted the Grothendieck's proof to deal with the case of positive characteristic and they obtained an analogous theorem for crystalline and de Rham cohomology. In this short paper [3], we can find a slightly different perspective on the comparison between infinitesimal cohomology and de Rham cohomology. In particular the theorem (3.1) is proven without using the notion of stratification, differential operators and linearizations. In fact in the article is treated the case of crystalline cohomology in characteristic $p > 0$, but the arguments that go into prove the comparison theorem also apply to reprove the Grothendieck's theorem, it is sufficient to translate the "crystalline" language into "infinitesimal language".

3.3 Modern interpretation of Grothendieck's argument

In this last part of the thesis we just want to sketch a more modern proof of the main theorem which can be found in [9], together with a slightly extension of Grothendieck's theorem (3.1).

Recall that from (2.5) a sheaf $\mathcal{E}$ on X_{inf} is the same thing as a compatible collection of Zariski sheaves $\{\mathcal{E}_T \in Sh(T) : T \in \text{Inf}(X)\}$. Suppose that $\mathcal{E}$ is a quasi-coherent sheaf on Sch/k, and that the natural map $j^*\mathcal{E}_{T'} \to \mathcal{E}_T$ is an isomorphism for every closed embedding $j : T \to T'$ in Inf(X) (i.e. $\mathcal{E}$ is a crystal). Given an affine scheme $X = \text{Spec}(A)$ and a presentation of the ring $A = S/I_1$ for a smooth $\mathbb{Q}$-algebra S, if we write I_ν for the kernel of the map $S^{\otimes\nu} \to A$, then each $S_\bullet^{\otimes\nu} = \{S^{\otimes\nu}/I_\nu^m\}_m$ is a tower of algebras, and $[v] \to \{S_\bullet^{\otimes\nu+1}\}$ is a cosimplicial tower of algebras. Now set $Y_m^\nu = \text{Spec}(S^{\otimes\nu}/I_\nu^m)$, we have showed in (2.23.1) that we can compute infinitesimal cohomology as:

$$H^\bullet(X_{inf}, \mathcal{E}) = H^\bullet \varprojlim_m \mathcal{E}(Y_m^{\bullet+1}) \tag{3.10.1}$$

Here we want to obtain the same isomorphism but with a more modern argument. First of all we know already that we can consider a forgetful functor:

$$u : \text{Inf}(X) \to Sch/k \qquad (U \hookrightarrow T) \to T$$

which is a is a morphism of sites $(Sch/k)_{zar} \to X_{inf}$, where if $\mathcal{E}$ is a Zariski sheaf over Sch/k its restriction to Inf(X) defines a sheaf $u^*\mathcal{E}$ which for our purpose we will denote again $\mathcal{E}$. Since the usual global section functor takes a sheaf $\mathcal{E}$ on X_{inf} to:

$$H^0(X_{inf}, \mathcal{E}) = \Gamma_{inf}(\mathcal{E}) = \varprojlim_{T \in inf(X)} \Gamma_{zar}(T, \mathcal{E}_T)$$

then Γ_{inf} factors as the forgetful functor Γ_{zar} from sheaves to presheaves, followed by the inverse limit functor . The forgetful functor preserves injectives as it is right adjoint to sheafification, so that it follows immediately from the Grothendieck's spectral sequence (B.10) that for every sheaf $\mathcal{E}$ on X_{inf} we have an isomorphism in the derived category:

$$\mathbb{H}(X_{inf}, \mathcal{E}) \cong \mathbb{R}\Gamma_{inf}(X_{inf}, \mathcal{E}) = \mathbb{R}\varprojlim{}_{T \in \text{inf}(X)} \mathbb{R}\Gamma_{zar}(T, \mathcal{E}) \cong \text{holim}_T \mathbb{H}_{zar}(T, \mathcal{E})$$

where $\mathbb{H}_{zar}(T, \mathcal{E}_T)$ is just $\mathbb{R}\Gamma_{zar}(T, \mathcal{E}_T)$ and following the argument in the appendix (B.35) we write holim$_T$ for $\mathbb{R}\varprojlim_{T \in \text{inf(X)}}$.

Lemma 3.11. *Given $X = Spec(A)$ with $A = S/I_1$ for a smooth algebra S as before, then the simplicial cotower $[v] \to \{Y_\bullet^{v+1}\}$ is right cofinal in inf(X). Right cofinality means that for each $T \in$ Inf(X), the slice category T/i is contractible, where i denotes the inclusion of $\{Y_\bullet^{v+1}\}$ into Inf(X).*

Proof. The proof of ([7],5.1) is valid in our case *mutatis mutandis* as sketched in ([9],1.4). □

Let $[\nu] \to C_\nu$ be a cosimplicial complex of abelian groups. From the discussion in the appendix (B.35) and from ([32],5.25) we know that the Bousfield-Kan total complex $\mathrm{holim}_{\nu\in\Delta} C_\nu^\bullet = Tot_\nu C_\nu^\bullet$ is just a specific Cartan-Eilenberg resolution of the associated total chain complex C^*.

Proposition 3.12. *Let $X = Spec(A)$, if $\mathcal{E}$ is a cochain complex of quasi-coherent sheaves on Sch/k.*

- *$holim_T \mathcal{E}(T) \cong Tot_\nu(\mathbb{R}\varprojlim_m)\mathcal{E}(Y_m^{\nu+1})$.*
- *Assume that $j^*\mathcal{E}_{T'} \to \mathcal{E}_T$ is onto for every closed embedding $j : T \to T'$ in Inf(X), then*

$$\mathbb{H}(X_{inf}, \mathcal{E}) \cong Tot_\nu(\varprojlim_m \mathcal{E}(Y_m^\nu))$$

Proof. The first point follows from the lemma (3.11) and the cofinality theorem ([5],XI.9.2), we know that holim_T is equivalent to $\mathrm{holim}_{N\times\Delta}=\mathrm{holim}_\Delta \mathrm{holim}_\mathbb{N} = Tot(\mathbb{R}\varprojlim_m)$. Under the assumption of (2) we also have that $\mathbb{R}\varprojlim_m \mathcal{E}(Y_m^\nu)$ can be replaced by $\varprojlim_m \mathcal{E}(Y_m^\nu)$. □

The sheaf Ω^p of p-differential forms is not a crystal for $p \neq 0$, however it satisfies the hypothesis of (3.12) so that we can compute $\mathbb{H}(X_{inf}, \Omega^p)$ as Tot_ν. With this last proposition we have obtained the same isomorphism as in (3.10.1) for affine, now with descent technique we can extend the result to non-affine.

Lemma 3.13. *Let $\mathcal{E}$ be a cochain complex of sheaves of abelian groups on Inf(X). The presheaf $V \to \mathbb{H}(V_{inf}, \mathcal{E})$ satisfies Zariski descent on X.*

Proof. [9] □

This said we have obtained a modern interpretation of Grothendieck's argument used to relate infinitesimal cohomology with Zariski cohomology. Moreover we can also give a sketch of the proof of the following theorem which is essentially an extension of (3.1).

Theorem 3.14. *The brutal truncations $\Omega^\bullet \to \Omega^{\leq i} \to \Omega^0 = \mathcal{O}$ induce homotopy equivalences:*

$$\mathbb{H}(X_{inf}, \Omega^\bullet) \cong \mathbb{H}(X_{inf}, \Omega^{\leq i}) \cong \mathbb{H}(X_{inf}, \mathcal{O})$$

Proof. Consider the hypercohomology spectral sequence: $E_1^{p,q}=H^q(X_{inf}, \Omega^p) \Longrightarrow \mathbb{H}^{p+q}(X_{inf}, \Omega^\bullet)$ the theorem follows essentially showing that $\mathbb{H}(X_{inf}, \Omega^p) = 0$ for $p > 0$. By induction on the size of a separated cover, using the lemma (3.13), we are reduced to the case in which X is separated and then affine. For the affine case $X = \mathrm{Spec}(A)$ consider as before a presentation $A = S/I$ for a smooth algebra S. If we write I_ν for the kernel of the map $S^{\otimes\nu} \to A$, then each $S_\bullet^{\otimes\nu} = \{S^{\otimes\nu}/I_\nu^m\}_m$ is a tower of algebras, and $[v] \to \{S_\bullet^{\otimes\nu+1}\}$ is a cosimplicial tower of algebras. Essentially in this situation we have to show that the complex $\{\Omega^p_{S^{\otimes\bullet}/I_\bullet}\}_\nu$ is acyclic, where S is a symmetric algebra. This complex is the inverse limit of the tower:

$$\{\Omega^p_{S^{\otimes\bullet}}/I_\bullet^m \Omega^p_{S^{\otimes\bullet}}\}_\nu$$

When $I = 0$ a chain contraction is given in ([8],7.9) and is defined by differential operators. Since a contraction of this kind must be continuous for the $I_\bullet$-adic topology, the result follows. □

A Algebraic de Rham Cohomology

In this section we will recall some fundamental definition and properties about the sheaf of differentials, the algebraic de Rham complex and we will study more in particular the concept of connection of a quasi-coherent module over a scheme.

A.1 Definition and first properties

Definition A.1. Fix a ring A. Let B be an A-algebra and M a B-module. An A derivation of B into M is an A-linear map $d : B \to M$ that verifies the Leibniz rule:

$$d(b_1 b_2) = b_1 db_2 + b_2 db_1 \quad b_i \in B$$

We denote the set of all the derivations as $\mathrm{Der}_A(B, M)$.

We can define an analog for differential forms in an algebraic settings as follows:

Definition A.2. The module of relative differential forms of B over A is a B-module $\Omega^1_{B/A}$ with an A derivation $d : B \to \Omega^1_{B/A}$ satisfying the following universal property: for any B-module M and for any A-derivation $d' : B \to M$ there exists a unique homomorphism of B-modules $\varphi : \Omega^1_{B/A} \to M$ such that $d' = \varphi \circ d$.

$$\begin{array}{ccc} B & \xrightarrow{d'} & M \\ {\scriptstyle d}\downarrow & \nearrow_{\varphi} & \\ \Omega^1_{B/A} & & \end{array}$$

Remark A.3. It's easy to see that the module of differential forms $(\Omega^1_{B/A}, d)$ exists and it is unique up to isomorphism, also thanks to the universal property we have an isomorphism of A-modules:

$$\mathrm{Hom}_B(\Omega^1_{B/A}, M) \to \mathrm{Der}_A(B, M) \quad \varphi \to \varphi \circ d$$

We can now enumerate some properties of modules of differentials.

Proposition A.4. *Given $\rho : B \to C$ a morphism of A-algebras, from the universal property we obtain canonical homomorphisms of C-modules:*

$$\alpha : \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A}, \quad \beta : \Omega^1_{C/A} \to \Omega^1_{C/B}$$

1. *(Base Change) For any A-algebra A', set $B' = B \otimes_A A'$. There exists a canonical isomorphism of B'-modules $\Omega^1_{B'/A'} \cong \Omega^1_{B/A} \otimes_B B'$*

2. *Let $B \xrightarrow{\rho} C$ be a homomorphism of A-algebras. From the universal property we obtain α, β canonical homomorphisms of C-modules, such that the following is an exact sequence:*

$$\begin{array}{ccccccc} \Omega^1_{B/A} \otimes_B C & \xrightarrow{\alpha} & \Omega^1_{C/A} & \xrightarrow{\beta} & \Omega^1_{C/B} & \longrightarrow & 0 \\ db \otimes c & \longrightarrow & cd\rho(b) & & & & \\ & & dc & \longrightarrow & dc & & \end{array}$$

3. *Let S be a multiplicative subset of B then $S^{-1}\Omega^1_{B/A} \cong \Omega^1_{S^{-1}B/A}$.*

4. *If $C = B/I$ we have an exact sequence:*

$$\begin{array}{ccccccc} I/I^2 & \xrightarrow{\delta} & \Omega^1_{B/A} \otimes_B C & \xrightarrow{\alpha} & \Omega^1_{C/A} & \longrightarrow & 0 \\ \bar{b} & \longrightarrow & db \otimes 1 & & & & \end{array}$$

Proof. [27] □

Since the module of differential forms is compatible with localization, we can construct on a general scheme X the sheaf of differentials simply defining them locally on affine open subschemes and then gluing. However there is an equivalent description of the module of differential forms which has a geometric interpretation that allow us to globalize immediately.

Proposition A.5. *[4] Let B be an A-algebra. Consider the multiplication map $\mu : B \otimes_A B \to B$ and let I be its kernel.*

$$\begin{array}{ccc} B \otimes_A B & \xrightarrow{\mu} & B \\ \sum_i b_i \otimes c_i & \longrightarrow & \sum_i b_i c_i \end{array}$$

Then the module of differential forms can be defined as $(\Omega^1_{B/A}, d) = (I/I^2, \delta)$, where the universal derivation is the homomorphism defined by $\delta(b) = 1 \otimes b - b \otimes 1$.

Proof. First of all we have the isomorphism $(B \otimes_A B)/I \to B$ whose inverse is induced from any of the two canonical maps:

$$i_1 : B \to B \otimes_A B \quad x \to x \otimes 1 \qquad i_2 : B \to B \otimes_A B \quad x \to 1 \otimes x$$

Due to this isomorphism the quotient I/I^2 has a structure of $B \otimes_A B$-module, and also of B-module by restriction of scalar via any of the map i_1, i_2. Using for example the structure given by i_1 it is easy to see that $I \subset B \otimes_A B$ is generated as a B-module by element of the form $1 \otimes b - b \otimes 1$ with $b \in B$. Indeed, if $\sum_{i=1}^n x_i \otimes y_i$ is such that $\sum_{i=1}^n x_i y_i = 0$ then:

$$\sum_{i=1}^n x_i \otimes y_i = \sum_{i=1}^n x_i y_i \otimes 1 + \sum_{i=1}^n (x_i \otimes 1)(1 \otimes y_i - y_1 \otimes 1) = \sum_{i=1}^n (x_i \otimes 1)(1 \otimes y_i - y_1 \otimes 1)$$

We can also see that $\delta = i_2 - i_1$ is a A-derivation of B in I/I^2, indeed δ is A-linear and respect the Leibniz rule:

$$\delta(xy) = (1 \otimes xy) - (xy \otimes 1)$$
$$x\delta(y) + y\delta(x) = x \otimes y - xy \otimes 1 + y \otimes x - xy \otimes 1$$

but modulo the ideal I^2 we have the identity:

$$1 \otimes xy = \delta(x)\delta(y) + x \otimes y + y \otimes x - xy \otimes 1$$

The choice of δ is, in some sense, canonical since both $i_1, i_2 : B \to B \otimes_A B$ are well defined sections of the multiplication map μ and they are also inverse map for the isomorphism $\bar{\mu}$. As a further motivation for the choice of δ, it can be shown ([4],8.1.8) that there is a bijection between the set of A-algebra morphisms $f : B \to (B \otimes_A B)/I^2$ satisfying $f = i_1 \bmod I/I^2$ and the set $\mathrm{Der}_A(B, I/I^2)$ of all A-derivations from B to I, and under this correspondence δ is the derivation which correspond to the map $f = i_2$ which is a canonical map as we have said before. Now that we have fixed the B-module structure induced by i_1 on $(B \otimes_A B)/I^2$, the last thing to check is the universal property which characterize $\Omega^1_{B/A}$. So for a B-module M and an A-derivation $d : A \to M$ consider the A-linear map:

$$\varphi : B \otimes_A B \to M \qquad x \otimes y \to xd(y)$$

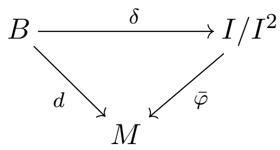
this map is also B-linear considering $B \otimes_A B$ with the B-module structure given by i_1. Notice that $I^2 \subset \mathrm{Ker}(\varphi)$ indeed we know that I/I^2 is generated as a B-module by element of the form:

$$(1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1) \qquad x, y \in B$$

and since d is a A-derivation:

$$\begin{aligned} \varphi((1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1)) &= \varphi(1 \otimes xy) - \varphi(y \otimes x) - \varphi(x \otimes y) + \varphi(xy \otimes 1) \\ &= d(xy) - yd(x) - xd(y) = 0 \end{aligned}$$

Hence the map φ induce a unique B-linear map $\bar{\varphi}$ such that the following diagram commute:



□

Now given a morphism of schemes $f : X \to Y$, we can define in a more general way the sheaf of differential forms of X relative to Y.

Definition A.6. Let $\Delta : X \to X \times_Y X$ be the diagonal morphism which is a locally closed immersion. In particular Δ defines a closed subscheme isomorphic to X in an open subset W of $X \times_Y X$. Let $\mathcal{I} = \mathrm{Ker}\Delta^\sharp \subset \mathcal{O}_W$ the sheaf of ideal defining the closed subset $\Delta(X)$ in W. Then we define the sheaf of relative differential forms of degree 1 of X over Y as the $\mathcal{O}_X$-module:

$$\Omega^1_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$$

The definition of $\Omega^1_{X/Y}$ does not depend on the choice of $W \subset X \times_Y X$ hence we can assume that W is the largest open with the required property, since then $\mathcal{I}$ is quasi-coherent, the same is true for $\mathcal{I}/\mathcal{I}^2$ and for $\Omega^1_{X/Y}$.
To motivate the definition and to understand why this is a generalization in the category of $\mathcal{O}_X$-modules of the module of relative differential forms we have to give a description of $\Omega^1_{X/Y}$ on local affine open subscheme of X.

Proposition A.7. *Let $f : X \to Y$ be an Y-scheme. Then, as a quasi-coherent $\mathcal{O}_X$-module, the sheaf of differential forms $\Omega^1_{X/Y}$ together with its exterior differential $d_{X/Y} : \mathcal{O}_X \to \Omega^1_{X/Y}$ is uniquely characterized up to canonical isomorphism by the following universal property. Given affine open subschemes $U = Spec(B) \subset X$ and $V = Spec(A) \subset Y$ satisfying $\sigma(U) \subset V$, there is a unique isomorphism $\Omega^1_{X/Y}|_U \cong \widetilde{\Omega^1_{B/A}}$ such that the restriction of $d_{X/Y}$ to U is induced from the exterior differential $d_{B/A} : B \to \Omega^1_{B/A}$.*

Proof. Consider $V = \mathrm{Spec}(A) \subset Y$ and $U = \mathrm{Spec}(B) \subset X$ such that $\sigma(U) \subset V$. Then the morphism $U \to V$ correspond to the ring morphism $A \to B$ so that B can be considered as an A-algebra. Let $J \subset B \otimes_A B$ be as before the kernel of the multiplication map $B \otimes_A B \to B$. Then from the previous discussion $\mathcal{I}|_{U \times_V U} = \tilde{J}$ and we obtain isomorphisms:

$$\Delta^*(\mathcal{I}/\mathcal{I}^2)|_U \cong (J/J^2 \otimes_{B\otimes_A B} B)^{\sim} \cong \widetilde{J/J^2} \cong \widetilde{\Omega^1_{B/A}}$$

where we have used the fact that B has an $B \otimes_A B$-module structure given by the multiplication map, and since $(B \otimes_A B)/J \xrightarrow{\cong} B$ then the A-module of the tensor product coincide with J/J^2. From the discussion in the affine case we can also see that the exterior differential $d_{B/A} : B \to \Omega^1_{B/A}$ is induced from the morphism of $\mathcal{O}_Y$-modules:

$$d_{X/Y} : \mathcal{O}_X \to \Omega^1_{X/Y} \qquad f \to p_2^*(f) - p_1^*(f)$$

where $p_1, p_2 : X \times_Y X \to X$ are the projections. For this reason also $d_{X/Y}$ is called the exterior differential of X over Y. □

We can therefore expect that we can translate in the new context the proposition (A.4) in this new context.

