UNIVERSITÀ DEGLI STUDI DI MILANO

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# GALOIS THEORY
## OF
# GROTHENDIECK


Supervisor: Prof. Luca BARBIERI VIALE

Thesis by:
Alessandro MONTRESOR
Student ID: 808469



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*To my parents and my brother.*
*Thanks for being next to me.*

# Contents

# INTRODUCTION

This thesis attempts to unify within the same work two topics which are central in the latest development of algebraic and arithmetic geometry.

The first topic is Grothendieck's notion of topology.
This concept arose to by-pass the problem of the Zariski topology being too coarse.
In fact, when we deal with complex algebraic varieties, the complex topology is available. This provides us all the method of classical algebraic topology and cohomology. When we consider an arbitrary base field, though, we just have the Zariski topology. This has much less open sets, hence in many cases cohomology groups fail to give interesting information.
The idea of Grothendieck was easy as much as brilliant: instead of considering coverings of open subsets we consider collection of morphisms with fixed codomain.
Not only Grothendieck topologies generalize the notion of topological space. They also allow, paralleling classical sheaf theory, to speak about sheaves on rather general categories (fibered product being the natural substitute of intersection).
Moreover Grothendieck proved that the category of sheaves for a Grothendieck topology has enough injective objects. This allows to define the $q$-th cohomology group of an object $U$ with values in a sheaf $\mathscr{F}$: this is $R^q\Gamma_U(\mathscr{F})$, the image of $\mathscr{F}$ via the right derived functors of the global section functor.
This is probably the most general approach to cohomology theory.

Galois Categories are the second topic.
This is the classical example of theory that bridges the gap among many different situations.
It widely clarifies the notion of fundamental group, which appears in nearly all fields of mathematics. For example the well known topological fundamental group and the absolute Galois group of a field are two manifestations of it (see [21] for a comprehensive treatment about fundamental groups). In both these situations we have a group (defined as the automorphism groups of some object) classifying some object (covers of a topological space in the first case, Galois extension of the field in the second).
This is brilliantly explained by the abstract theory of Galois categories: the group is the automorphisms group of a functor with images in the category of finite sets; the classification is given by an equivalence induced by this functor.

We treat this topics in the abstract framework of categories.
This is not mere desire of generalization: dealing with general categories is often easier then considering more concrete objects.
Just think about the definition of morphism of schemes: there are involved two topological spaces $Y$ and $X$, two sheaves of rings $\mathcal{O}_Y$ and $\mathcal{O}_X$ which locally looks like the spectrum of a ring, a continuous map $f : Y \to X$, a morphism of sheaves from $\mathcal{O}_X$ to the direct image of $\mathcal{O}_Y$ via $f$.
In categorical language this is just an arrow between two objects.
It does sound easier.

In the last chapter we carry over the abstract theory to the category of schemes.
Firstly we give a general method to construct Grothendieck topologies in this category. The idea is that we relax the condition of being an open immersion and we consider wider class of morphisms to form coverings.
Secondly we prove that the category of finite étale morphisms over a connected scheme is a Galois category.
For example this gives a new insight into Galois cohomology: this is just étale cohomology when we choose the base scheme to be the spectrum of a field.
Finally we revisit a classical result, Hilbert theorem 90, and we state and prove it in a much more general form using étale cohomology.
This is yet another example of an arithmetical result which has a deep geometric interpretation.

# PREREQUISITES AND CONVENTION

The (real) basics of commutative algebra, scheme theory and category theory are assumed to be known. Any classical books on the topic will do.
For example: for commutative algebra [2]; for scheme theory [7]; for category theory [16].

We denote with $\underline{\mathbf{Set}}$, $\underline{\mathbf{FSet}}$, $\underline{\mathbf{Ab}}$ and $\underline{\mathbf{Rng}}$ the category of sets, finite sets, abelian groups and rings respectively.

All rings are assumed to be commutative Noetherian with identity.
All schemes are assumed to be locally Noetherian.