Proposition A.8. *Let $f : X \to Y$ a morphism of schemes.*

1. *(Base Change) Let Y' be a Y-scheme. Let $X' = X \times_Y Y'$ and let $p : X' \to X$ be the first projection. Then $\Omega^1_{X'/Y'} \cong p^*\Omega^1_{X/Y}$.*
2. *Given $Y \to Z$ a morphism of schemes, we have an exact sequence:*

$$f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$$

3. *Let U be an open subset of X; then $\Omega^1_{X/Y}|_U \cong \Omega^1_{U/Y}$.*
4. *If Z is a closed subscheme of X defined by a quasi-coherent sheaf of ideals $\mathcal{I}$, we have a canonical exact sequence:*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to \Omega^1_{Z/Y} \to 0$$

Despite these properties, in general it is difficult to study in detail the sheaf of differential 1-forms. However, in the case of some specific families of morphisms (unramified, smooth, étale [4]) much more con be said, in particular:

Proposition A.9. *Let $f : X \to Y$ be smooth of relative dimension r at a point $x \in X$. Then $\Omega^1_{X/Y}$ is locally free of rank r at x.*

Definition A.10. The sheaves of differential forms of higher degree are defined as $\Omega^n_{X/Y} = \bigwedge^i \Omega^1_{X/Y}$, the n-th exterior power of $\Omega^1_{X/Y}$. Sometimes we will denote $\Omega^0_{X/Y} = \mathcal{O}_X$.

Remark A.11. In particular if $\Omega^1_{X/Y}$ is locally free, then so is each $\Omega^n_{X/Y}$.

Definition A.12. The algebraic de Rham complex $\Omega^\bullet_{X/Y}$ is the sequence defined as:

$$\Omega^0_{X/Y} \xrightarrow{d^0} \Omega^1_{X/Y} \xrightarrow{d^1} \Omega^2_{X/Y} \to \ldots \to \Omega^i_{X/Y} \to \ldots$$

Where the differentials $d^i : \Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y}$ are differential operators of order 1, calculated as:

$$d^i(\sum f dx_{j_0\ldots j_i}) = \sum df \wedge dx_{j_0\ldots j_i}$$

Consequently we can define the algebraic de Rham cohomology of X over Y as the hypercohomology of the algebraic de Rham complex:

$$H^\bullet_{dR}(X/Y) = \mathbb{H}^\bullet(X, \Omega^\bullet_{X/Y}) = \mathbb{R}^\bullet\Gamma(X, \Omega^\bullet_{X/Y})$$

Proposition A.13 (Poincaré Lemma). *Let k be a field of characteristic zero. Let $R = k[x_1, \ldots, x_n]$. Then the following sequence is an exact sequence of k-vector spaces.*

$$0 \to k \to R \xrightarrow{d} \Omega^1_{R/k} \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n_{R/k} \to 0$$

hence if $X = \mathbb{A}^n_k$ then $H^0_{dR}(X/k) = k$ and all other cohomology groups are zero.

Proof. [20] We proceed by induction on n, the case $n = 0$ is clear. Take $\omega \in \Omega^p_{R/k}$ with $d\omega = 0$, we want to find some $\eta \in \Omega^{p-1}_{R/k}$ such that $d\eta = \omega$. We can always write $\omega = \omega' dx_1 + \omega''$ where in ω', ω'' does not appear dx_1. Now we define

$$\eta' = \int \omega' dx_1 \quad \text{where} \int x_1^r dx_1 = \frac{1}{r+1} x_1^{r+1}$$

Then of course $\eta' \in \Omega^{p-1}_{R/k}$, and $d\eta' = \omega' dx_1 + \omega'''$ where ω''' does not involve dx_1. Now replacing ω with $\omega - d\eta'$ we can reduce ourselves to the case where ω does not involve dx_1, hence:

$$\omega = \sum f_{i_1\ldots i_p} dx_{i_1} \ldots dx_{i_p} \text{ with } i_j > 1 \qquad d\omega = 0 \implies \frac{\partial f_{i_1,\ldots,i_p}}{\partial x_1} = 0$$

This means that the polynomials $f_{i_1\ldots i_p}$ do not involve x_1 so we have reduced to the situation of $k[x_2, \ldots, x_n]$ where the result is true by induction. □

Remark A.14. With respect to the previous proposition, we can note the following interesting facts:

- In the case char$k = p > 0$ the sequence can't be exact indeed $d(x^p) = 0$.
- The same proof can be generalized in two directions. We can replace k with any ring A containing the rational numbers. Secondly, instead of taking R to be a polynomial ring, we can take R to be formal power series, or germs of holomorphic functions at the origin, in case $k = \mathbb{C}$.

The first fundamental result that we state is due to Grothendieck [15], who proved a comparison theorem between the algebraic de Rham cohomology of a non singular scheme over $\mathbb{C}$ and the usual de Rham cohomology of the associated analytic space. Indeed we have to remember that if X is a scheme of finite type over $\mathbb{C}$ there is a natural way to associate X with an analytic space X^{an}. We can summarize the construction as follows, take a covering $X = \bigcup U_i$ of open affine subsets, and embed every $U_i \xrightarrow{j_i} \mathbb{A}^n_{\mathbb{C}}$ as a closed subscheme, and take the polynomials generators $f_1, \dots, f_r$ of its ideal. These polynomials define a closed analytic subspace of $\mathbb{C}^n$, which we call $(U_i)^{an}$. We can now glue together the spaces $(U_i)^{an}$ to obtain X^{an}. In this way we obtain also a morphism of ringed spaces $j : (X^{an}, \mathcal{O}_X^{an}) \to (X, \mathcal{O}_X)$.

Proposition A.15. *If X is a scheme of finite type over $\mathbb{C}$, then the analytification of the scheme X, denoted by $(X^{an}, \mathcal{O}^{an})$ is a complex analytic space with a morphism of ringed spaces $j : X^{an} \to X$ such that the map over the underlying space is the inclusion on closed points.*

Originally, the foundations for many relations between the theory of algebraic and analytic geometry were put in place by Serre who proved in [30] a slightly less general version of the following theorem:

Theorem A.16. *[18] Let X a proper scheme over $\mathbb{C}$, then the category of coherent algebraic sheaves on X and the category of coherent analytic sheaves on the corresponding analytic space X^{an} are equivalent, more in detail:*

- *Any analytic coherent sheaf $\mathcal{E}^{an}$ on X^{an} is the analytification of an algebraic coherent sheaf $\mathcal{E}$ on X.*
- *For any coherent sheaves $\mathcal{E}, \mathcal{F}$ on X, any morphism $\mathcal{E}^{an} \to \mathcal{F}^{an}$ is induced by a morphism $\mathcal{E} \to \mathcal{F}$.*
- *For any coherent sheaf $\mathcal{E}$ on X, with analytification $\mathcal{E}^{an}$ on X^{an}, the following natural maps are bijections: $H^i(X, \mathcal{E}) \xrightarrow{\cong} H^i(X^{an}, \mathcal{E}^{an})$.*

Now we are able to state the comparison result from algebraic to analytic de Rham cohomology:

Theorem A.17. *[15] Let X be a smooth scheme of finite type over $\mathbb{C}$. Then we have a functorial isomorphism:*

$$H^\bullet_{dR}(X/\mathbb{C}) \xrightarrow{\cong} H^\bullet_{dR^{an}}(X^{an}) \cong H^\bullet(X^{an}, \mathbb{C})$$

where $H^\bullet(X^{an}, \mathbb{C})$ is the usual Betti cohomology of X^{an}.

Remark A.18. If X is a smooth scheme of finite type over $\mathbb{C}$, the analytic de Rham cohomology of the related analytic space X^{an} is defined as $H^\bullet_{dR^{an}}(X^{an}) = \mathbb{H}^\bullet(X^{an}, \Omega^\bullet_{X^{an}/\mathbb{C}})$, but here thanks to the Poincaré's lemma the complex of sheaves $\Omega^\bullet_{X^{an}/\mathbb{C}}$ is just a resolution of the constant sheaf $\mathbb{C}$, so that the hypercohomology of this complex is just the usual cohomology $H^\bullet(X^{an}, \mathbb{C})$.

The last useful theorem that is true only in zero characteristic is the following:

Theorem A.19. *Let k be a field of characteristic zero. Let X be a smooth and proper variety over k. Then the spectral sequence:*

$$E_1^{p,q} = H^p(X, \Omega^q_{X/k}) \implies H^{p+q}_{dR}(X/k)$$

degenerates at the E_1 page, and thus we have a natural filtration (Hodge filtration), giving isomorphism:

$$\bigoplus_{p+q=n} H^p(X, \Omega^q_{X/k}) \cong H^n_{dR}(X/k)$$

which shows for instance that if X is affine then:

$$H^\bullet_{dR}(X/k) = H^\bullet(\Gamma(X, \Omega^\bullet_{X/k}))$$

this was originally established using analytic techniques, but can also be proved algebraically as in [12], the same theorem is still true in the relative situation of a smooth and proper morphism of schemes $f : X \to S$ with S a scheme of characteristic 0.

Example A.20.

- Let K be a field of characteristic zero and $P(x) = x^3 + ax + b \in K[x]$ such that has no repeated roots, and call $R = K[x, y]/(y^2 - P(x))$. Consider now the elliptic curve over K given by Spec$(R) = X$. It's easy to show that $H^0_{dR}(X/K) = K$ and $H^i_{dR}(X/K) = 0$ for $i \geq 2$ since X is a curve and hence dim$X$=dim$R = 1$. We have then only to study $H^1_{dR}(X/K)$, which is exactly the cokernel of the derivation $d : R \to \Omega^1_{R/K}$, note that since we are in the affine case we can identify $\Omega^1_{X/k}$ with $\Omega^1_{R/k}$. First of all we obtain some relations:

  $$0 = d(y^2 - P(x) = 2ydy - P'dx \qquad 2ydy = P'dx$$

  Since we have $\Delta \neq 0$ and P has no repeated roots then P, P' are coprime hence exists polynomials $A, B \in K[x]$ such that $AP + BP' = 1$. We can write $dx = (AP + BP')dx$ from this consider:

  $$\omega = Aydx + 2Bdy \longrightarrow y\omega = dx \quad dy = \frac{1}{2}P'\omega$$

  so that we can represent every element η of $\Omega_{R/K}$ as $\eta = (C + Dy)\omega$, with $C, D \in K[x]$. Now we have to study when a 1-form is exact, i.e. if exist $E, F \in K[x]$ such that:

  $$(C + Dy)\omega = d(E + Fy) = E'dx + F'ydx + Fdy$$
  $$C\omega + Dy\omega = (\frac{1}{2}P'F + F'P)\omega + E'y\omega$$

  This means that $Dy\omega$ is always exact. While for $C\omega$, if F has leading term cx^d, then $\frac{1}{2}P'F + F'P$ has term $(\frac{3}{2} + d)cx^{d+2}$:

  $$(\frac{1}{2}3x^2)(cx^d) + (cdx^{d-1})x^3 = \frac{3}{2}cx^{d+2} + cdx^{d+2} = (\frac{3}{2} + d)cx^{d+2}$$

  hence we can choose c and d so that we remove the leading term of C. Repeating this process we can write $C\omega$ as an exact differentiation plus a K-linear combination of $\omega, x\omega$, and this is a basis of $H^1_{dR}(X/K)$.

- Similar computation in the case $R = k[x, y]/(y^2 - x^3)$, and $X =$ Spec(R), give us as result $H^0_{dR}(X/k) = k$ and $H^i_{dR}(X/k) = 0$ for $i > 1$, but if $k = \mathbb{C}$ then this is an example where the algebraic de Rham cohomology of a non-smooth scheme is problematic since the first Betti cohomology group of the complex curve $y^2 - x^3 = 0$ is zero.

- In [25] we can find a explicit computation for a complete elliptic curve.

A.2 Connections

The concept of connection for a morphism of schemes $f : X \to S$ was a notion of central importance in the first part of the thesis because led us to the definition of stratification. So in this section we expose the main result that we need about connections following [1], [11] and [22]. In fact we want to give a complete algebraic definition of a concept which is a very important tool in differential geometry, so first of all we recall the classical definition. In general, a section of a vector bundle generalises the notion of a function on a manifold, in the sense that a standard vector-valued function $f : M \to \mathbb{R}^n$ can be viewed as a section of the trivial vector bundle $M \times \mathbb{R}^n \to M$. It is therefore natural to ask how it is possible to differentiate a section in analogy to how one differentiates a vector field. The answer to this question if provided by the notion of connection

Definition A.21. [26] Let $\pi : E \to M$ be a vector bundle over a manifold M, let $\mathcal{E}(M)$ denote the space of smooth sections of E and $\mathcal{T}(M)$ denote the space of smooth section of the tangent bundle TM. A connection in E is a map:

$$\nabla : \mathcal{T}(M) \times \mathcal{E}(M) \to \mathcal{E}(M)$$

written $(X, Y) \to \nabla_X Y$, satisfying the following properties:

- ∇ is linear over $C^\infty(M)$ in X:

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y \quad \text{for } f, g \in C^\infty(M)$$

- ∇ is linear over $\mathbb{R}$ in Y:

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2 \quad \text{for } a, b \in \mathbb{R}$$

- ∇ satisfies the following product rule:

$$\nabla_X(fY) = f\nabla_X Y + X(f)Y \quad \text{for } f \in C^\infty(M)$$

In particular a connection is called linear if it is a connection on M in TM.

Given a connection we can measure the change of a vector field along a curve $\gamma : I \to M$ with respect to the connection ∇ over the tangent bundle.

Definition A.22. A vector field along a curve $\gamma : I \to M$ is a smooth map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for every $t \in I$. We let $\mathcal{T}(\gamma)$ denote the space of vector fields along γ.

Proposition A.23. *Let ∇ be a linear connection, for each curve $\gamma : I \to M$, then ∇ determines a unique operator called covariant derivative $D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$ which satisfies similar condition as before:*

- *Linearity over $\mathbb{R}$*
- *Product rule: $D_t(fV) = \dot{f}V + fD_t(V)$ for $f \in C^\infty(I)$.*
- *For any extension $\hat{V}$ of V, we have $D_t(V)(t) = \nabla_{\dot{\gamma(t)}}(\hat{V})$.*

With this definitions is possible to define the notion of acceleration and geodesics on a differentiable manifolds. Another construction involving covariant differentiation along curves is parallel translation.

Definition A.24. Let M be a manifold with a linear connection ∇. A vector field V along a curve γ is said to be parallel along γ with respect to ∇ if $D_t V \equiv 0$.

Definition A.25. Given a curve $\gamma : I \to M$, $t_0 \in I$ and a vector $V_0 \in T_{\gamma(t_0)}M$, the parallel translate of V_0 along γ is the unique parallel vector field V along γ such that $V(t_0) = V_0$.

The existence and uniqueness of the parallel translate of a vector of the tangent space along a curve is related to the theorem of existence and uniqueness of solutions for linear ODEs.

Remark A.26. If $\gamma : I \to M$ is a curve and $t_0, t_1 \in I$, parallel translation defines an operator:

$$P_{t_0 t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M \quad P_{t_0 t_1}(V_0) = V(t_1)$$

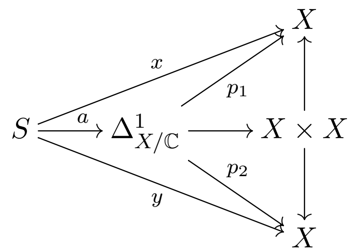
Where V is the parallel translate of V_0 along γ. It is easy to check that $P_{t_0 t_1}$ is a linear isomorphism between the different tangent space, this is the sense in which a connection "connects" nearby tangent spaces. Given a linear connection ∇ on M it is even possible to recover the covariant differentiation along a curve γ by this operators.

This last remark give us an idea of how we can adapt the concept of connection in a more algebraic setting, but to bridge the gap from the differential to the algebraic definition given in the first chapter of the thesis (1.11) we can consider the situation of a smooth scheme X over $\mathbb{C}$ as it has been done in [10]. Now recall the notation and the definitions of the first section of the thesis (1.1), consider the diagonal morphism $\Delta : X \to X \times X$ and let $(\Delta^n_{X/\mathbb{C}}, \mathcal{P}^n_{X/\mathbb{C}})$ be the n-th infinitesimal neighborhood of the diagonal. Given now two morphisms of schemes $x, y : S \to X$, we say that x, y are infinitesimally close of the first order if the map $(x, y) : S \to X \times X$ factorize through the first infinitesimal neighborhood of the diagonal. Note that if $x, y : S \to X$ are infinitesimally close points, then they induce the same map of topological spaces from S to X, the only difference is what happens with sheaves of functions, this is one sense in which x and y can be really regarded as "close". We can now translate the concept of differential connection in this situation as follows:

Definition A.27. Let V be a $\mathcal{O}_X$-module locally free of finite rank over X a smooth complex variety, a holomorphic connection γ on V consists of the following data:

- For each pair of map x, y infinitesimally close of first order, a morphism: $\gamma_{x,y} : x^*V \to y^*V$.
- For each morphism $x : S \to X$ then $\gamma_{x,x} = id$.
- For every map $f : T \to S$ and each pair of maps $x, y : S \to X$ infinitesimally close of first order, we have that: $f^*(\gamma_{y,x}) = \gamma_{yf,xf}$.

From this definition which is similar to the notion of parallel translation we will arrive to a new definition of connection that can be generalized in the relative situation of a morphism of schemes $f : X \to S$ with V a quasi-coherent sheaf of $\mathcal{O}_X$-module. First consider two maps infinitesimally close of first order $x, y : S \to X$, then for definition the following diagram is commutative:



Note that given an isomorphism γ_{p_1,p_2} we could obtain a holomorphic connection from the pull-back $\gamma_{x,y} = a^*(\gamma_{p_1,p_2})$, conversely since from the definition $\gamma_{id,id} = id$, then every connection gives us an isomorphism $\gamma_{p_1,p_2} \in \text{Hom}(p_1^* V, p_2^* V)$ such that reduces to the identity over the diagonal $\Delta(X) \cong \Delta^0_{X/S} \cong X$, and by adjunction we obtain a morphism $D_\gamma : V \to p_{1*}p_2^* V$. Now define the jet bundle as $J^1(V) = p_{1*}p_2^*(V)$, notice that for (1.2.2) it is canonically isomorphic to the sheaf of principal parts of order one:

$$J^1(V) = p_{1*}p_2^*(V) \cong \mathcal{P}^1_{X/\mathbb{C}}(V)$$

We can therefore interpret a holomorphic connection as a $\mathcal{O}_X$-linear morphism $D_\gamma : V \to \mathcal{P}^1_{X/\mathbb{C}}(V)$ such that the composition $V \xrightarrow{D_\gamma} \mathcal{P}^1_{X/\mathbb{C}}(V) \xrightarrow{p} V$ is the identity, where the map p is just the canonical projection induced by $X \xrightarrow{\cong} \Delta(X) \to \Delta^1_{X/\mathbb{C}}$. From this discussion we can say that a connection can be interpreted essentially as a section for the projection morphism p.

Now given the characterization of the jet bundle $J^1(V) = \mathcal{P}^1_{X/\mathbb{C}}(V)$ we can also consider the canonical differential operator of (1.2.4):

$$j^1 : V \to J^1(V) \qquad x \to 1 \otimes x$$

notice that j^1 is not $\mathcal{O}_X$-linear, in fact given $x \in X$ then $J^1(V)_x = (\mathcal{O}_{X\times X} \otimes V)_x / \mathcal{I}^2_x$ and we can compute:

$$j^1_x(as) = 1 \otimes 1 \otimes as = 1 \otimes a \otimes s + a \otimes 1 \otimes s - a \otimes 1 \otimes s = a j^1_x(s) + (1 \otimes a - a \otimes 1) \otimes s = a j^1_x(s) + da \otimes s$$

notice that also in this case we have that $p \circ j^1 = id$, so we can define a map ∇:

$$V \underset{D_\gamma}{\overset{j^1}{\rightrightarrows}} \mathcal{P}^1_{X/\mathbb{C}}(V) \xrightarrow{p} V$$

$$\begin{array}{rcl} \nabla : \; V & \longrightarrow & \Omega^1_{X/\mathbb{C}}(V) \\ x & & j^1(x) - D_\gamma(x) \end{array}$$

indeed for every section x of V we can consider $\nabla(x)$ which by definition is a section of the kernel of the map p:

$$\begin{aligned} \text{Ker}(\mathcal{P}^1_{X/\mathbb{C}} \otimes_{\mathcal{O}_X} V \xrightarrow{p} \mathcal{O}_X \otimes V) &= \text{Ker}((\mathcal{O}_{X\times X}/\mathcal{I}^2) \otimes V \xrightarrow{p} (\mathcal{O}_{X\times X}/\,\mathcal{I}) \otimes V) \\ &\cong (\mathcal{I}/\mathcal{I}^2) \otimes V \cong \Omega^1_{X/\mathbb{C}} \otimes V = \Omega^1_{X/\mathbb{C}}(V) \end{aligned}$$

Remark A.28. We have managed to define the differential of a section of V starting from a holomorphic connection that compares two fiber of V infinitesimally close, and it is easy to see that if a is a section of $\mathcal{O}_X$, and x a section of V:

$$\nabla(ax) = j^1(ax) - D_\gamma(ax) = aj^1(x) + da \otimes x - aD(x) = a\nabla(x) + da \otimes x$$

More importantly the converse is also true, we can recover D_γ and hence also γ starting with a map ∇, but if we want to obtain the linearity of D_γ we have also to ask that ∇ satisfies the following identity:

$$\nabla(ax) = da \otimes x + a.\nabla x \tag{A.28.1}$$

We can then conclude that is equivalent to have a holomorphic connection γ on V or a $\mathbb{C}$-linear morphism $\nabla : V \to \Omega^1_X(V)$ satisfying the previous condition (A.28.1).

With this last remark in mind we obtain the definition, due to Koszul, of a connection in algebraic term, and from the following proposition we can see that this new definition of connection is equivalent to the more abstract definition given in the first chapter of the thesis (1.11).

Proposition A.29. *Let $f{:}X{\to}S$ be a morphism of schemes and V a quasi-coherent $\mathcal{O}_X-$module. A connection $\epsilon : \mathcal{P}^1_{X/S} \otimes V \to V \otimes \mathcal{P}^1_{X/S}$ with the definition of (1.11) on V is equivalent to an additive $f^{-1}(\mathcal{O}_S)$-linear morphism:*

$$\nabla : V \to V \otimes \Omega^1_{X/S} \quad \textit{such that} \quad \nabla(ax) = a\nabla x + x \otimes da$$

where x is a section of V and a is a section of $\mathcal{O}_X$.

Proof. The equivalence can bee seen by setting:

$$\epsilon \to \nabla_\epsilon(x) = \epsilon \circ d_{1,V} - x \otimes \mathrm{id}_{\mathcal{P}} = \epsilon(1 \otimes 1 \otimes x) - x \otimes (1 \otimes 1)$$

Since ϵ reduces to the identity modulo $\Omega^1_{X/S}$ then $\nabla(x) \in V \otimes \Omega^1_{X/S}$. Also if we call $\epsilon \circ d_{1,V} = \lambda$ then a calculation shows that:

$$\begin{aligned}
\nabla_\epsilon(ax) &= (1 \otimes a)\lambda(x) - (a \otimes 1)(x \otimes (1 \otimes 1)) \\
&= (1 \otimes a)\lambda(x) - (1 \otimes a)(x \otimes (1 \otimes 1)) + (1 \otimes a)(x \otimes (1 \otimes 1)) - (a \otimes 1)(x \otimes (1 \otimes 1)) \\
&= (1 \otimes a)\nabla_\epsilon(x) + da(x \otimes 1) = a\nabla_\epsilon(x) + x \otimes da
\end{aligned}$$

Where we have to remember that $d_{1,V}$ is linear for the right module structure of $\mathcal{P}^1_{X/S}$.

Conversely, if we have a map ∇ on V we can consider $\lambda(x) = \nabla(x) + x \otimes (1 \otimes 1)$ and from the previous calculation we already know that $\lambda : V \to V \otimes \mathcal{P}^1_{X/S}$ is linear for the right $\mathcal{O}_X$-module structure of $\mathcal{P}^1_{X/S}$, then simply by extension of scalar we obtain a $\mathcal{P}^1_{X/S}$-linear isomorphism $\epsilon : \mathcal{P}^1_{X/S} \otimes V \to V \otimes \mathcal{P}^1_{X/S}$ which is the identity modulo $\Omega^1_{X/S}$.
So starting with a connection ∇ we have a $\mathcal{P}^1_{X/S}$-linear isomorphism:

$$\epsilon((a \otimes b) \otimes x) = \nabla(x) \otimes (1 \otimes 1) + (x \otimes (a \otimes b))$$

and the associated connection is exactly ∇ again, indeed:

$$\nabla_\epsilon(x) = \epsilon((1 \otimes 1) \otimes x) - x \otimes (1 \otimes 1) = \nabla(x) \otimes (1 \otimes 1) + x \otimes (1 \otimes 1) - x \otimes (1 \otimes 1) = \nabla(x)$$

□

Definition A.30. Given a connection ∇ on an $\mathcal{O}_X$-module V it is possible to extend a connection to a $f^{-1}(\mathcal{O}_S)$-linear morphism of $\mathcal{O}_X$-modules:

$$\nabla_i : \Omega^i_{X/S}(V) \to \Omega^{i+1}_{X/S}(V)$$

satisfying the identity:

$$\nabla_i(\omega \otimes x) = d\omega \otimes x + (-1)^i \omega \wedge \nabla(x)$$

with ω section of $\Omega^i_{X/S}$ and x section of V, over an open of X and where $\omega \wedge \nabla(x)$ denotes the image of $\omega \otimes \nabla(x)$ under the canonical mapping:

$$\begin{array}{ccc} \Omega^i_{X/S} \otimes_{\mathcal{O}_X} (\Omega^1_{X/S} \otimes_{\mathcal{O}_X} V) & \longrightarrow & \Omega^{i+1}_{X/S} \otimes_{\mathcal{O}_X} V \\ \omega \otimes \tau \otimes x & & (\omega \wedge \tau) \otimes x \end{array}$$

Note that of course with this notation $\nabla_0 = \nabla$.

Definition A.31. Given a connection ∇ over an $\mathcal{O}_X$-module V, the curvature of the connection $K = K(V, \nabla)$ is the $\mathcal{O}_X$-linear map:

$$K := \nabla_1 \nabla : V \to \Omega^2_{X/S} \otimes_{\mathcal{O}_X} V$$

when X/S is smooth we can consider the curvature as:

$$K \in \mathrm{Hom}_{\mathcal{O}_X}(V, \Omega^2_{X/S} \otimes V) \cong \mathrm{End}(V) \otimes \Omega^2_{X/S}$$

Note that even though neither ∇ nor ∇_1 are $\mathcal{O}_X$-linear, it is easy to check that the composition is an $\mathcal{O}_X$-linear map, in fact it is true even more:

$$\begin{aligned} \nabla_{i+1}\nabla_i(\omega \otimes x) &= \nabla_{i+1}(d\omega \otimes x + (-1)^i \omega \wedge \nabla(x)) \\ &= d^2\omega \otimes x + (-1)^{i+1} d\omega \wedge \nabla(x) + (-1)^i d\omega \wedge \nabla(x) + \omega \wedge K(x) = \omega \wedge K(x) \end{aligned}$$

so that $K(V, \nabla)$ can be thought as an element of $\mathrm{End}(V) \otimes \Omega^2_{X/S}$ or a 2-differential form with coefficient in $\mathrm{End}(V)$.

Definition A.32. We say that a connection ∇ over V is integrable if $K(V, \nabla) = 0$.

Definition A.33. Given an integrable connection ∇ over V, we can define the the de Rham complex of (V, ∇) as:

$$0 \to V \xrightarrow{\nabla} \Omega^1_{X/S} \otimes_{\mathcal{O}_X} V \xrightarrow{\nabla_1} \Omega^2_{X/S} \otimes_{\mathcal{O}_X} V \xrightarrow{\nabla_2} \dots$$

which will be denoted by $(\Omega^\bullet_{X/S}(V), \nabla)$. Note that this complex does not depend only on V but also from his connection ∇.

On [11] there is an equivalent definition of integrable connection, but to state it we have to establish a correspondence between connections and derivation. So suppose that X/S is smooth and consider $\mathrm{Der}(X/S)$ the sheaf of germs of S-derivation of $\mathcal{O}_X$ into itself, and $\mathcal{E}nd_S(V)$ the sheaf of $f^{-1}(\mathcal{O}_S)$-linear endomorphism of V, note that both of them are naturally sheaf of

$f^{-1}(\mathcal{O}_S)$-Lie algebras. Given a connection ∇ over V we can obtain an $\mathcal{O}_X$-linear mapping between them:

$$\bar{\nabla} : \mathrm{Der}(X/S) \to \mathcal{E}nd_S(V)$$

where recall that we have an $\mathcal{O}_X$-linear isomorphism:

$$\begin{array}{ccc} \mathrm{Der}(X/S) & \xrightarrow{\cong} & \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X) \\ D & \longrightarrow & D' \end{array} \tag{A.33.1}$$

and $\bar{\nabla}(D)$ is given by the following composition:

$$V \xrightarrow{\nabla} \Omega^1_{X/S} \otimes_{\mathcal{O}_X} V \xrightarrow{D'\otimes 1} \mathcal{O}_X \otimes_{\mathcal{O}_X} V \cong V$$

note that $\bar{\nabla}(D)$ is indeed $f^{-1}(\mathcal{O}_S)$-linear since:

$$\bar{\nabla}(D)(ax) = D(a)x + a\bar{\nabla}(D)(x) \tag{A.33.2}$$

where D, a, x are section of $\mathrm{Der}(X/S), \mathcal{O}_X$ and V respectively over an open subset of X. Conversely, since X/S is smooth, it is possible to show that given $\bar{\nabla} : \mathrm{Der}(X/S) \to \mathcal{E}nd_S(V)$ which satisfies the rule (A.33.2) then $\bar{\nabla}$ arises from a unique connection ∇ over V.

Lemma A.34. *[22] The connection ∇ is integrable exactly when the mapping $\bar{\nabla} : Der(X/S) \to \mathcal{E}nd_S(V)$ is also a Lie-algebra homomorphism. So we can also see the curvature of the connection ∇ as a measure of the failure of $\bar{\nabla}$ to commute with the Lie bracket operation.*

Proof. We have to prove the following general identity:

$$[\bar{\nabla}(D_1), \bar{\nabla}(D_2)] - \bar{\nabla}([D_1, D_2]) = (D'_1 \wedge D'_2)K$$

where the right hand side is the composite map:

$$V \xrightarrow{K} \Omega^2_{X/S} \otimes V \xrightarrow{(D'_1\wedge D'_2)\otimes 1} \mathcal{O}_X \otimes_{\mathcal{O}_X} V$$

To prove this identity let D_1, D_2 be two derivation, then we have two $\mathcal{O}_X$-linear morphisms $D'_1, D'_2 : \Omega^1_{X/S} \to \mathcal{O}_X$, and we can prove the following identity:

$$((D'_1 \wedge D'_2) \otimes id) \circ \nabla_1 = \bar{\nabla}(D_1) \circ (D_2 \otimes id) - \bar{\nabla}(D_2) \circ (D_1 \otimes id) - ([D_1, D_2]' \otimes id) \tag{A.34.1}$$

Where the left hand side of the identity is the composition:

$$\Omega^1_{X/S} \otimes V \xrightarrow{\nabla_1} \Omega^2_{X/S} \otimes V \xrightarrow{(D'_1\wedge D'_2)\otimes id} \mathcal{O}_X \otimes V \quad \text{with} \quad D'_1 \wedge D'_2(a \wedge b) = D'_1(a)D'_2(b) - D'_2(a)D'_1(b)$$

We can prove (A.34.1) for element of form $d(a) \otimes x$ with a section of $\mathcal{O}_X$ and x section of V, since $\Omega^1_{X/S}$ is generated as an $\mathcal{O}_X$-module by elements da for every section a of $\mathcal{O}_X$, so that we have:

$$\begin{aligned} \mathrm{LHS} &= ((D'_1 \wedge D'_2) \otimes id)\nabla_1(da \otimes x) = ((D'_1 \wedge D'_2) \otimes id)(da \wedge \nabla(x)) = D_1(a)\bar{\nabla}(D_2)(x) - D_2(a)\bar{\nabla}(D_1)(x) \\ \mathrm{RHS} &= \bar{\nabla}(D_1)(D_2(a)x) - \bar{\nabla}(D_2)(D_1(a)x) - ([D_1, D_2]' \otimes id) \\ &= (D_1 \circ D_2(a))x + D_2(a)\bar{\nabla}(D_1)(x) - (D_2 \circ D_1(a))x - D_1(a)\bar{\nabla}(D_2)x - (D_1 \circ D_2(a)) + (D_2 \circ D_1(a)) \\ &= D_1(a)\bar{\nabla}(D_2)(x) - D_2(a)\bar{\nabla}(D_1)(x) \end{aligned}$$

Now for the proof of the lemma we can simply compose with ∇ the identity (A.34.1) to obtain:

$$[\bar{\nabla}(D_1), \bar{\nabla}(D_2)] - \bar{\nabla}([D_1, D_2]) = (D_1 \wedge D_2)K$$

□

A.2.1 Gauss-Manin connection

An example of connection which is particularly important in the algebraic framework is the so called Gauss-Manin connection, which is a way to differentiate cohomology classes with respect to parameters. The basic idea behind the Gauss-Manin connection is actually very simple. Suppose that $f : X \to B$ is a proper map between manifolds, with dim$X >$dim$B$. In general the fibers $X_b = f^{-1}(b)$ are smooth compact manifolds, and moreover by the Ehresmann fibration theorem they will be diffeomorphic, in particular they have isomorphic homology and cohomology. Now suppose that $\alpha in \Omega^k(X)$ such that the restriction $i_b^*\alpha \in \Omega(X_b)$ is closed. Then this gives a family of cohomology classes: $[i_b^*\alpha] \in H^k(X_b)$. Let $b_1, \dots, b_n$ be a set of local coordinates in B. Then consider the classes of the form:

$$\left[i_b^* \left(\frac{\partial^{i_1 + \dots i_n} \alpha}{\partial b_1^{i_1} \dots \partial b_n^{i_n}} \right) \right] \in H^k(X_b)$$

by taking higher and higher derivatives if necessary, eventually the number of classes of this form will wxceed the k-th Betti number of X_b. Then necessarily , some linear combination of these classes must equal zero, i.e. the family of classes $[i_b^*\alpha]$ satisfies a linear PDE. This is the PDE encoded by the Gauss-Manin connection.

The Gauss-Manin connection was introduced by Y. Manin (1958) for curves and A. Grothendieck (1966) in higher dimensions. For a proper and smooth morphism $f : X \to S$ of $\mathbb{C}$-schemes the de Rham cohomology of the fibers of f is described by the locally free $\mathcal{O}_S$-modules $H^n_{dR}(X/S) = R^n f_*(\Omega^\bullet_{X/S})$, the relative de Rham cohomology sheaves. Then, by the comparison theorem (A.17) for algebraic and analytic de Rham cohomology, we have a canonical isomorphism between the sheaf defined over S^{an} by the coherent algebraic sheaf $R^i f_*(\Omega^\bullet_{X/S})$ on S, and the sheaf:

$$R^i f_*^{an}(\mathbb{C}_{X^{an}}) \otimes_{\mathbb{C}_{S^{an}}} \mathcal{O}_{S^{an}}$$

There is a canonical absolute integrable connection on this analytic sheaf, namely the canonical connection on the tensor product:

$$\nabla(\alpha \otimes f) = a \otimes df \in R^i f_*^{an}(\mathbb{C}_{X^{an}}) \otimes_{\mathbb{C}_{S^{an}}} \Omega^1_{S^{an}} \tag{A.34.2}$$

where f is a section of $\mathcal{O}_{S^{an}}$ and α is a section of $R^i f_*^{an}(\mathbb{C}_{X^{an}})$. This connection is characterized by the fact that its horizontal section are those of the subsheaf:

$$R^i f_*^{an}(\mathbb{C}_{X^{an}}) = R^i f_*^{an}(\mathbb{C}_{X^{an}}) \otimes_{\mathbb{C}_{S^{an}}} \mathbb{C}_{S^{an}}$$

Definition A.35. The above integrable connection is the analytic version of the Gauss-Manin connection on X^{an}/S^{an}.

In the projective case, known theorems relating algebraic and analytic cohomology enable one to construct the Gauss-Manin connection on the relative algebraic de Rham cohomology, but this approach is unsatisfactory for several reasons.For example it is already difficult in practice to make concrete the locally constant section decomosition as in (A.34.2) In this section we will see that this transcendental connection is induced by an algebraic connection over the algebraic de Rham cohomology sheaf $R^i f_*(\Omega^\bullet_{X/S})$, and this latter connection can be defined by purely algebraic means. In particular we will give two different construction for the Gauss-Manin connection of a smooth morphism $f : X \to S$ that can be applied also in characteristic $p > 0$. Firstly following [17] we will sketch a construction for a canonical integrable connection, in the sense of derived categories, on the de Rham cohomology $Rf_*(\Omega^\bullet_{X/S})$, secondly we will give a more explicit construction of the same connection following [24] Since this connection on the sheaves $R^if_*(\Omega^\bullet_{X/S})$ will be integrable, when working in characteristic 0, for (1.18) we would obtain also a canonical stratification, and hence for (2.11) this shows that $Rf_*(\Omega^\bullet_{X/S})_{strat}$ is a crystal of $\mathcal{O}_{X_{strat}}$-modules, i.e. it extends canonically to all infinitesimal neighborhoods of S, and not just those with retractions onto S.

To build such a connection is equivalent to constructing , for each diagram:

$$\begin{array}{ccccc} X & & X' & & X'' \\ \downarrow f & & & \searrow f' \quad f'' \swarrow & \\ S & \underset{q_1}{\overset{q_2}{\leftleftarrows}} \!\!\to & & S' & \end{array} \tag{A.35.1}$$

where $S \xrightarrow{i} S'$ is a closed immersion defined by a square zero ideal, q_1, q_2 are two retractions of i and the two maps f', f'' are smooth liftings of $f : X \to S$ given by base change by q_1, q_2, an isomorphism:

$$Rf'_*(\Omega^\bullet_{X'/S'}) \xrightarrow{\cong} Rf''_*(\Omega^\bullet_{X''/S'}) \tag{A.35.2}$$

satisfying the natural condition of transitivity for a third retraction $q_3 : S' \to S$. To understand why this is true we need to use the following base change formula for the de Rham cohomology.

Proposition A.36. *If X is smooth on S, the de Rham cohomology sheaves commute with base change, provided we work with derived categories (B.3).*

Proof. Consider the diagram:

$$\begin{array}{ccc} X' = X \times_S S' & \xrightarrow{\quad p \quad} & X \\ \downarrow f' & & \downarrow f \\ S' & \xrightarrow{\quad g \quad} & S \end{array}$$

since X/S is smooth then $\Omega^\bullet_{X/S}$ is locally free and hence is flat, so that we can use the Kunneth formula:

$$Lg^*(Rf_*(\Omega^\bullet_{X/S})) \xrightarrow{\cong} Rf'_*(p^*(\Omega^\bullet_{X/S}))$$

but we also know that in general it is true that: $p^*(\Omega^\bullet_{X/S}) \cong \Omega^\bullet_{X'/S'}$ then we obtain:

$$Lg^*(Rf_*(\Omega^\bullet_{X/S})) \xrightarrow{\cong} Rf'_*(\Omega^\bullet_{X'/S'})$$

which is the assertion that the De Rham cohomology commutes with base change in the sense of derived categories. □

Now if we manage to build the isomorphisms (A.35.2) as in [17], we can apply the base change formula (A.36) to obtain a canonical isomorphism:

$$Lq_1^*(Rf_*(\Omega^\bullet_{X/S})) \xrightarrow{\cong} Lq_2^*(Rf_*(\Omega^\bullet_{X/S}))$$

which is exactly the definition of a connection in the sense of derived categories. Notice that this construction is independent from the retractions q_1, q_2 and is valid for any two smooth liftings f_1, f_2. From this argument we would like to obtain that if $f{:}X{\to}S$ is a smooth morphism of schemes, then the de Rham cohomology sheaves $R^i f_*(\Omega^\bullet_{X/S})$ are fortified with a canonical stratification structure, but this is not true as these sheaves do not commute with base change. For example if $f : X{\to}S$ is a smooth and proper morphism of schemes of characteristic 0, then the $R^i f_*(\Omega^\bullet_{X/S})$ are locally free and do commute with base change, but in general we have to develop this idea in the contest of derived category where we have the requested base change properties and we can show that indeed $Rf_*(\Omega^\bullet_{X/S})$ is fortified with an absolute stratification.

Another completely algebraic construction, which uses spectral sequences, has been given by N.M. Katz and T. Oda [24]. To do this we want to build a category of module with connection

Definition A.37. Let (V, ∇) and (V', ∇') be quasi-coherent $\mathcal{O}_X$-module with connection, an $\mathcal{O}_X$-linear map $F : V \to V'$ is called horizontal if:

$$F(\bar{\nabla}(D)(x)) = \bar{\nabla}'(D)(F(x))$$

for D a section of Der(X/S) and x section of V over an open subset of X.

Definition A.38. Let MC(X/S) be the abelian category whose objects are given by pairs (V, ∇) and whose morphisms are the horizontal map $F : (V, \nabla) \to (V', \nabla')$.

In the category MC(X/S) we can define a tensor product where:

$$\begin{aligned}(V, \nabla) \otimes (V', \nabla') &= (V \otimes V', \nabla'') \\ \nabla''(D)(x \otimes y) &= \bar{\nabla}(D)(x) \otimes y + x \otimes \bar{\nabla}'(y)\end{aligned}$$

Furthermore, if V is locally of finite presentation then we can also define an internal Hom functor:

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_X}((V, \nabla), \_) : \text{MC}(X/S) &\longrightarrow \text{MC}(X/S) \\ (V', \nabla') &\qquad (\mathcal{H}om_{\mathcal{O}_X}(V, V'), \nabla''') \\ (\nabla'''(D)(F))(x) &= \nabla'(D)(F(x)) - F(\nabla(D)(x))\end{aligned}$$

Definition A.39. We denote MIC(X/S) the full abelian subcategory of MC(X/S) consisting of sheaves of quasi-coherent $\mathcal{O}_X$-modules with integrable connections. Note that this category is stable under formation of tensor product and internal Hom.

As we have said before (A.33), in the category MIC(X/S) for every object (V, ∇) we can build its de Rham complex $\Omega^\bullet_{X/S}(V)$, and, since MIC(X/S) has enough injectives, we can then define its de Rham cohomology as the right hyper derived functors of $R^0 f_*$, that is:

$$H^q_{dR}(X/S, V, \nabla) = R^q f_*(\Omega^\bullet_{X/S} \otimes_{\mathcal{O}_X} V)$$

it can be proven as in [17] that these functors $H^q_{dR}(X/S; \_)$ are the right derived functors of the left exact functor:

$$H^0_{dR}(X/S, \_) : \text{MIC}(X/S, \_) \to \text{MIC}(S/S, \_)$$

Where MIC$(S/S, _)$ is simply the category of quasi-coherent sheaves on S.
It is possible to go further in this direction as in [22], and if we suppose that $\pi : Y \to X$ is a smooth morphism, then we have a natural forgetful functor:

$$\begin{aligned} U:\ \text{MIC}(Y/S) &\longrightarrow \text{MIC}(Y/X) \\ (V, \nabla) &\longrightarrow (V, \nabla|\text{Der}(Y/X)) \end{aligned}$$

so that we can define the de Rham complex of $(V, \nabla|\text{Der}(Y/X))$ denoted by $\Omega^{\bullet}_{Y/X} \otimes_{\mathcal{O}_Y} V$, and we can define its de Rham cohomology as:

$$H^q_{dR}(Y/X, (V, \nabla)) = H^q_{dR}(Y/X, V, \nabla|\text{Der}(Y/X)) = R^q\pi_*(\Omega^{\bullet}_{Y/X} \otimes_{\mathcal{O}_Y} V) \tag{A.39.1}$$

In this situation we can consider the canonical filtration of $\Omega^{\bullet}_{Y/S} \otimes_{\mathcal{O}_Y} V$ by locally free sheaves:

$$F^i(\Omega^{\bullet}_{Y/S} \otimes_{\mathcal{O}_Y} V) = F^i(\Omega^{\bullet}_{Y/X}) \otimes_{\mathcal{O}_Y} V \qquad F^i(\Omega^{\bullet}_{Y/S}) = \text{Im}(\pi^*(\Omega^i_{X/S}) \otimes_{\mathcal{O}_Y} \Omega^{\bullet - i}_{Y/S} \to \Omega^{\bullet}_{Y/S})$$

and we get a spectral sequence associated to this filtered object, a direct calculation shows that:

$$E_1^{p,q} = \Omega^p_{X/S} \otimes_{\mathcal{O}_X} H^q_{dR}(Y/X, (V, \nabla))$$

so that the de Rham complex of $H^q_{dR}(Y/X, V, \nabla)$ is the complex $(E_1^{\bullet,q}, d_1^{\bullet,q})$ and the differential map $d_1^{0,q}$ is then a canonical integrable S-connection called Gauss-Manin connection on the quasi coherent $\mathcal{O}_X$-module $H^q_{dR}(Y/X, V, \nabla)$. The functors $H^q_{dR}(Y/X, \_) : \text{MIC}(Y/S) \to \text{MIC}(X/S)$ can be now interpreted as an exact connected sequence of cohomological functors.

Remark A.40. In particular choosing $V = \mathcal{O}_X$ and ∇ the trivial connection given by the differential $d : \mathcal{O}_X \to \Omega^1_{X/S}$, so that $(V, \nabla) = (\mathcal{O}_X, d)$ then we have that:

$$H^q_{dR}(X/S, \mathcal{O}_X, d) = H^q_{dR}(X/S) = R^q f_*(\Omega^{\bullet}_{X/S})$$

and we have obtained a canonical way to associate an integrable connection on the relative de Rham cohomology $H^q_{dR}(X/S)$.

This is a definition for the Gauss-Manin connection, which being an integrable connection, in characteristic 0 give rise to a stronger stratification structure. However, in characteristic $p > 0$ has curvature $K = 0$ but do not prolong to a stratification, actually this fact is explained in [23] by relating the p-curvature of the Gauss-Manin connection to the Kodaira-Spencer map.

B Homological Algebra

Through this section we will follow [14], [19] and [33], we assume the basic notion of homological algebra.

B.1 Spectral Sequences

Spectral sequence are useful to compute cohomology groups, we will give some definition and basic properties trying to highlight their usefulness with some important example. First we have to fix some notation and introduce the right terminology.

Definition B.1. Let $\mathcal{A}$ be an abelian category, recall that a double complex is a collection $\{C^{p,q}, d_h^{p,q}, d_v^{p,q}\}_{p,q\in\mathbb{Z}}$ with $C^{p,q} \in \text{Ob}(\mathcal{A})$, and maps as in the following diagram:

$$\begin{array}{ccccccccc}
 & & \dots & & \dots & & \dots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \rightarrow & C^{p-1,q+1} & \rightarrow & C^{p,q+1} & \rightarrow & C^{p+1,q+1} & \rightarrow & \dots \\
 & & \uparrow & & \uparrow{\scriptstyle d_v^{p,q}} & & \uparrow & & \\
\dots & \longrightarrow & C^{p-1,q} & \longrightarrow & C^{p,q} & \xrightarrow{d_h^{p,q}} & C^{p+1,q} & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \rightarrow & C^{p-1,q-1} & \rightarrow & C^{p,q-1} & \rightarrow & C^{p+1,q-1} & \rightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \dots & & \dots & & \dots & &
\end{array}$$

such that $d_h \circ d_h = 0 = d_v \circ d_v$ and $d_h^{p,q+1} \circ d_v^{p,q} + d_v^{p+1,q} \circ d_h^{p,q} = 0$. We can associate to a double complex the so called total complex given by

$$\text{Tot}(C^{\bullet\bullet})^n = \bigoplus_{p+q=n} C^{p,q} \qquad d_h + d_v : \text{Tot}(C^{\bullet\bullet})^n \to \text{Tot}(C^{\bullet\bullet})^{n+1}$$

To understand better the definition and the motivation for the construction of a spectral sequence we can consider a double complex $\{E_0^{\bullet\bullet}\}$, considering only the vertical differential we can obtain a second double complex $\{E_1^{pq}\}$ where $E_1^{pq} = H^q(E_0^{p,\bullet})$ considering now only the horizontal arrow we obtain a third double complex $\{E_2^{pq}\}$ with $E_2^{pq} = H^p(E_1^{\bullet,q})$. As we will state soon, there is a a relation between the cohomology of these new double complexes and the cohomology of the total complex $\text{Tot}(C^{\bullet\bullet})$, for the moment we can think to the element of $E_2^{p,q}$ as a second order approximation of the cohomology of $\text{Tot}(C^{\bullet\bullet})$. A spectral sequence is nothing but a generalization of this process.

Definition B.2. A cohomological spectral sequence in $\mathcal{A}$ is a family of object $E = \{E_r^{pq}\}$ with $r \geq 0$, where $E_r^{\bullet\bullet}$ are called r-pages with maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r \circ d_r = 0$ and with the condition that there are isomorphism between E_{r+1}^{pq} and the cohomology of $E_r^{\bullet\bullet}$ at the spot (p,q), i.e.:

$$\alpha_r^{p,q} : E_{r+1}^{p,q} \xrightarrow{\cong} H^{p,q}(E_r) = \ker d_r^{p,q} / \text{Im} d_r^{p-r,q+r-1}$$

In the application the importance of the spectral sequence is the existence of the limit objects $E_\infty^{p,q}$, if indeed for any pair (p,q) there exists $r_0(p,q)$ such that $d_r^{p,q} = 0, d_r^{p+r,q-r+1} = 0$ for $r \geq r_0(p,q)$ then the $\alpha_r^{p,q}$ identify all $E_r^{p,q}$ for $r \geq r_0$ and we will denote this object by $E_\infty^{p,q}$. The simplest and most common way to ensure this properties is the following condition:

Definition B.3. A spectral sequence $E = \{E_r^{p,q}, d_r\}$ is bounded if for every n there are only finitely many non zero terms $E_r^{p,q}$ such that $p+q=n$.

Definition B.4. Let $\{H^n\}$ be a collection of object of $\mathcal{A}$ and $E = \{E_r^{p,q}, d_r\}$ a spectral sequence. We say that E converges to $\{H^n\}$ if exists a regular finite decreasing filtration $\dots \subset F^{p+1}H^n \subset F^pH^n \subset \dots$ on each H^n and isomorphisms:

$$\beta^{p,q} : E_\infty^{p,q} \xrightarrow{\cong} F^pH^{p+q}/F^{p+1}H^{p+q}$$

where a filtration is said to be regular if $\bigcap_p F^pH^n = \{0\}$, and $\bigcup_p F^pH^n = H^n$. The traditional symbolic way of describing such a bounded convergence is:

$$E_\infty^{p,q} \Rightarrow H^{p+q}$$

The larger the number of zero object $E_r^{p,q}$ and of zero morphisms $d_r^{p,q}$ the better the sequence may serve as a computational tool. In particular in the majority of the applications of spectral sequences we have an even more strong property that simplify the convergence of a spectral sequence.

Definition B.5. A cohomological sequence E collapses at the $E_r^{\bullet\bullet}$ page if there is exactly one non zero row or column, i.e. $E_r^{p,q} = 0$ if $p \neq k$ or $q \neq k$ for some integer k. In this case since $E_r^{p,q}$ is the unique object with $p+q=n$ then if E converges to $\{H^n\}$ then $E_r^{p,q} = H^{p+q}$.

In general [33] a filtration over a cochain complex C naturally determines a spectral sequence which, under suitable conditions, converges to the cohomology of the complex $H^\bullet(C)$. In particular there are two filtrations associated to the total complex of every double complex, playing the related spectral sequences off against each other is an easy way to calculate the cohomology for the total complex.

Proposition B.6. *Let $C^{\bullet\bullet}$ be a first quadrant double complex, i.e. $C^{p,q} = 0$ for $p,q < 0$, and consider the total complex T=Tot($C^{\bullet\bullet}$). There are two decreasing filtrations over T:*

$$F_I^q(T^n) = \bigoplus_{i+j=n}^{i\geq q} C^{i,j} \quad F_{II}^p(T^n) = \bigoplus_{i+j=n}^{j\geq p} C^{i,j},$$

Since these filtrations over T are bounded, they induce two spectral sequences converging to the common limit $H^\bullet(T)$. We can compute the first and second pages of these spectral sequences directly as follows:

$$^{I}E_1^{p,q} = H^q(C^{p\bullet}) \qquad ^{II}E_1^{pq} = H^q(C^{\bullet p})$$
$$^{I}E_2^{pq} = H_h^p(H_v^q(C^{\bullet\bullet})) \qquad ^{II}E_2^{pq} = H_v^p(H_h^q(C^{\bullet\bullet}))$$

B.2 Hypercohomology

In what follows we will assume that $\mathcal{A}, \mathcal{B}$ are abelian categories with enough injectives. We will write $Ch^+(\mathcal{A})$ for the category of bounded below chain complexes in $\mathcal{A}$ and $\mathcal{K}^+(\mathcal{A})$ for the homotopy category of bounded below chain complexes.

Proposition B.7. *Let $A^\bullet \in Ch^+(\mathcal{A})$ be a bounded below complex in $\mathcal{A}$, then it exists always an upper half plane double complex $(I^{\bullet\bullet}, d_h, d_v)$ with a morphism of complexes called augmentation $\epsilon : A^\bullet \to I^{\bullet,0}$ with the following properties:*

- *If $A^p = 0$ then $I^{p,\bullet} = 0$ and $I^{p,q} = 0$ for $q < 0$.*
- *The complex $I^{p,\bullet}$ is an injective resolution of A^p.*

The double complex $I^{\bullet\bullet}$ is called Cartan-Eilenberg resolution and has several good properties namely:

- *There is a quasi-isomorphism of complexes induced by ϵ, i.e. $H^n(A^\bullet) = H^n(\mathrm{Tot}(I^{\bullet\bullet}))$.*
- *Any morphism $A^\bullet \to B^\bullet$ in $\mathcal{K}^+(\mathcal{A})$ could be extended to a morphism of any Cartan-Eilenberg resolution of $A^\bullet, B^\bullet$, and this extension is unique up to homotopy of double complex.*
- *If two morphism $A^\bullet \to B^\bullet$ are homotopic, then any two their extensions to morphisms of Cartan-Eilenberg resolutions are also homotopic.*

In other words, taking a Cartan-Eilenberg resolution determines a functor from $\mathcal{K}^+(\mathcal{A}) \to \mathcal{K}^+(\mathcal{I}_\mathcal{A})$.

Definition B.8. Given a left exact functor $F : \mathcal{A} \to \mathcal{B}$, let $A^\bullet \in \mathrm{Ch}^+(\mathcal{A})$ and take a Cartan-Eilenberg resolution $I^{\bullet\bullet}$ of $A^\bullet$, then we define the hyper-derived functor of F as:

$$\mathbb{R}^i F : \mathrm{Ch}^+(\mathcal{A}) \to \mathcal{B} \qquad \mathbb{R}^i F(A^\bullet) = H^i(\mathrm{Tot}(F(I^{\bullet\bullet})))$$

If $f : A^\bullet \to B^\bullet$ is a chain map that induces $\tilde{f} : I^{\bullet\bullet} \to J^{\bullet\bullet}$ a map between their C-E resolutions, than the $\mathbb{R}^i F : \mathbb{R}^i F(A^\bullet) \to \mathbb{R}^i F(B^\bullet)$ is just the map $H^i(\mathrm{Tot}\tilde{f})$.

Remark B.9. If F is again a left exact functor and if we consider an object $A \in \mathcal{A}$ as a complex $A^\bullet$ i.e. $A^n = 0$ if $n \neq 0$ and $A^0 = A$, then $\mathbb{R}^i F(A^\bullet) = R^i F(A)$, in this sense hyper-derived functors are a generalization of derived functors.

Example B.10 (Grothendieck Spectral Sequence)**.** Grothendieck introduced this spectral sequence associated to the composition of two functors. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories such that $\mathcal{A}, \mathcal{B}$ have enough injectives, suppose then that we have a left-exact functor $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{B}$, if G sends injective object of $\mathcal{A}$ to F-acyclic objects of $\mathcal{B}$, then there is a convergent first quadrant cohomological spectral sequence for each A in $\mathcal{A}$:

$${}^I E_2^{pq} = (R^p F)(R^q G)(A) \Rightarrow R^{p+q}(FG)(A)$$

To construct the spectral sequence chose an injective resolution of $A \to I^\bullet$, apply G to the complex and then using a C-E resolution of $G(I^\bullet)$ we can form the hyper derived functors $\mathbb{R}^n F(GI^\bullet)$. There are two spectral sequence converging to these hyper derived functors, the first:

$${}^I E_2^{pq} = H^p((R^q F)(GI^\bullet) \Rightarrow (\mathbb{R}^{p+q} F)(GI^\bullet)$$

but we have that $(R^qF)(G(I^s)) = 0$ for $q > 0$ since $G(I^s)$ are $F-$acyclic, so this spectral sequence collapse to yield:

$$(\mathbb{R}^n F)(GI^\bullet) \cong H^n(FG(I^\bullet)) = R^n(FG)(A)$$

Conversely if $p + q = n$ then the second spectral sequence is converging:

$$^{II}E_2^{pq} = (R^pF)H^q(GI^\bullet) = (R^pF)(R^qG)(A) \Rightarrow R^n(FG)(A)$$

Example B.11 (Leray Spectral Sequence)**.** Let $f : X \to Y$ be a continuous map of topological spaces. The direct image sheaf functor $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ preserves injectives and it is left exact since has a left adjoint $(f^{-1} \dashv f_*)$. If $\mathcal{F}$ is a sheaf of abelian groups on X and $\Gamma(Y, f_*\mathcal{F}) = \mathcal{F}(f^{-1}Y) = \mathcal{F}(X) = \Gamma(X, \mathcal{F})$, thus:

$$\begin{array}{ccc} \mathrm{Sh}(X) & \xrightarrow{f_*} & \mathrm{Sh}(Y) \\ & {}_{\Gamma}\searrow \quad \swarrow_{\Gamma} & \\ & \mathrm{Ab} & \end{array}$$

The Grothendieck spectral sequence in this case is called Leray spectral sequence and

$$E_2^{pq} = H^p(Y, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

Example B.12. [31] Using again the Grothendieck spectral sequence we can obtain relation between Čech cohomology and sheaf cohomology. Consider the composition of functor:

$$\mathrm{AbSh}(X) \xrightarrow{U} \mathrm{AbPSh}(X) \xrightarrow{\check{H}^0(\mathcal{U};\_)} \mathrm{Ab}, \qquad \Gamma : \mathrm{AbSh}(X) \to \mathrm{Ab}$$

Where U is the forgetful functor and $\check{H}^0$ is the 0-th Čech cohomology group with respect to a covering $\mathcal{U}$ of X. Since U has a left adjoint, namely sheafification, it sends injectives to injectives hence, applying the Grothendieck spectral sequence, for any cover $\mathcal{U}$ of X and any abelian sheaf $\mathcal{F}$ we have:

$$E_2^{pq} = \check{H}^p(\mathcal{U}, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

where $\mathcal{H}^q(\mathcal{F})$ is the presheaf $V \to H^q(V, \mathcal{F})$. We have hence functorial homomorphism:

$$\check{H}^p(\mathcal{U}; \mathcal{F}) \to H^p(X, \mathcal{F})$$

Note that if for all finite intersections of a covering the cohomology vanishes, the sequence collapse at the 2-page and we obtain an isomorphism of Čech cohomology for the covering to sheaf cohomology. This is useful for example if $\mathcal{F}$ is a quasi-coherent module over a Noetherian separated scheme X and each element of $\mathcal{U}$ is an open affine subscheme such that all finite intersection are again affine.
However more in general given a sheaf $\mathcal{F}$ of abelian group over a topological space X the Čech cohomology groups are defined as the inductive limit over all covering $\mathcal{U}$ in X: $\check{H}^\bullet(X, \mathcal{F}) = \varinjlim \check{H}^\bullet(\mathcal{U}, \mathcal{F})$, from the universal property of colimits we obtain an homomorphism:

$$\check{H}^n(X; \mathcal{F}) \to H^n(X, \mathcal{F})$$

which in general is not an isomorphism, however if X is paracompact and Haudorff is an isomorphism for all n. For the thesis we will need a similar result to relate Čech cohomology to sheaf cohomology over a site, but a more general comparison of Čech and sheaf cohomology in more modern term can be found in [6].

B.3 Derived Categories

Spectral sequences are very useful to compute hyper-derived functors but in general it can get very complicated to calculate the different terms. In this section we will introduce derived category and functors between them, which turn out to be a very convenient generalization of hyper-derived cohomology and allows us to show some important result.
In what follows $\mathcal{A}$ will be an abelian category, Ch($\mathcal{A}$) is the category of cochain complex over $\mathcal{A}$ and $\mathcal{K}(\mathcal{A})$ is the category of complex modulo cochain homotopy. We will use the following decorations $^{+,-,b}$ to denote the full subcategory on the complexes bounded above, below or bounded respectively.

Definition B.13. The derived category $D(\mathcal{A})$ of the abelian category $\mathcal{A}$ is such that exists a functor $Q_{\mathcal{A}} : \mathcal{K}(\mathcal{A}) \to D(\mathcal{A})$ with the following properties:

a $Q_{\mathcal{A}}(f)$ is an isomorphism for any quasi-isomorphism f.

b Any functor $F : \mathcal{K}(\mathcal{A}) \to \mathcal{D}$ transforming quasi-isomorphism to isomorphism can be uniquely factorized through $D(\mathcal{A})$, namely it exists a unique functor $G : D(\mathcal{A}) \to \mathcal{D}$ with $F = G \circ Q_{\mathcal{A}}$.

In other terms $D(\mathcal{A})$ is the homotopy category of $\mathcal{K}(\mathcal{A})$ with respect to quasi-isomorphism as weak-equivalence, this means that the derived category $D(\mathcal{A})$ is the localization of $\mathcal{K}(\mathcal{A})$ by quasi-isomorphisms.

Remark B.14. Thanks to this characterization, if we denote $\mathcal{S}_{\mathcal{A}}$ the class of quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$, to prove the existence of the derived category we have only to prove that the localization $\mathcal{K}(\mathcal{A})[\mathcal{S}_{\mathcal{A}}^{-1}]$ exists. To prove this it plays a key role the fact that $\mathcal{K}(\mathcal{A})$ is a triangulated category.

In general the derived category $D(\mathcal{A})$ is not abelian but it is still additive and since it is a localization of $\mathcal{K}(\mathcal{A})$ is still triangulated. In the application where we deal with bounded above or below complex we can use an even more intuitive characterization of the derived category and functors between them.

Definition B.15. Given $\mathcal{A}, \mathcal{B}$ abelian category and a left (right) exact functor $F : \mathcal{A} \to \mathcal{B}$. A class of object $\mathcal{R}$ is said to be adapted to F if it is stable under direct sums and satisfies the following two condition:

- F maps any acyclic complex (i.e. $H^i(\_) = 0$) from $\mathcal{K}^{+(-)}(\mathcal{R})$ into acyclic complex.
- Any object of $\mathcal{A}$ is a subobject (quotient) of an object from $\mathcal{R}$.

Proposition B.16. *Let $\mathcal{R} \subset \mathcal{A}$ a class of object adapted to a left exact functor $F : \mathcal{A} \to \mathcal{B}$ and let $\mathcal{S}_{\mathcal{R}}$ be the class of quasi-isomorphisms in $\mathcal{K}^+(\mathcal{R})$. Then the localization category $\mathcal{K}^+(\mathcal{R})[\mathcal{S}_{\mathcal{R}}]$ exists and the canonical functor:*

$$\mathcal{K}^+(\mathcal{R})[\mathcal{S}_{\mathcal{R}}^{-1}] \to D^+(\mathcal{A})$$

is an equivalence of categories.
A dual statement is true for a class of object adapted to a right exact functor G and replacing $\mathcal{K}^+, D^+$ with $\mathcal{K}^-, D^-$

Proof. To prove this proposition we have to verify that:

i) The localization $\mathcal{K}^+(\mathcal{R})[\mathcal{S}_{\mathcal{R}}^{-1}]$ exists.

ii) The category $\mathcal{K}^+(\mathcal{R})[\mathcal{S}_{\mathcal{R}}^{-1}]$ is a full subcategory of $\mathcal{K}^+(\mathcal{A})[\mathcal{S}_{\mathcal{A}}^{-1}] = D^+(\mathcal{A})$.

iii) For any object $C^\bullet \in D^+(\mathcal{A})$ there exist a complex $R^\bullet \in \mathcal{K}^+(\mathcal{R})$ and a quasi-isomorphism $t : C^\bullet \to R^\bullet$.

□

Remark B.17. In the specific case where we consider an abelian category $\mathcal{A}$ with enough injectives, then the first two step (i,ii) of the proof are clear. We have also seen that given a complex $A^\bullet$ then we can choose a Cartan Eilenberg resolution $I^{\bullet\bullet}$ of $A^\bullet$ and we have said that there is a quasi isomorphisms between $A^\bullet$ and $\mathrm{Tot}(I^{\bullet\bullet})$ so that the third step of the proof is true. In this special case is true even more, if we denote the class of injectives object $\mathcal{I}$, and if $s : I^\bullet \to K^\bullet$ is a quasi isomorphism between an object of $\mathcal{K}^+(\mathcal{I})$ and an object of $\mathcal{K}^+(\mathcal{A})$, then there is a morphism of complexes $t : K^\bullet \to I^\bullet$ such that $t \circ s$ is homotopic to id_I. So that we have an equivalence of categories:

$$\mathcal{K}^+(\mathcal{I}) \cong \mathcal{K}^+(\mathcal{I})[\mathcal{S}_{\mathcal{I}}] \cong D^+(\mathcal{A})$$

In this context we can see that the class $\mathcal{I}$ is adapted to any left exact functor F. Indeed we have only to verify that if $I^\bullet$ is an acyclic complex, then $F(I^\bullet)$ also is. Consider the zero morphism $0 : I^\bullet \to I^\bullet$, this is a quasi-isomorphism and from what we have said before is homotopic to $\mathrm{id}_{\mathcal{I}^\bullet}$. Hence the zero morphism of $F(\mathcal{I}^\bullet)$ is homotopic to $\mathrm{id}_{F(\mathcal{I}^\bullet)}$ so that indeed $F(I^\bullet)$ is acyclic. So is true that in general injectives (projective) object are a class adapted to every functor left (right) exact, but in some situation it could happen that we don't have enough injectives or projectives, this is why it is important to consider also other class of object adapted to a functor like flasque sheaves for the global section functor Γ or flat sheaves for the tensor product functor.

Now given $\mathcal{A}, \mathcal{B}$ abelian categories and an additive functor $F : \mathcal{A} \to \mathcal{B}$ in general applying element wise F we have an obvious functor between triangulated categories $F : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$. If for example F is exact, then it is easy to see that F sends quasi-isomorphism to quasi isomorphism and so factorize with the localization that defines the derived category and induce a functor $F : D(\mathcal{A}) \to D(\mathcal{B})$. However in general F does not send q.i. to q.i. so we can ask if there is a derived functor from $D(\mathcal{A})$ to $D(\mathcal{B})$ which is an extension of F or at least that is close to F. The main important fact in the following part is that we can derive every additive functor $F : \mathcal{A} \to \mathcal{B}$, but we have to expect that if we derive a left (right) exact functor then we have to obtain a sort of generalization of the hyper-derived functor $\mathbb{R}(F)$ ($\mathbb{L}(F)$).

Definition B.18. Let $\mathcal{A}$ be an abelian category, $\mathcal{K}^\dagger(\mathcal{A})$ be a triangulated subcategory of $\mathcal{K}(\mathcal{A})$ and let $\mathcal{S}_\dagger$ be the class of quasi-isomorphisms in $\mathcal{K}^\dagger(\mathcal{A})$. We say that $\mathcal{K}^\dagger(\mathcal{A})$ is a localizing subcategory of $\mathcal{K}(\mathcal{A})$ if the natural functor:

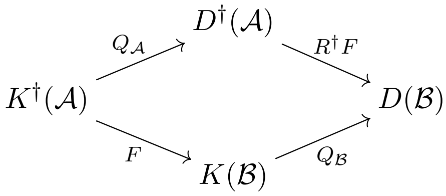
$$\mathcal{K}^\dagger(\mathcal{A})[\mathcal{S}_\dagger^{-1}] \to D(\mathcal{A}) = \mathcal{K}(\mathcal{A})[\mathcal{S}^{-1}]$$

is fully faithful, and in this case we write $D^\dagger(\mathcal{A}) = \mathcal{K}^\dagger(\mathcal{A})[\mathcal{S}_\dagger^{-1}]$.

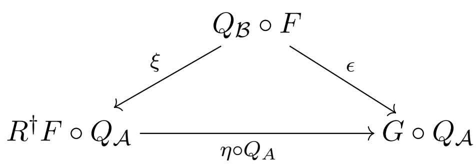
Example B.19. The subcategory of bounded complex $\mathcal{K}^+(\mathcal{A}), \mathcal{K}^-(\mathcal{A}), \mathcal{K}^b(\mathcal{A})$ are localizing subcategory, also for every class $\mathcal{R}$ of adapted object to a left exact functor, we have that $\mathcal{K}^+(\mathcal{R})$ is a localizing subcategory, and dually if F is right exact then $\mathcal{K}^-(\mathcal{R})$ is a localizing subcategory.

Definition B.20. Let $F : \mathcal{K}^\dagger(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$ be morphism of triangulated category with $\mathcal{K}^\dagger(\mathcal{A})$ a localizing subcategory of $\mathcal{K}(\mathcal{A})$. Let $Q_\mathcal{A} : \mathcal{K}^\dagger(\mathcal{A}) \to D^\dagger(\mathcal{A})$ and $Q_\mathcal{B} : \mathcal{K}(\mathcal{B}) \to D(\mathcal{B})$ denote the localization functors. The right derived functor of F is a pair of a functor of triangulated categories and a natural transformation $(R^\dagger F, \xi)$:

$$R^\dagger F : D^\dagger(\mathcal{A}) \to D(\mathcal{B}) \qquad\qquad \xi : Q_\mathcal{B} \circ F \to R^\dagger F \circ Q_\mathcal{A}$$



satisfying the following universal property, if $G : D^\dagger(\mathcal{A}) \to D(\mathcal{B})$ is a functor of triangulated categories with a natural transformation $\epsilon : Q_\mathcal{B} \circ F \to G \circ Q_\mathcal{A}$ then there exists a unique natural transformation $\eta : R^\dagger F \to G$ such that the following diagram is commutative:



Dually the left derived functor of F is a pair $(L^\dagger F, \xi)$:

$$L^\dagger F : D^\dagger(\mathcal{A}) \to D(\mathcal{B})$$
$$\xi : L^\dagger F \circ Q_\mathcal{A} \to Q_\mathcal{B} \circ F$$

and the dual universal property ensures the existence of natural transformations $\eta : G \to L^\dagger F$.

Thanks to the universal property if the derived functor exists then has to be unique.

Theorem B.21. *Let $F : \mathcal{K}^\dagger(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$ as before. Suppose that there is a triangulated subcategory $\mathcal{C} \subset \mathcal{K}^\dagger(\mathcal{A})$ such that:*

(A) Every object of $K^\dagger(\mathcal{A})$ admits a quasi-isomorphism into an object of $\mathcal{C}$.

(B) If $C^\bullet \in \mathcal{C}$ is acyclic, then $F(C^\bullet)$ is also acyclic.

Then F has a right derived functor $(R^\dagger F, \xi)$. Furthermore for any $C^\bullet \in \mathcal{C}$ the map:

$$\xi(C^\bullet) : Q_\mathcal{B} \circ F(C^\bullet) \xrightarrow{\cong} R^\dagger F \circ Q_\mathcal{A}(C^\bullet)$$

is an isomorphism in $D(\mathcal{B})$.

Proof. We will construct in this setting the derived functor, the full proof can be found in [19]. First of all the restriction of F to $\mathcal{C}$ takes q.i. to q.i. indeed if $s : C_1^\bullet \to C_2^\bullet$ is a q.i., let $C_3^\bullet$ be the third side of a triangle built on s. Then $C_3^\bullet$ is acyclic so that $F(C_3^\bullet)$ also is for the property (B), and $F(s)$ is hence a q.i. This means that if we denote $\mathcal{S}_\mathcal{C}$ the class of q.i. in $\mathcal{C}$ then F factorizes with the localization to give a functor:

$$\bar{F} : \mathcal{C}[\mathcal{S}_\mathcal{C}^{-1}] \to D(\mathcal{B})$$

with the property that $\bar{F}Q_{\mathcal{C}} = Q_{\mathcal{B}}F$ on $\mathcal{C}$. Secondly using the property (A) we can see that the natural functor:

$$T : \mathcal{C}[\mathcal{S}_{\mathcal{C}}^{-1}] \to D^{\dagger}(\mathcal{A}) = \mathcal{K}^{\dagger}(\mathcal{A})[\mathcal{S}_{\dagger}^{-1}]$$

is an equivalence of categories. Let U be a quasi-inverse of T, then define:

$$R^{\dagger}F = \bar{F} \circ U$$

□

This theorem is very general and allows us to derive any functor every time we have a triangulated subcategory $\mathcal{C} \subset \mathcal{K}^{\dagger}(\mathcal{A})$ as in the theorem.

Example B.22.

- If F is left exact and there is a class of object $\mathcal{R}$ adapted to F. Then the localizing subcategory $\mathcal{K}^{+}(\mathcal{R}) \subset \mathcal{K}^{+}(\mathcal{A})$ has exactly the property required in the theorem, hence there exists the right derived functor $R^{+}F$.
- If $\mathcal{A}$ has enough injectives then since the class of injectives objects $\mathcal{I}$ is adapted to every left exact functor F, then every such functor has a derived functor $R^{+}F$.
- Dually if F is right exact and there is a class of object $\mathcal{R}$ adapted to F. Then the localizing subcategory $\mathcal{K}^{-}(\mathcal{R}) \subset \mathcal{K}^{-}(\mathcal{A})$ has exactly the property required in the theorem, hence there exists the left derived functor $L^{-}F$.
- If $\mathcal{A}$ has enough projectives then since the class of projectives objects $\mathcal{I}$ is adapted to every right exact functor F, then every such functor has a left derived functor $L^{-}F$.

It is important to see that we don't have to ask that F are left or right exact, but if $\mathcal{A}$ has enough injectives then we can always define $R^{+}F$, whereas if $\mathcal{A}$ has enough projectives object then exists $L^{-}F$. However it is easy to see that if F is left (right) exact we have the following.

Corollary B.23. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories:*

- *If $\mathcal{A}$ has enough injectives, the hyper-derived functor $\mathbb{R}^{i}F(A^{\bullet})$ are the cohomology of $RF(A^{\bullet})$, i.e. $\mathbb{R}^{i}F(A^{\bullet}) = H^{i}R^{+}(F(I^{\bullet}))$.*
- *If $\mathcal{A}$ has enough projectives, the hyper derived functors $\mathbb{L}_{i}F(A^{\bullet})$ are the cohomology of $LF(A^{\bullet})$, i.e. $\mathbb{L}_{i}F(A^{\bullet}) = H^{-i}L^{-}(F(I^{\bullet}))$.*

Recall that to define any hyper-derived functor of F it is not necessary for F to be left or right exact. However the assumption that F be left (right) exact is needed to ensure that, if A is a complex concentrated in the zero degree, the $\mathbb{R}^{i}F(A)$ ($\mathbb{L}_{i}F(A)$) are the ordinary derived functor $R^{i}F(A)$ ($L_{i}F(A)$). This means that in the case of a left (right) exact functor, then the derived functor in the sense of the derived categories is strictly related to the ordinary derived functor.

To see that in some cases there are many advantage to work with derived categories instead of spectral sequence we can prove again the Grothendieck spectral sequence for a composition of functor.

Definition B.24. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories with two additive left exact functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$. Let $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{R}_{\mathcal{B}}$ be class of adapted object to F and G respectively. Assume that $F(\mathcal{R}_{\mathcal{A}}) \subseteq \mathcal{R}_{\mathcal{B}}$. Then the right derived functors $RF, RG, R(G \circ F) : D^+(\bullet) \to D^+(\bullet)$ exist and the natural morphism of functor $R(G \circ F) \to RG \circ RF$ is an isomorphism.

Proof. The condition of the theorem ensures the existence of the right derived functors, in particular $\mathcal{R}_{\mathcal{A}}$ is adapted either to F and to $G \circ F$. The morphism $\eta : R(G \circ F) \to RG \circ RF$ is defined by the universal property. Since we know that any object of $D^+(\mathcal{A})$ is isomorphic to an object of $\mathcal{K}^+(\mathcal{R})$ we must prove that if $K^\bullet \in \mathcal{K}^+(\mathcal{R})$ then $\eta(K^\bullet) : R(G \circ F) \to RG \circ RF$ is an isomorphism, but this is trivially true from how we construct the derived category. □

Conceptually, the composition of functors $R(G \circ F) = RG \circ RF$ is much more simpler than the original spectral sequence. Of course we can recover the old spectral sequence from this proposition by taking cohomology and using the spectral sequence of a double complex. In general the derived categories provide some easy way to prove important formulas and relations in various contexts. The following are some examples in the category of sheaves of abelian groups over a topological space, especially in [19](Ch.II.5) we can find other relationships among the derived functors in the context of schemes and quasi-coherent modules.

Example B.25.

- For $f : X \to Y$ continuous map between topological space, consider $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^-(\text{SAb}_Y)$ where SAb_Y is the category of sheaf of abelian group on Y. We can prove that:

$$f^{-1}(\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet) = f^{-1}\mathcal{F}^\bullet \overset{L}{\otimes} f^{-1}\mathcal{G}^\bullet$$

  To prove this we can replace $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ with their flat resolution and the identity follows from the general fact that $f^{-1}(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet) = f^{-1}\mathcal{F}^\bullet \otimes f^{-1}\mathcal{G}^\bullet$ because the same is true for the stalk $(\mathcal{F} \otimes \mathcal{G})_y = \mathcal{F}_y \otimes \mathcal{G}_y$.

- For $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^-(\text{SAb}_X)$ and $\mathcal{H}^\bullet \in D^+(\text{SAb}_X)$ we have:

$$R\text{Hom}(\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet, \mathcal{H}^\bullet) = R\text{Hom}(\mathcal{F}^\bullet, R\text{Hom}(\mathcal{G}^\bullet, \mathcal{H}^\bullet))$$

  Indeed we have only to prove the corresponding identity for sheaf of abelian group and then replace $\mathcal{G}$ with a flat resolution and $\mathcal{H}$ with an injective resolution.

B.4 Simplicial Methods in Homological Algebra

Definition B.26. Let Δ be the category whose objects are the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for integers $n \geq 0$, and whose morphisms are the non-decreasing monotone functions. If $\mathcal{A}$ is any category, a simplicial object A in $\mathcal{A}$ is a contravariant functor from Δ to $\mathcal{A}$, i.e. $A : \Delta^{op} \to \mathcal{A}$. We will write A_n for $A([n])$. Similarly, a cosimplicial object C in $\mathcal{A}$ is a covariant functor $C : \Delta \to \mathcal{A}$, and we write $C^n = C([n])$. A morphism of simplicial objects is just a natural transformation, and the category $\mathcal{SA}$ of all simplicial object in $\mathcal{A}$ is just the functor category $\mathcal{A}^{\Delta^{op}}$.

Definition B.27. Let A be a simplicial object in an abelian category $\mathcal{A}$. The unnormalized chain complex $C = C(A)$ has $C_n = A_n$, and its boundary morphism $d : C_n \to C_{n-1}$ is the alternating sum of the face operators $\partial_i : C_n \to C_{n-1}$:

$$d = \partial_0 - \partial^1 + \cdots + (-1)^n \partial^n$$

The normalized chain complex $N(A)$ is the chain subcomplex of $C(A)$ where:

$$N_n(A) = \bigcap_{i=0}^{n-1} \ker(\partial_i : A_n \to A_{n-1})$$

Theorem B.28. *[33] For any abelian category $\mathcal{A}$ the normalized chain complex functor N is an equivalence of categories between $\mathcal{S}\mathcal{A}$ and $Ch_{\geq 0}(\mathcal{A})$:*

$$\mathcal{S}\mathcal{A} \xrightarrow{N} Ch_{\geq 0}(\mathcal{A})$$

Under this correspondence, simplicial homotopy corresponds to homology $\pi_*(A) \cong H_*(N(A))$ and simplicially homotopic morphism correspond to chain homotopic maps.

For our purpose we need some generalizations of this theorem, in particular we want to generalize the situation from simplicial abelian groups to abelian groups spectra. To understand better what we want to obtain we need to give some definitions and basic notions about simplicial sets and spectra.
Let S^n be the standard simplicial n-sphere, defined as an n-simplex with its boundary collapsed to a point for $n \geq 1$, and as two points for $n = 0$. Given pointed simplicial set X, Y let $Map_*(X, Y)$ be the function complex of based maps $X \to Y$. For X a pointed simplicial set, let ΣX be $S^1 \wedge X$, and ΩX be $Map_*(S^1, X)$. It is well-known that Σ is left adjoint to Ω.

Definition B.29. A spectrum E is a sequence of pointed simplicial sets:

$$E = \{E^0, E^1, \ldots, \}$$

together with structure maps $S^1 \wedge E^k \to E^{k+1}$ for $k \geq 0$. A map of spectra $f : E \to F$ is a sequence of maps $f^k : E^k \to F^k$ compatible with the structure maps:

$$\begin{array}{ccc} S^1 \wedge E^k & \longrightarrow & E^{k+1} \\ \downarrow & & \downarrow \\ S^1 \wedge F^k & \longrightarrow & F^{k+1} \end{array}$$

we will denote Spt for the resulting category of spectra.

Proposition B.30. *There is a pair of adjoint functors:*

$$Spt \underset{\Sigma^\infty}{\overset{R}{\rightleftarrows}} \mathcal{S}_*$$

where if X is a simplicial set and E is a spectrum,$\Sigma^\infty X = \{n \to S^n \wedge X\}$ is the suspension with right adjoint $R(E) = E^0$ the zeroth space.

Definition B.31. Let E be a spectrum. The stable homotopy groups of E are defined as:

$$\pi_q E = \varinjlim_k \pi_{q+k} E^k$$

where the colimit is over the maps $\pi_{q+k}E^k \to \pi_{q+k}\Omega E^{k+1} \cong \pi_{q+k+1}E^{k+1}$ for $k > -q$. Homotopy groups define a functor from spectra to abelian groups.

Definition B.32. A map of spectra $f : E \to F$ is a stable equivalence if it induces an isomorphism on stable homotopy groups. Similarly with what we did for the derived category of an abelian category, it makes sense to invert in Spt all stable equivalences, and the resulting category is called the stable homotopy category $HoSpt$.

Remark B.33. [13] There is a close connection between chain complexes and spectra. Indeed if we want to represent an arbitrary chain complex C by means of chain complexes concentrated in non-negative degree $C^0, C^1, \ldots$, we can consider the following sequence:

$$\begin{array}{ccccc}
\cdots & \cdots & \cdots & \cdots & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
C_2^0 & C_1^1 & C_0^2 & C_{-1}^3 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
C_1^0 & C_0^1 & C_{-1}^2 & C_{-2}^3 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
C_0^0 & C_{-1}^1 & C_{-2}^2 & C_{-3}^3 & \cdots
\end{array}$$

together with isomorphisms $C_i^0 \cong C_i$ for $i \geq 0$, $C_i^{-i} = C_i$ for $i < 0$ and $C_j^i \cong C_{j+1}^{i+1}$ for arbitrary i, j. Homology in arbitrary degree is given by:

$$H_j(C) \cong \varinjlim_n H_{n+j}(C^n)$$

Notice that if we let $\mathbb{Z}[1]$ be the chain complex concentrated in degree 1 with a single $\mathbb{Z}$, then the isomorphism $C_j^i \cong C_{j+1}^{i+1}$ can be expressed as a map $\mathbb{Z}[1] \otimes C^i \to C^{i+1}$ which is an isomorphism in positive degrees, where the tensor product is defined as $(C \otimes D)_n = \oplus_{p+q=n} C_p \otimes D_q$.

In particular if we replace in the definition of spectra the category of simplicial sets $\mathcal{S}$, the smash product $\wedge$ and the circle S^1 with:

- the category of simplicial abelian group $\mathcal{S}Ab$, with degree-wise tensor, and $\tilde{\mathbb{Z}}[S^1] = \mathbb{Z}[S^1]/\mathbb{Z}[*]$ the simplicial abelian group generated by S^1.
- the category of chain complexes concentrated in non-negative degrees $Ch^{\geq 0}$, with tensor of chain complexes and $N(\mathbb{Z}[S^1]) = \mathbb{Z}[1]$.
- the category of chain complexes Ch, with tensor of chain complexes and $\mathbb{Z}[1]$.

then we obtain the definition of the categories $Spt(\mathcal{S}Ab), Spt(Ch^{\geq 0})$ and $Spt(Ch)$.

Theorem B.34. *The relation of Spt to Ch is given by a sequence of adjunction. The first couple of adjoint is induced by the usual free-forgetful functors, the second is a consequence of the Dold-Kan correspodence, the third couple is induced by the natural inclusion and truncation and the last one is just the same couple as in (B.30):*

$$Spt \underset{\tilde{\mathbb{Z}}[\_]}{\rightleftarrows} Spt(\mathcal{S}Ab) \overset{N}{\rightleftarrows} Spt(Ch^{\geq 0}) \underset{\tau}{\overset{i}{\rightleftarrows}} Spt(Ch) \overset{R}{\rightleftarrows} Ch$$

The main important point is the fact that all the maps to the left of $Spt(\mathcal{S}Ab)$ are inducing equivalences on the associated homotopy category. Up to homotopy, looping and delooping an abelian group spectrum corresponds to shifting the gradings of the chain complex up or down. The homotopy groups of the abelian group spectrum are isomorphic to the homology groups of the corresponding chain complex.

Remark B.35. Notice that from the previous theorem we obtain that the homotopy category obtained inverting stable equivalences of abelian group spectra is equivalent to the category formed by inverting quasi-isomorphism of chain complexes, i.e. the derived category of abelian groups. This means that the homotopy category of spectra of simplicial abelian groups is a triangulated category and we can generalize the common construction of homological algebra for chain complexes with homotopy theory for spectra. In particular following ([32],5.32), let F be a functor from a small category $\mathcal{C}$ to the category of fibrant spectra (i.e. spectra E such that the structure map induce equivalences $E^n \to \Omega E^{n+1}$). For each non negative n one has a diagram of n-th spaces of the spectra F_n. For each object C of $\mathcal{C}$ we know that $F_n(C)$ is a fibrant simplicial set and following ([5],XI) we can form the homotopy limit of this diagram $holimF_n$, one has to see that there exist weak equivalences $holimF_n \to \Omega holimF_{n+1}$ so that we obtain a fibrant spectrum $holimF$. Now if we consider the correspondence from $Spt(\mathcal{S}Ab)$ to Ch then $holimF$ correspond to the evaluation at F of the total right derived functor of the limit along C, i.e. $\mathbb{R}\varprojlim_{C\in\mathcal{C}} F(C)$. Similarly there is a generalization of the total functor of homological algebra. In particular dually to the theorem that the category of simplicial objects in an abelian category is equivalent to the category of non-negative chain complexes, one has that the category of cosimplicial objects in an abelian category is equivalent to that of non-negative cochain complexes. The category of cosimplicial abelian group spectra is equivalent to a category of bicomplexes. The functor Tot ([32]5.24) from cosimplicial spectrum to spectrum corresponds to a functor sending a bicomplex to its total complex, and there is a spectral sequence which in the case of bicomplex is just the usual spectral sequence for the homology.

In fact what we really use in the sketch of the modern interpretation of Grothendieck's proof is a sheaf analogue of the correspondence described above.

Theorem B.36. *[6] Let X be a site:*

- *Let $Sh(X, \mathcal{S}Ab)$ be the category of simplicial abelian sheaves, and let $Sh(X, Ch^{\geq 0})$ be the category of sheaves on X with values in non-negatively graded chain complexes of abelian groups. The normalized chain complex extends object-wise to a functor:*

$$Sh(X, \mathcal{S}Ab) \xrightarrow{\cong} Sh(X, Ch^{\geq 0})$$

  which is an equivalence of categories. Moreover, both these categories are naturally categories with weak equivalences: the weak equivalences in $Sh(X, \mathcal{S}Ab)$ are the stalk wise weak equivalences of simplicial sets and the weak equivalences in $Sh(X, Ch^{\geq 0})$ are the quasi-isomorphisms. The normalized chain complex functor preserves these weak equivalences.

- *The homotopy category of sheaves of abelian group spectra is equivalent via the normalization functor to the homotopy category of complexes of abelian sheaves. In particular corresponding to any abelian sheaf F and any integer q there is a stable Eilenberg-Mac Lane sheaf of spectra $K(F, q)$, which by definition is the sheaf of abelian group spectra whose normalization consists of F concentrated in degree $-q$.*

C Grothendieck Topos

In this section we will give some general result on the theory of sites and Grothendieck's topos, which arise and can be used in different branches of mathematics. There are several equivalent way of defining them, in our case a site will be a category with a Grothendieck (pre)-topology and it will play the role of a topological space; in this context a topos will be the category of sheaves over a site. We will focus on Grothendieck's topos, because they allows us to develop a sheaf theory and a cohomology theory for sheaves not only over some topological spaces as usual, but also over a category with some additional data. The main reference for what will follow is [28], [29] [21] and [31]. From now on the category are assumed to have all finite limits.

C.1 Grothendieck topologies

Definition C.1. Let $\mathcal{C}$ be a category, a covering sieve S_C over $C \in \text{Ob}(\mathcal{C})$ is a subfunctor of the functor $\text{Hom}_{\mathcal{C}}(\_, C)$. This means that if $g : C' \to C$ is in S_C and if $f : C'' \to C'$ is any other arrow, then $S(f)(g) = g \circ f$ is in S_C.

If S_C is a sieve over C and $f : D \to C$ is a morphism, then left composition by f gives a sieve on D denoted f^*S_C and called the pullback of S_C along f:

$$f^*S_C(E) = S_C \times_{\text{Hom}(E,C)} \text{Hom}(E, D) = \{g : E \to D \text{ such that } f \circ g \in S_C(E)\} \qquad \forall E \in \text{Ob}(\mathcal{C})$$

Definition C.2. A Grothendieck topology $\mathcal{T}$ on a category $\mathcal{C}$ is a collection, for each object $C \in \text{Ob}(\mathcal{C})$ of distinguished sieve on C, denoted $\mathcal{T}(C) \subset \text{Sieves}(C)$ called covering sieves of C. This selection is subject to certain axioms:

- For any object C in $\mathcal{C}$, then $\text{Hom}(\_, C)$ is a covering sieve on C.
- If S_C is a covering sieve on C, and $f : D \to C$ is a morphism, then the pullback f^*S_C is a covering sieve on D.
- Let S_C be a covering sieve on C, and let T_C be any sieve on C. Suppose that for each object D of $\mathcal{C}$ and each arrow $f : D \to C$ in $S_C(D)$, the pullback sieve f^*T_C is a covering sieve on D. Then T_C is a covering sieve on C.

Definition C.3. A category $\mathcal{C}$ together with a Grothendieck topology is called a site. If $\mathcal{C}$ is a site we will denote $\text{Cat}(\mathcal{T})$ the underlying category.

Now we have to define a sheaf over a site. In general the notion of a presheaf does not require a topology on a category, but just like in classical topology, the notion of a sheaf should allow gluing and so we need a topology to define them.

Definition C.4.

- A presheaf of sets (groups, rings) on a site $\mathcal{C}$ is a functor $\mathcal{F}$:

$$\mathcal{F} : \text{Cat}(\mathcal{C})^{op} \to \text{Set} \ \ (\text{Grp}, \text{Ring})$$

- A sheaf on a site $\mathcal{C}$ is a presheaf $\mathcal{F}$ such that for all object $C \in \mathrm{Obj}(\mathcal{C})$ and all covering sieves S_C on C, the natural map φ induced by the inclusion of S into $\mathrm{Hom}(\_, C)$ is a bijection:
$$\mathrm{Hom}(\mathrm{Hom}(\_, C), \mathcal{F}) \xrightarrow{\varphi} \mathrm{Hom}(S_C, \mathcal{F})$$
- A morphism of presheaves is just a morphism of functor, namely a natural transformation. A morphisms of sheaves is a morphism of presheaves.

Remark C.5. The sheaf condition over $\mathcal{F}$ is equivalent to ask that for each natural transformation: $x : S_C \to \mathcal{F}$ we can find a unique natural transformation $a : \mathrm{Hom}(\_, C) \to \mathcal{F}$ such that the following diagram commute:

$$\begin{array}{ccc} S_C & \xrightarrow{x} & \mathcal{F} \\ \big\downarrow & \nearrow_{\exists! a} & \\ \mathrm{Hom}(\_, C) & & \end{array}$$

This means that for every object $C \in \mathrm{Obj}(\mathcal{C})$ for every covering sieve S_C, and for every matching family of elements of $\mathcal{F}$ i.e $\{x_f \in \mathcal{F}(\mathrm{dom}_f)\}_{f \in S_C}$ such that for all $f \in S_C$ and all morphism $g : D \to dom_f$, then $\mathcal{F}(g)(x_f) = x_{f \circ g}$; it exists an element $x \in \mathcal{F}(C)$ such that $\mathcal{F}(f)x = x_f$ for all $f \in S_C$.

Since with this definition it is sometimes difficult to define a topology and to describe the sheaves, then it is more useful the following description:

Definition C.6. Let $\mathcal{C}$ be a category, a Grothendieck pre-topology on $\mathcal{C}$ is an assignment to each object U of $\mathcal{C}$ of a collection of families $\{U_i \to U\}_{i \in I}$ of morphisms, called covering families satisfying the following axiom:

- Every family consisting of a single isomorphism $\{V \xrightarrow{\cong} U\}$ is a covering family for U.
- For any covering $\{U_i \to U\}_{i \in I}$ and any morphism $V \to U$ in $\mathcal{C}$, the fiber products $U_i \times_U V$ exist, and $\{U_i \times_U V \to V\}_{i \in I}$ is a covering of V.
- If $\{U_i \to U\}_{i \in I}$ is a covering of U, and if for each $i \in I, \{V_{ij} \to U_i\}_{j \in J_i}$ is a covering of U_i, then the family $\{V_{ij} \to U\}_{i \in I, j \in J_i}$ is a covering of U.

On this pre-topology we can define a sheaf as a presheaf $\mathcal{F}$ such that the first arrow of the following diagram is an equalizer, for every object U and for every covering $\{U_i \to U\}$:

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i_0, i_1) \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \tag{C.6.1}$$

this means that given a covering $\{U_i \xrightarrow{i} U\}_{i \in I}$ and a set of elements $a_i \in \mathcal{F}(U_i)$ such that $\mathcal{F}(p_0)(a_{i_0}) = \mathcal{F}(p_1)(a_{i_1}) \in \mathcal{F}(U_{i_0} \times_U U_{i_1})$ then there is a unique section $a \in \mathcal{F}(U)$ such that $\mathcal{F}(i)(a) = a_i$.

Remark C.7. For any pre-topology, the collection of all sieves which contain a covering family from the pre-topology is defined as the Grothendieck topology generated by the pre-topology. Very different pre-topologies can define the same topology. The point is that sheaf on a pre-topology are also sheaf on the induced topology. In fact is it true even more, if $\mathcal{T}, \mathcal{T}'$ are two pre-topologies on $\mathcal{C}$ we say that $\mathcal{T} \prec \mathcal{T}'$ if every sieves belonging to $\mathcal{T}$ also belongs to $\mathcal{T}'$, in this way we can describe the family of pre-topologies as a pre-order category and two pre-topologies are equivalent if and only if they have the same sieves. In this situation it's not difficult to prove that if $\mathcal{T} \prec \mathcal{T}'$ then every sheaf in $\mathcal{T}'$ is also a sheaf on $\mathcal{T}$, so that two equivalent pre-topologies induce two topologies which have the same sheaves. This means that sheaf theory only depends on the topology and not on the pre-topology.

Example C.8.
- (Zariski Site) Let X be a topological space and consider X_{Zar} the category where object are the open subset of X and morphisms are just the inclusions. Then given an open $U \subset X$, the topological covering $\{U_i\}_{i\in I}$ such that $\cup_i U_i = U$, induce a covering $(U_i \to U)_{i\in I}$ on the related category. The family of all coverings over the open subset of X form a Grothendieck pre-topology on X_{Zar}, where for different open subsets V, V' of U we have the fibered product. $V \times_U V' = V \cap V'$. In this situation then the definition of sheaves over X_{Zar} is equivalent to the usual definition over the topological space X.
- (Etale site) Let X be a noetherian scheme and let Et/X be the category of étale morphism $U \to X$ with morphisms given by X-morphisms. Then a covering family is given by a family of étale morphism $\{f_i : U_i \to U\}$ such that the U_i cover U, i.e. $U = \bigcup_i \varphi_i(U_i)$. With this covering we obtain the so called étale site X_{et}.

Let $\mathcal{F}$ be a presheaf of sets on a site $\mathcal{C}$. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. We can always consider the equalizer of the diagram (C.6.1) and we will denote it with $\mathcal{F}(\mathcal{U})$, note that from the definition we have always a canonical map $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$. If we have a morphism of coverings $\mathcal{V} \to \mathcal{U}$ we will obtain as well a natural map $\mathcal{F}(\mathcal{V}) \to \mathcal{F}(\mathcal{U})$ compatible with the map $\mathcal{F}(V) \to \mathcal{F}(U)$. Let $\mathcal{J}_U$ be the category of all coverings of U, which has as objects the coverings of U in $\mathcal{C}$, and as morphism the refinements. We can define a presheaf:

$$\begin{aligned} \mathcal{F}^+ :\ & \mathcal{C} \longrightarrow \text{Sets} \\ & U \longrightarrow \varinjlim_{\mathcal{J}_U^{op}} \mathcal{F}(\mathcal{U}) \end{aligned} \tag{C.8.1}$$

It is not difficult to see that the diagram $\mathcal{J}_U^{op} \to$ Set is filtered, hence the colimit $\mathcal{F}^+(U)$ can be described as usual explicitly. With this method we have defined a functor $\mathcal{F} \to (\mathcal{F} \to \mathcal{F}^+)$.

Proposition C.9. *Consider the canonical map $\theta : \mathcal{F} \to \mathcal{F}^+$. For every object U of $\mathcal{C}$, and every section $s \in \mathcal{F}^+(U)$ there exists a covering $\{U_i \to U\}$ such that $s_{|U_i}$ is in the image of $\theta : \mathcal{F}(U_i) \to \mathcal{F}^+(U_i)$.*

Definition C.10. A presheaf of sets $\mathcal{F}$ on a site $\mathcal{C}$ is separated if, for all coverings of $\{U_i \to U\}$ the map $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective.

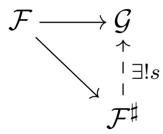
Proposition C.11. *Let $\mathcal{F}$ be a presheaf, then $\mathcal{F}^+$ is separated. If $\mathcal{F}$ is separated then $\mathcal{F}^+$ is a sheaf and the canonical map θ is injective. If $\mathcal{F}$ is a sheaf then the map θ in an isomorphism.*

Proposition C.12. *Let $\mathcal{C}$ be a site and consider the natural inclusion functor:*

$$i : Shv(\mathcal{C}) \to Pshv(\mathcal{C})$$

then there exists an exact functor $(\_)^{++} = (\_)^\sharp : Pshv(\mathcal{C}) \to Shv(\mathcal{C})$ called sheafification functor such that it is the left adjoint to the inclusion functor $(\_)^\sharp \dashv i$. In particular given any presheaf $\mathcal{F}$

on $\mathcal{C}$ then we will call $\mathcal{F}^{++} = \mathcal{F}^\sharp$ the sheaf associated to $\mathcal{F}$ and by adjunction we have a canonical map $\mathcal{F} \to \mathcal{F}^\sharp$ such that for any map $\mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ a sheaf on $\mathcal{C}$ then there exists a unique map $s : \mathcal{F}^\sharp \to \mathcal{G}$ making the following diagram commutative:



Proposition C.13. *Let $\mathcal{F} : \mathcal{I} \to Shv(\mathcal{C})$ be a diagram. Then $\varprojlim_{\mathcal{I}} \mathcal{F}$ exists and is equal to the limit in the category of presheaves, whereas $\varinjlim_{\mathcal{I}} \mathcal{F}$ exists and is the sheafification of the colimit in the category of presheaves.*

An important properties of presheaves over a topological spaces is functoriality i.e. given a morphisms of spaces $X \xrightarrow{f} Y$ we can define the inverse and direct image functors. Now we want to do the same in a general contest where our spaces are site.

Definition C.14. Let $\mathcal{C}, \mathcal{D}$ be two sites, a functor $\bar{f} : \mathcal{C} \to \mathcal{D}$ is continuous if, for every covering family $\{U_i \to U\}$ of any object U of $\mathcal{C}$ we have the following:

- the family $\{\bar{f}(U_i) \to \bar{f}(U)\}$ is a covering family of $\bar{f}(U)$.
- If $V \to U$ is a morphism, then $\bar{f}(V \times_U U_i) \cong \bar{f}(V) \times_{\bar{f}(U)} \bar{f}(U_i)$ is an isomorphism.

Proposition C.15. *Given a functor $\bar{f} : \mathcal{C} \to \mathcal{D}$ between sites, if $\mathcal{F}$ is a presheaf on $\mathcal{D}$ then we can define a presheaf $f_*\mathcal{F}$ on $\mathcal{C}$ simply as:*

$$f_*\mathcal{F}(V) = \mathcal{F}(\bar{f}(V)) \qquad \textit{for all } V \in \textit{Obj}(\mathcal{C})$$

If $\bar{f}$ is a continuous functor then for every sheaf $\mathcal{F}$ on $\mathcal{D}$ then $f_*\mathcal{F}$ is a sheaf on $\mathcal{C}$. We have then a functor:

$$f_* : \textit{PSh}(\mathcal{D}) \to \textit{PSh}(\mathcal{C})$$

such that commutes with all limits.

Proof. Of course $f_*\mathcal{F}$ is a presheaf. If $\{V_i \to V\}$ is a covering in $\mathcal{C}$ then $\{\bar{f}(V_i) \to \bar{f}(V)\}$ is a covering in $\mathcal{D}$. From the isomorphism $\bar{f}(V \times_U U_i) \cong \bar{f}(V) \times_{\bar{f}(U)} \bar{f}(U_i)$ we see that the sheaf condition for the sheaf $\mathcal{F}$ and the covering $\{\bar{f}(V_i) \to \bar{f}(V)\}$ is the same as the sheaf condition for the presheaf $f_*\mathcal{F}$ and $\{V_i \to V\}$. The functor f_* commutes with limits because in general, for each object $U \in \mathrm{Obj}(\mathcal{C})$ we have that the functor:

$$\Gamma(\_, U) : \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sets} \qquad \mathcal{F} \to \mathcal{F}(U)$$

commutes with limits and colimits. □

Now starting from a continuous functor between sites $\bar{f} : \mathcal{C} \to \mathcal{D}$ by restriction we have obtained a functor $f_* : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$ between sheaves on sites. We want now to find a left adjoint to f_*.

Proposition C.16. *[31] Let $\bar{f} : \mathcal{C} \to \mathcal{D}$ be a functor between sites, let $V \in \mathcal{D}$ denote $\mathcal{I}_V^f$ the category given by:*

$$Obj(\mathcal{I}_V^f) = \{(U, \phi) | U \in Obj(\mathcal{C}), \phi : V \to \bar{f}(U)\}$$
$$Hom_{\mathcal{I}_V^f}((U, \phi), (U', \phi')) = \{s : U \to U' \mid \bar{f}(s) \circ \phi = \phi'\}$$

Given a presheaf $\mathcal{F}$ on $\mathcal{C}$, for every object $V \in \mathcal{D}$ there are presheaves:

$$\mathcal{F}_V : \mathcal{I}_V^{f\ op} \to Sets \quad \mathcal{F}_V(U, \phi) = \mathcal{F}(U)$$

From this we can define a presheaf in the following way:

$$\begin{aligned} f'(\mathcal{F}) : \ \mathcal{D}^{op} &\longrightarrow Sets \\ V &\longrightarrow \varinjlim_{\mathcal{I}_V^{op}} \mathcal{F}_V \end{aligned}$$

Now we will denote $f^{-1}(\mathcal{F})$ the sheaf associated to the presheaf $f'(\mathcal{F})$.

Lemma C.17. *Given a continuous functor between sites $\bar{f} : \mathcal{C} \to \mathcal{D}$ then the two functor that we have defined as:*

$$f^{-1} : Sh(\mathcal{C}) \to Sh(\mathcal{D}) \qquad f_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$$

are adjoint $f^{-1} \dashv f_*$. The functor f^{-1} is called pull-back functor, and f_* is called push-forward functor.

Definition C.18. A morphism of sites $f : \mathcal{D} \to \mathcal{C}$ is given by a continuous functor $\bar{f} : \mathcal{C} \to \mathcal{D}$ such that f^{-1} is an exact functor.

Remark C.19. Notice that a morphism of site $f : \mathcal{C} \to \mathcal{D}$ is given by a functor in the opposite direction. To understand the use of our notation it is useful to keep in mind the following example. Consider $f : X \to Y$ a continuous map of topological spaces. From the previous example we have sites X_{Zar}, Y_{Zar}. Now consider the functor $\bar{f} : Y_{Zar} \to X_{Zar}$ defined by the usual topological pre-image $V \to f^{-1}(V)$, note that this functor is continuous because inverse image of open covering are open coverings. It is easy to see that the functor f_* is indeed the usual push-forward functor of topology. Now from the property of adjoint functor the topological pullback functor coincide with f^{-1}, also we know that in the topological case f^{-1} is exact and so we have obtained a morphisms of sites in the direction that we wanted $f : X_{Zar} \to Y_{Zar}$.

Now that we have described some basic result about sites we turn our attention to topos.

Definition C.20. A category $\mathcal{T}$ is a Grothendieck topos if it is equivalent to the category of sheaves on a site $\mathcal{C}$ i.e. $\mathcal{T} \cong \mathrm{Sh}(\mathcal{C})$.

- Let $\mathcal{C}, \mathcal{D}$ be sites. A morphism of topos $f : \mathrm{Shv}(\mathcal{D}) \to \mathrm{Pshv}(\mathcal{C})$ is given by a pair of functor:

  $$f_* : \mathrm{Shv}(\mathcal{D}) \to \mathrm{Shv}(\mathcal{C}) \quad f^{-1} : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$$

  such that we have an adjunction $f^{-1} \dashv f_*$ and f^{-1} preserve all finite limits.

- Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be sites. Given morphism of topos $f : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$ and $g : \mathrm{Sh}(\mathcal{E}) \to \mathrm{Sh}(\mathcal{D})$ the composition $f \circ g$ is the morphism of topos defined by the functors $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Remark C.21. After these definitions, it is useful to make some remarks:

- For what we have seen it is obvious that a morphism between sites $f : \mathcal{D} \to \mathcal{C}$ induce also a morphism between the corresponding topos $(f^{-1}, f_*) : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$.

- Different sites can give us equivalent topos. For example if we take the one point discrete category $\{*\}$, the only existing covering is the identity and hence $\{*\}$ form a site. Since over $\{*\}$ every sheaf is a presheaf and a presheaf is a choice of set then we have equivalence:
$$\mathrm{Shv}(\{*\}) \cong \mathrm{Pshv}(\{*\}) \cong \mathrm{Set}^{\{*\}} \cong \mathrm{Set}$$
but in [31] there are other examples of sites $\mathcal{C}$ not equivalent to the singleton $\{*\}$ such that $\mathrm{Sh}(\{*\}) = \mathrm{Set} \cong \mathrm{Sh}(\mathcal{C})$.

- If we consider the site X_{Zar} of open subset on a topological space X, then of course the topos $\mathrm{Shv}(X_{Zar})$ is equivalent to the category of sheaves over X.

In general topos can have some good properties that the site over which they are defined doesn't have e.g. a topos is always complete and cocomplete (C.13); so for example every topos $\mathcal{T}$ has a final object which is the sheaf associated to te presheaf: $U \to \{*\}$. However one of the most important difference between sites and topos is the definition of morphisms. In particular there are some morphisms of topos that are not induced by a morphism between the underlying sites, in fact this is the case for the morphism of infinitesimal topos induced by a scheme morphism as we have seen in (2.12). Another simpler example is the following:

Example C.22. Let X be a scheme. Let $\mathcal{S}$ be the site of open subschemes with the Zariski topology, and let $\mathcal{S}'$ be the site of open affine subschemes with the Zariski topology. Let $\mathcal{T}$ and $\mathcal{T}'$ be the associated topos. We can obtain a morphism of topos $f : \mathcal{T} \to \mathcal{T}'$ thanks to the exact functor $f^{-1} : \mathcal{T}' \to \mathcal{T}$ such that for every affine open U, then $f^{-1}(U) = U$. Since open affines form a base for the topology, then f must be an equivalence, and if g is its inverse, then g^* does not restrict to a map of sites. Indeed if V is an open subscheme, then $g^{-1}(V) = \mathrm{Hom}_X(U, V)$ but if V is not affine then $g^{-1}(V)$ is not represented by any object of $\mathcal{S}'$.

However we have to say that it can be shown [Exposé IV, Proposition 4.9.4, [28]], that any morphism of topos is equivalent to a morphism of topos which comes from a morphism of sites, but to do that we have to change sites, more precisely:

Proposition C.23. *let $\mathcal{C}$ and $\mathcal{D}$ be sites, Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topos. Then there exists a site $\mathcal{C}'$ and a diagram of functors:*
$$\mathcal{C} \xrightarrow{v} \mathcal{C}' \xleftarrow{u} \mathcal{D}$$
such that:

- *The functor v is a special cocontinuous functor such that it induces an equivalence between the related topos.*

- *The functor u commutes with fiber products, it is continuous and defines a morphism of sites $\mathcal{C}' \to \mathcal{D}$.*

- *The morphism of topos f agrees with the composition of morphism induced by v and u:*

$$Sh(\mathcal{C}) \xrightarrow{\cong} Sh(\mathcal{C}') \to Sh(\mathcal{D})$$

In complete analogy with the topological case, it remains to give some basic ideas and definitions about ringed spaces and ringed topos.

Definition C.24.

- A ringed topos is a pair $(\mathrm{Sh}(\mathcal{C}), \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$, The sheaf $\mathcal{O}$ is called the structure sheaf of the ringed site.
- Let $(\mathrm{Sh}(\mathcal{C}), \mathcal{O})$ and $(\mathrm{Sh}(\mathcal{D}), \mathcal{O}')$ be ringed topos. A morphism of ringed topos:

$$(f, f^\sharp) : (\mathrm{Sh}(\mathcal{C})\mathcal{O}) \to (\mathrm{Sh}(\mathcal{D}), \mathcal{O}')$$

  is given by a morphism of topos $f : \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D})$ together with a map of sheaves ring $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$ which by adjunction is equivalent to a map of sheaves of rings $f^\sharp : \mathcal{O}' \to f_*\mathcal{O}$.
- Let $(f, f^\sharp) : (\mathrm{Sh}(\mathcal{C}_1), \mathcal{O}_1) \to (\mathrm{Sh}(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (\mathrm{Sh}(\mathcal{C}_2), \mathcal{O}_2) \to (\mathrm{Sh}(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topos, then we define the composition of morphisms of ringed topos by the rule:

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp)$$

  where the composition of morphism of topos is already defined and $f^\sharp \circ g^\sharp$ indicates the morphism of sheaves of rings:

$$\mathcal{O}_3 \xrightarrow{g^\sharp} g_*\mathcal{O}_2 \xrightarrow{g_*f^\sharp} g_*f_*\mathcal{O}_1 = (g \circ f)_*\mathcal{O}_1$$

Definition C.25. Let $\mathcal{C}$ be a site, and let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$:

- A sheaf of $\mathcal{O}$-modules is given by an abelian sheaf $\mathcal{F}$ together with a map of sheaves of sets: $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ such that for every object U of $\mathcal{C}$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an $\mathcal{O}(U)$-module over $\mathcal{F}(U)$.
- A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}$-modules is a morphism of abelian sheaves such that the following diagram commute:

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ {\scriptstyle id \times \varphi} \downarrow & & \downarrow \\ \mathcal{O} \times \mathcal{G} & \longrightarrow & \mathcal{G} \end{array}$$

Proposition C.26. *Let $\mathcal{C}$ be a site, the category $\mathcal{O}_\mathcal{C}$-Mod of sheaves of $\mathcal{O}_\mathcal{C}$-modules over the ringed topos $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$, is an abelian category. The forgetful functor $\mathcal{O}_\mathcal{C}$-Mod$\to$ AbSh$(\mathcal{C})$ is exact, hence kernels, cokernels and exactness of $\mathcal{O}_\mathcal{C}$-modules correspond to the same notions for abelian sheaves. This is important because among other things this last fact implies that $\mathcal{O}_\mathcal{C}$-Mod has always enough injectives, and there exists a functorial injective embedding.*

So given a site $\mathcal{C}$ we have obtained a well behaved category $\mathcal{O}_{\mathcal{C}}$-Mod, in complete analogy with the standard topological case, we can also definea couple of adjoint functors:

Tensor Product	$U \to (\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_{\mathcal{C}(U)}} \mathcal{G}(U)$
Internal Hom	$U \to \mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{C}(U)}}(\mathcal{F}(U), \mathcal{G}(U))$

where $U \in$ Obj($\mathcal{C}$) and $\mathcal{F}, \mathcal{G} \in \mathcal{O}_{\mathcal{C}}$-Mod. The last important fact is that we have a couple of adjoint functor associated to each morphism of ringed topos.

Proposition C.27. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topos. We can define two functors:*

$$f_* : \mathcal{O}_{\mathcal{C}}{-}Mod \to \mathcal{O}_{\mathcal{D}}{-}Mod \qquad f^* : \mathcal{O}_{\mathcal{D}} v Mod \to \mathcal{O}_{\mathcal{C}}{-}Mod$$

such that they are adjoint $f^* \dashv f_*$ and we can define them as follows:

- *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{\mathcal{C}}$-modules. We define the pushforward of $\mathcal{F}$ as the sheaf of $\mathcal{O}_{\mathcal{D}}$-modules which as sheaf of abelian groups is equal to $f_*\mathcal{F}$ and where the module structure is given by the restriction via $f^\sharp : \mathcal{O}_{\mathcal{D}} \to f_*\mathcal{O}_{\mathcal{C}}$ of the canonical $f_*\mathcal{O}_{\mathcal{C}}$-module structure on $f_*\mathcal{F}$:*

$$\mathcal{O}_{\mathcal{D}} \times f_*\mathcal{F} \to f_*\mathcal{O}_{\mathcal{C}} \times f_*\mathcal{F} \to f_*\mathcal{F}$$

- *If $\mathcal{G}$ is a sheaf of $\mathcal{O}_{\mathcal{D}}$-modules, recall that $f^{-1}(\mathcal{G})$ have a natural structure of $f^{-1}(\mathcal{O}_{\mathcal{D}})$-module, and thanks to the map $f^\sharp : f^{-1}\mathcal{O}_{\mathcal{D}} \to \mathcal{O}_{\mathcal{C}}$ we can define the pullback of $\mathcal{G}$ as the sheaf of $\mathcal{O}_{\mathcal{C}}$-modules defined by the formula:*

$$f^*\mathcal{G} = \mathcal{O}_{\mathcal{C}} \otimes_{f^{-1}(\mathcal{D})} f^{-1}\mathcal{G}$$

C.2 Cohomology

Here we want to introduce some general notion of cohomology of sheaves, in particular we want to define and compare the usual sheaf cohomology defined as a derived functor and Čech cohomology. Recall that the category of sheaves of abelian group over a site $\mathcal{C}$ is in general an abelian category and it does have enough injectives, however it is not true in general that has enough projectives ([21],8.1). It follows that we can construct right derived functors of additive functors Sh($\mathcal{C}$) $\to \mathcal{A}$, where $\mathcal{A}$ is any abelian category, by the usual method of forming injective resolutions. The cohomology theory of topos consists essentially of the study of these derived functors. We can give now the basics definitions:

Definition C.28. Let $\mathcal{C}$ be a site and $\mathscr{E} =$ Sh($\mathcal{C}$), we define the the n-th cohomology group of an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ as the n-th right derived functor of the left exact global section functor, so that the family of functor $H^n(\mathscr{E}, \_)$ forms a universal δ-functor ShAb($\mathcal{C}$) $\to$ Ab, more explicitly:

$$R^n\Gamma(\mathscr{E}, \mathcal{F}) = H^n(\mathscr{E}, \mathcal{F}) = H^n(\Gamma(\mathscr{E}, \mathcal{I}^\bullet))$$

where $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$.

This definition is essentially the same as in the topological case, however, in that case, given a sheaf $\mathcal{F}$ on a topological space X we can define the global section functor as $\mathcal{F}(X)$ i.e the value of $\mathcal{F}$ in the terminal object of the site X_{Zar}. In general it might happen that our site $\mathcal{C}$ does not have a final object, hence we need a more general definition for the global section functor and for the cohomology groups.

Definition C.29. Let $(\mathscr{E}, \mathcal{O})$ a ringed topos where $\text{Sh}(\mathcal{C}) = \mathscr{E}$. Let $\mathcal{F}, \mathcal{G}$ two $\mathcal{O}$-modules. We will denote the value at $\mathcal{G}$ of the n-th right derived functor of $\text{Hom}_{\mathcal{O}}(\mathcal{F}, \_)$ as:

$$\text{Ext}^n_{\mathcal{O}}(\mathscr{E}, \mathcal{F}, \mathcal{G}) = R^n\text{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

The functors $\{\text{Ext}^n_{\mathcal{O}}(\mathscr{E}, \mathcal{F}, \mathcal{G})\ \}_{n\geq 0}$ define a covariant universal δ-functor in the variable $\mathcal{G}$ or controvariant in $\mathcal{F}$, where $\text{Ext}^0_{\mathcal{O}}(\mathscr{E}, \mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.

Definition C.30. [28] Let $(\mathscr{E}, \mathcal{O})$ a ringed topos where $\mathscr{E} = \text{Sh}(\mathcal{C})$. For every object X of $\text{Sh}(\mathcal{C})$ we denote $X_{\mathcal{O}}$ the free $\mathcal{O}$-module generated by X. We can define:

$$H^q(X, \mathcal{F}) = \text{Ext}^q_{\mathcal{O}}(\mathscr{E}, X_{\mathcal{O}}, \mathcal{F})$$

so that $H^q(X, \_)$ is the n-th right derived functor of $\text{Hom}_{\mathcal{O}}(X_{\mathcal{O}}, \_) = \text{Hom}_{\mathscr{E}}(X, \_)$ which we will denote as $\Gamma(X, \_)$. In this new situation even if $\mathcal{C}$ does not have a final object we can define the global section functor as:

$$\text{Hom}_{\mathcal{O}}(\mathcal{O}, \_) = \text{Hom}_{\mathscr{E}}(e, \_) = \Gamma(\mathscr{E}, \_)$$

where e is the final object of $\text{Sh}(\mathcal{C})$, so that finally we can define the n-th cohomology groups of a $\mathcal{O}$-module over $\mathcal{C}$ as the n-th right derived functor of the global section functor:

$$\text{Ext}^n_{\mathcal{O}}(\mathscr{E}, \mathcal{O}, \mathcal{F}) = H^n(\mathscr{E}, \mathcal{F})$$

Example C.31.

- Let X be a topological space and consider the topos $\mathscr{E} = \text{Sh}(X_{Zar})$, then the cohomology groups $H^n(\mathscr{E}, F)$ are just the cohomology groups of X with coefficient in an abelian sheaf F.
- If X is a projective object of a topos $\mathscr{E}$, the functor $\text{Hom}_{\mathscr{E}}(X, \_) : \text{Ab}(\mathscr{E}) \to \text{Ab}$ preserves epimorphism, hence is exact, so that $H^n(\mathscr{E}, X, F) = 0$ for all $F \in \text{Ab}(\mathscr{E})$ and $n > 0$. This applies in particular if the topos is equivalent to the category of presheaves over some small category $\mathscr{E} = \text{Set}^{C^{op}}$. In this case for a representable presheaf $\tilde{U} = \text{Hom}_C(\_, U)$ where U is an object of C, then the global section functor $\Gamma(\tilde{U}, A) = \text{Hom}_{\mathscr{E}}(\tilde{U}, A) = A(U)$ is an exact functor, so that $H^n(\mathscr{E}, \tilde{U}, A) = 0$ for all $n > 0$, and the free abelian sheaf generated by $\tilde{U}$ is projective.

As a general fact, everything that can be said about cohomology of abelian sheaves it is true also for cohomology of sheaves of modules, indeed note that in the notation $H^q(\mathscr{E}, \mathcal{F})$ the sheaf of ring $\mathcal{O}$ does not appear, this because these functors commute with restrictions of scalar. In particular we have the following:

Lemma C.32. *Let $\mathcal{C}$ be a site and $\mathscr{E} = Sh(\mathcal{C})$, let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a $\mathcal{O}$-module and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian group. Then we have isomorphisms:*

$$H^i(\mathscr{E}, \mathcal{F}_{ab}) = H^i(\mathscr{E}, \mathcal{F})$$

also, for every object U in $\mathscr{E}$ we have:

$$H^i(U, \mathcal{F}_{ab}) = H^i(U, \mathcal{F})$$

Proof. The δ-functor $(\mathcal{F} \to H^p(U,\mathcal{F}))_{p\geq 0}$ is universal. Since the forgetful functor $\mathcal{O}$-Mod$\to$ Ab($\mathcal{C}$) is exact, also $(\mathcal{F} \to H^p(U,\mathcal{F}_{ab}))_{p\geq 0}$ is a δ-functor. So we want to conclude by uniqueness of universal δ-functor but we have to show that also the second δ-functor is universal, and this is true since for any injective object $\mathcal{I}$ of $\mathcal{O}$-Mod, then $H^i(U,\mathcal{I}_{ab}) = 0$ and $\mathcal{O}$-Mod has enough injectives. □

We shall now recall the functoriality properties of cohomology groups with respect to geometric morphisms.

Proposition C.33. *Let $f : \mathscr{F} = Sh(\mathcal{C}) \to Sh(\mathcal{D}) = \mathscr{G}$ be a geometric morphism between Grothendieck topos. Then:*

- *If $\mathcal{G}$ is a sheaf of abelian group over $\mathcal{D}$, we have an homomorphism:*

$$H^n(\mathscr{G},\mathcal{G}) \xrightarrow{f^*} H^n(\mathscr{F}, f^*\mathcal{G}) \qquad \textit{for each } n \geq 0$$

  which are functorial in f and natural in $\mathcal{G}$.

- *If $\mathcal{F}$ is a sheaf of abelian group in $\mathcal{C}$, we have a spectral sequence (the Leray spectral sequence):*

$$H^p(\mathscr{G}, R^q f_*(\mathcal{F})) \implies H^{p+q}(\mathscr{F},\mathcal{F}) \tag{C.33.1}$$

Proof.
- Consider $\mathscr{F} \xrightarrow{\Gamma(\mathscr{F},\_)} \text{Set} \xleftarrow{\Gamma(\mathscr{G},\_)} \mathscr{G}$, let α be the unit of the adjunction $f^{-1} \dashv f^*$. Since f^{-1} is exact, the sequence of functors $\{R^n(\Gamma(\mathscr{G}, f^{-1}\_) : \text{Ab}(\mathscr{G}) \to \text{Ab}, n \geq 0\}$ is exact and connected, hence by the universal property of derived functors, the natural transformation $\Gamma(\mathscr{F},\_)(\alpha) : \Gamma(\mathscr{F},\_) \to \Gamma(\mathscr{F},\_)f_*f^{-1} \cong \Gamma(\mathscr{G},\_)f^{-1}$ extends uniquely to a natural transformation $R^n(\Gamma(\mathscr{F},\_)) \to R^n(\Gamma(\mathscr{G}, f^{-1}\_))$, and evaluating the natural transformation over an abelian sheaf $\mathcal{F}$ gives us the desired homomorphism.

- This is just the Grothendieck spectral sequence for the composition:

$$(\text{Ab}(\mathscr{F}) \xrightarrow{f_*} \text{Ab}(\mathscr{G}) \xrightarrow{\Gamma(\mathscr{G},\_)} \text{Ab}) = (\text{Ab}(\mathscr{F}) \xrightarrow{\Gamma(\mathscr{F},\_)} \text{Ab})$$

  to prove the equality we can see that since f^{-1} is exact then preserves limit, in particular final object:

$$\Gamma(\mathscr{F}, f_*F) = \text{Hom}_{\mathscr{F}}(e, f_*F) = \text{Hom}_{\mathscr{G}}(f^{-1}e, F) = \text{Hom}_{\mathscr{G}}(e', F) = \Gamma(\mathscr{G}, F)$$

  Now we can apply the Grothendieck spectral sequence coming from the composition since f_* take injectives to injectives because its left adjoint f^{-1} is exact.

□

From this last definition it can be convenient to have an explicit description of the functors $R^q f_*$ and this is provided by the following proposition:

Proposition C.34. *Let $f : \mathscr{F} = Sh(\mathcal{C}) \to Sh(\mathcal{D}) = \mathscr{G}$ be a geometric morphism between Grothendieck topos. Then for any sheaf of abelian group $\mathcal{F}$ on $\mathcal{C}$ and any $n \geq 0$ then $R^n f_*(\mathcal{F})$ is the sheaf associated to the following presheaf:*

$$U \longrightarrow H^n(\mathscr{F}, f^*\tilde{U}, \mathcal{F})$$

The existence of enough injectives in the category of abelian sheaves over a site provides us with an existence theorem for derived functors, but as we can imagine computing the cohomology in a Grothendieck topos can be difficult, here we develop a useful technique that has been used before. Consider a fixed topos $\mathscr{E} = \mathrm{Sh}(\mathcal{C})$ and call $\mathscr{P}$ the corresponding presheaf topos $\mathrm{Set}^{\mathcal{C}^{op}}$. Suppose that $\mathcal{C}$ has pullbacks, let $\mathcal{U} = \{U_i \xrightarrow{i} U\}_{i\in I}$ be a family of morphisms of $\mathcal{C}$ with common codomain, for each multi-index $\sigma = (i_0, \dots, i_n) \in I^{n+1}$ define $U_\sigma = U_{i_0} \times_U \cdots \times_U U_{i_n}$. Then we have a simplicial object in $\mathscr{P}$ of the form:

$$\cdots \quad \coprod_{\tau\in I^3} h_{U_\tau} \rightrightarrows \coprod_{\sigma\in I^2} h_{U_\sigma} \rightrightarrows \coprod_{i\in I} h_{U_i} \longrightarrow h_U$$

where the face map are obtained by combining the canonical projections. Applying the free functor $\mathscr{P} \xrightarrow{Z} \mathrm{Ab}(\mathscr{P})$ we obtain a simplicial object in $\mathrm{Ab}(\mathscr{P})$, and as usual taking alternating sums of the face map we obtain a chain complex:

$$N_\bullet(\mathcal{U}) \to Z(h_U)$$

the complex $N_\bullet(\mathcal{U})$ is exact in $\mathrm{Ab}(\mathscr{P})$.

Definition C.35. Let $\mathcal{F}$ be an abelian presheaf, we define the cochain complex of abelian group:

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) = \mathrm{Hom}_{\mathrm{Ab}(\mathscr{P})}(N_\bullet(\mathcal{U}), \mathcal{F})$$

Since the groups in the complex $N_\bullet(\mathcal{U})$ are free we can write the cochain complex as:

$$\prod_{i\in I} \mathcal{F}(U_i) \to \prod_{\sigma\in I^2} F(U_\sigma) \to \prod_{\tau\in I^3} \mathcal{F}(U_\tau) \to$$

This construction is functorial in $\mathcal{F}$ and hence we have obtained a functor $\check{\mathcal{C}}^\bullet(\mathcal{U}, \_)$ from the category of abelian presheaves on $\mathcal{C}$ to the category of bounded below complex of abelian groups. We will call the n-th Čech cohomology group of $\mathcal{U}$ with coefficient in $\mathcal{F}$ the abelian group $H^n(\mathcal{U}, \mathcal{F})$ given by the n-th cohomology group of the complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \_)$. Note that $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U)$ if and only if $\mathcal{F}$ satisfies the sheaf axiom.

Proposition C.36. *With the previous notation $H^n(\mathcal{U}; \_)$ is the n-derived functor of $H^0(\mathcal{U}, \_)$.*

Proof. Representable presheaves are projective in the presheaf topos $\mathscr{P}$, this means that also the abelian presheaf in $N_\bullet(\mathcal{U})$ are projective and so $\mathcal{F} \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is an exact functor from $\mathrm{Ab}(\mathscr{E})$ to cochain complexes of abelian groups. So the functors $H^q(\mathcal{U}, \_)$ form an exact connected sequence, and hence we have obtained a δ-functor. if $\mathcal{I}$ is injective, then $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) = \mathrm{Hom}_{\mathrm{Ab}(\mathscr{P})}(N_\bullet(\mathcal{U}), \mathcal{I})$ is exact in positive degree so that the cohomology groups vanish for $n > 0$, hence we have obtained a universal δ-functor and the theorem follows from the uniqueness theorem for derived functors. □

Proposition C.37. *Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be two families of morphism of $\mathcal{C}$, a refinement $\mathcal{V} \to \mathcal{U}$ is a functor $r : J \to I$ together with a family of factorization:*

$$\begin{array}{ccc} V_j & \xrightarrow{r(j)} & U_{r(j)} \\ & \searrow & \downarrow \\ & & U \end{array}$$

for all $j \in J$ so that we have a map $r_\bullet : N_\bullet(\mathcal{V}) \to N_\bullet(\mathcal{U})$. If $r, s : \mathcal{V} \to \mathcal{U}$ are two refinements maps, then the induced chain maps $r_\bullet, s_\bullet$ are chain-homotopic.

Proposition C.38. *Let $\mathcal{U} = \{U_i \xrightarrow{i} U\}_{i \in I}$ be a family of morphism over a site $\mathcal{C}$ and consider the inclusion $R \to h_U$ of the sieve on U generated by $\mathcal{U}$. Then for any abelian presheaf $\mathcal{F}$ we have a canonical isomorphism:*

$$H^n(\mathcal{U}, \mathcal{F}) = H^n(R, \mathcal{F})$$

Proof. Since R is a subobject of h_U, we get that $N_\bullet(\mathcal{U})$ is a projective resolution of the free abelian presheaf generated by R. From this the cohomology groups of the complex $C^\bullet(\mathcal{U}, \mathcal{F})$ are canonical isomorphic to $\mathrm{Ext}^n(\mathscr{P}, Z(R), \mathcal{F})$, where $\mathscr{P}$ is the topos of presheaves over $\mathcal{C}$. More in particular since every morphism in R factors through one of the U_i, there exists a refinement map $R \to \mathcal{U}$; but the inclusion map is clearly a refinement map $\mathcal{U} \to R$. So by the previous proposition (C.37) the complexes $N_\bullet(\mathcal{U})$ and $N_\bullet(R)$ are chain homotopy equivalent; hence so are $C^\bullet(\mathcal{U}; \mathcal{F})$ and $C^\bullet(R; \mathcal{F})$, and so their cohomology groups are isomorphic. Moreover, this isomorphism does not depend on the choice of refinement $R \to \mathcal{U}$, since the two different choices induce cochain maps $C^\bullet(\mathrm{U}; \mathcal{F}) \to C^\bullet(R; \mathcal{F})$ which differ by a homotopy. The only thing to check is that $C^\bullet(R; A)$ is a projective resolution of R, but this is true, as specified on the proof of (C.36), because in the presheaf topos $\mathscr{P}$ representable presheaves are projective and coproducts of projectives objects are projectives. □

Definition C.39. Let $(\mathcal{C}, \mathscr{T})$ a site such that $\mathcal{C}$ has pullbacks, U an object of $\mathcal{C}$ and $\mathcal{F}$ a presheaf of abelian group on $\mathcal{C}$. We define the n-th Čech cohomology group of U with coefficient in $\mathcal{F}$ to be the filtered colimit:

$$\check{H}^q(U, \mathcal{F}) = \varinjlim_{R \in \mathscr{T}(U)} H^q(R, \mathcal{F})$$

Remark C.40. Since $\varinjlim_{R \in \mathscr{T}(U)}$ is exact, the functors $\check{H}^n(U, \_)$ form an exact connected sequence on $\mathrm{Ab}(\mathscr{P})$, and they also vanish on injectives for $q > 0$. Note that:

$$\check{H}^0(U, \mathcal{F}) \cong \varinjlim_{R \in \mathscr{T}(U)} \mathrm{Hom}_{\mathscr{P}}(R, \mathcal{F}) = A^+(U)$$

so that the functor $\mathrm{Ab}(\mathscr{P}) \to \mathrm{Ab}(\mathscr{P})$ which sends $\mathcal{F}$ to the presheaf $U \to \check{H}^n(U, \mathcal{F})$ is the n-th derived functor of the functor $(\_)^+$ defined in (C.8.1).

Usually it is more useful to consider Čech cohomology groups with coefficient in a sheaf, but the inclusion functor $i_* : \mathrm{Ab}(\mathscr{E}) \to \mathrm{Ab}(\mathscr{P})$ is not exact, hence this functors do not provide an exact connected sequence in $\mathrm{Ab}(\mathscr{E})$. However, if $\mathcal{F}$ is a sheaf of abelian groups, for every object X of $\mathcal{C}$ we have:

$$i_*(\mathcal{F})(X) = H^0(\tilde{X}, \mathcal{F}) = \mathcal{F}(X)$$

By definition the functor i_* is left exact hence we will denote $R^n i_*$ its right derived functors. Since for all X in $\mathcal{C}$ the functor $\mathcal{G} \to \mathcal{G}(X)$ is exact in the category of presheaves of abelian groups, we have:

$$R^n i_*(\mathcal{F})(X) = H^n(\tilde{X}, \mathcal{F})$$

for all X in $\mathcal{C}$ and every sheaf of abelian group $\mathcal{F}$, this means that:

$$R^n i_*(\mathcal{F}) = \underline{H}^n(\mathcal{F}) : X \to H^n(\tilde{X}, \mathcal{F})$$

With this notation if $\mathcal{F}$ is a sheaf of abelian groups then we can denote its Čech cohomology groups as $H^n(\mathcal{U}, i_*(\mathcal{F}))$.

Lemma C.41. *[28] Let $(\mathcal{C}, \mathscr{T})$ be a site, let R be a presheaf of sets over $\mathcal{C}$, and $\mathcal{F}$ a sheaf of abelian groups. There exists a spectral sequence, functorial in R and in $\mathcal{F}$:*

$$H^{p+q}(R^\dagger, \mathcal{F}) \Leftarrow E_2^{p,q} = H^p(R, R^q i_*(\mathcal{F}))$$

In particular if $\mathcal{U} = \{U_i \xrightarrow{i} X\}_{i \in I}$ is a covering family, we have a spectral sequence:

$$H^{p+q}(\tilde{X}, \mathcal{F}) \Leftarrow E_2^{p,q} = H^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

Proof. This is just a Grothendieck spectral sequence, by definition of associated sheaf we have a canonical isomorphism: $H^0(R^\dagger, \mathcal{F}) \cong H^0(R, i_*(\mathcal{F}))$, also we know that i_* transform injective objects in injective objects. If $R \to h_X$ is the sieves generated by $\mathcal{U}$, since $\mathcal{U}$ is a covering family, the sheaf associated to R is the same as the sheaf $\tilde{X}$ associated to h_X. $\square$

References

[1] P. Berthelot. *Cohomologie Cristalline des Schemas de Caracteristique p>0*, volume 407. Springer, 2006.

[2] P. Berthelot and A. Ogus. *Notes on crystalline cohomology*. Princeton University Press, 2015.

[3] B. Bhatt and A. J. de Jong. Crystalline cohomology and de rham cohomology. *arXiv preprint arXiv:1110.5001*, 2011.

[4] S. Bosch. *Algebraic Geometry and Commutative Algebra*, volume 407. Springer-Verlag London, 2013.

[5] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*, volume 304. Springer Science & Business Media, 1972.

[6] K. S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Transactions of the American Mathematical Society*, 1973.

[7] G. Cortinas. Infinitesimal k-theory. *arXiv preprint math/0001138*, 2000.

[8] G. Cortinas. De rham and infinitesimal cohomology in kapranov's model for non commutative algebraic geometry. *Compositio Mathematica*, 2003.

[9] G. Cortinas, C. Haesemeyer, and C. A. Weibel. Infinitesimal cohomology and the chern character to negative cyclic homology. *arXiv preprint math/0703133*, 2007.

[10] P. Deligne. Théorème de lefschetz et critères de dégénérescence de suites spectrales. *Publications Mathématiques de l'IHÉS*, 35:107–126, 1968.

[11] P. Deligne. *Équations différentielles à points singuliers réguliers*, volume 163. Springer, 2006.

[12] P. Deligne and L. Illusie. Relevements modulop 2 et décomposition du complexe de de rham. *Inventiones mathematicae*, 89, 1987.

[13] B. I. Dundas, M. Levine, P. A. Ostvær, O. Röndigs, and V. Voevodsky. *Motivic homotopy theory: lectures at a summer school in Nordfjordeid, Norway*. Springer Science & Business Media, 2007.

[14] S. I. Gelfand and Y. I. Manin. *Methods of homological algebra*. Springer Science & Business Media, 2013.

[15] A. Grothendieck. On the de rham cohomology of algebraic varieties. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 29(1):95–103, 1966.

[16] A. Grothendieck. Éléments de géométrie algébrique (rédigés avec la collaboration de jean dieudonné): Iv. étude locale des schémas et des morphismes de schémas, quatrieme partie. *Publications mathématiques de l'IHES*, 32:5–361, 1967.

[17] A. Grothendieck. Crystals and the de rham cohomology of schemes. *Dix exposés sur la cohomologie des schémas*, 1968.

[18] A. Grothendieck and M. Raynaud. *Séminaire de Géométrie Algébrique du Bois Marie - Revetements étales et groupe fondamental*. 2002.

[19] R. Hartshorne. *Residues and duality*, volume 20. Springer, 1966.

[20] R. Hartshorne. On the de rham cohomology of algebraic varieties. *Publications Mathématiques de l'IHÉS*, 1975.

[21] P. T. Johnstone. *Topos theory*. Courier Corporation, 2014.

[22] N. M. Katz. Nilpotent connections and the monodromy theorem: Applications of a result of turrittin. *Publications mathématiques de l'IHES*, 1970.

[23] N. M. Katz. Algebraic solutions of differential equations (p-curvature and the hodge filtration). *Inventiones mathematicae*, 18, 1972.

[24] N. M. Katz, T. Oda, et al. On the differentiation of de rham cohomology classes with respect to parameters. *Journal of Mathematics of Kyoto University*, 8(2):199–213, 1968.

[25] K. S. Kedlaya. p-adic cohomology: from theory to practice.

[26] J. M. Lee. *Riemannian manifolds: an introduction to curvature*, volume 176. Springer Science & Business Media, 2006.

[27] Q. Liu. *Algebraic geometry and arithmetic curves*, volume 6. Oxford University Press on Demand, 2002.

[28] A. G. e. J.-L. V. Michael Artin. *Séminaire de Géométrie Algébrique du Bois Marie - Théorie des topos et cohomologie étale des schémas*. Springer-Verlag Berlin, New York, 1972.

[29] J. S. Milne. Lectures on étale cohomology. *Available on-line at http://www. jmilne. org/math/CourseNotes/LEC. pdf*, 1998.

[30] J.-P. Serre. Géométrie algébrique et géométrie analytique. In *Annales de l'institut Fourier*, volume 6, 1956.

[31] T. Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2020.

[32] R. W. Thomason. Algebraic k-theory and étale cohomology. In *Annales scientifiques de l'École Normale Supérieure*, volume 18, 1985.

[33] C. A. Weibel. *An introduction to homological algebra*. Number 38. Cambridge university press, 1995.