# 1 GROTHENDIECK TOPOLOGIES

Roughly speaking a site on a category is a way to see its object as open subsets of a topological space. In fact the prototype of a site is the category $\mathcal{O}(X)$ of open subsets of a topological space $X$, with inclusions as arrows.
More precisely, instead of usual coverings $\{U_i \subseteq U\}_{i\in I}$ of open subsets we consider collection of arrows $\{U_i \to U\}_{i\in I}$ with fixed codomain. A site on a category consists of a collection of coverings for each object, satisfying some axioms.
The point in doing so is that it allows to define (abelian) sheaves in a natural way, i.e. as presheaves satisfying the sheaf condition on any covering. Of course, in order to state such a condition, we have to consider fibered products instead of intersections (this make sense since the fibered product in $\mathcal{O}(X)$ is intersection).
Clearly the collection of sheaves on a site with morphisms of presheaves as arrows form a category. This turn out to have all the nice properties of the usual category of sheaves on a topological space. In fact we will prove that it is a Grothendieck category (see the Tohoku paper [8]).
In particular this implies it has enough injective object and this allows to speak about cohomology groups of an object with value on a sheaf $\mathscr{F}$. These are the images of $\mathscr{F}$ via the right derived functors of the section functor on $U$.
Finally we will introduce the canonical topology, which is the finer such that all representable presheaves are sheaves. In particular we will consider the category of sets endowed with a continuous action of a profinite group, which will be of central importance in the rest of the thesis.

The notion of site was invented by Artin and Grothendieck to formalize the latter's intuition about étale cohomology.
Artin was probably the first teaching and writing about the topic in the abstract categorical form ([1]). Afterwards the theory was systematically treated in SGA III ([4]).
A modern book dealing with the categorical theory of sites, which approach I will mostly follow here, is [22]; another precious source was [27].

## 1.1 Sites, sheaves, cohomology

In this section we define sites. The prototype of a site is the category $\mathcal{O}(X)$ of open subsets of a given topological space, with the inclusions as arrows. While we give the axioms defining sites we look at their meaning in this category, to keep always clear in mind how they came from geometric intuition.

For simplicity, in the whole section we assume categories to have fibered products.

**Definition 1.1.1** *We call* **covering** *of an object $U \in \mathcal{C}$ a collection of morphism $\{U_i \to U\}_{i \in I}$.*
*We call* **coverage** *on $\mathcal{C}$ a collection of coverings for each object $U \in \mathcal{C}$ satisfying the following condition*

**(T1)** *for any covering $\{U_i \to U\}_{i \in I}$ and any morphism $V \longrightarrow U$, the collection $\{U_i \times_U V \to V\}_{i \in I}$ is a covering for $V$.*

If $\mathcal{T}$ is a coverage on a category, we will denote with $\mathbf{Cov}_{\mathcal{T}}(U)$ (or simply with $\mathbf{Cov}(U)$ if the coverage is clear from the context) the collection of coverings of $U$ in $\mathcal{T}$.

**Remark 1.1.2** In $\mathcal{O}(X)$ axiom (**T1**) means that if $\{U_i \subseteq U\}_{i \in I}$ is a covering for $U$ and $V \subseteq U$, then $\{U_i \cap V \subseteq V\}_{i \in I}$ is a covering for V.
Of course usual coverings of open subsets define a coverage on $\mathcal{O}(X)$.

This definition is already enough to define sheaves:

**Definition 1.1.3** *Let $\mathcal{T}$ be a covering on a category $\mathcal{C}$.*
*We say that a presheaf $\mathscr{F}$ (of set, abelian group, ring etc.) on $\mathcal{C}$ is a* **sheaf** *for $\mathcal{T}$ (of set, abelian group, ring etc.) if for any $U \in \mathcal{C}$ and any covering $\{U_i \to U\}_{i \in I} \in \mathcal{T}$ the following sequence is exact*

$$\mathscr{F}(U) \longrightarrow \prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \times_U U_j)$$

*(the arrows to be understood).*
*We denote the category of (abelian, if not mentioned) sheaves for a coverage $(\mathcal{C}, \mathcal{T})$ with $\mathcal{S}(\mathcal{T})$.*

Of course applied to $\mathcal{O}(X)$ this correspond to the usual definition of sheaf.

To obtain the definition of site we need to add two axioms. These corresponds to the idea that "composition" of coverings is again a covering and that any isomorphism is itself a covering.

**Definition 1.1.4** *A* **site** *(or* **topology***) on a category $\mathcal{C}$ is a coverage satisfying the following extra conditions*

**(T2)** *if $\{U_i \to U\}$ is a covering and for all $i$ $\{U_{ij} \to U_i\}$ is a covering, then $\{U_{ij} \to U_i \to U\}$ is a covering.*

**(T3)** *any isomorphism form a covering itself.*

**Remark 1.1.5** We rephrase these condition in $\mathcal{O}(X)$.
(**T2**) means that if $\{U_i \subseteq U\}$ is a covering for $U$ and for all $i$ $\{U_{i,j} \subseteq U_i\}$ is a covering for $U_i$, then $\{U_{i,j} \subseteq U\}$ is a covering for $U$.
(**T3**) correspond to the fact that any open set cover itself.
Once again $\mathcal{O}(X)$ together with usual coverings satisfies (**T2**) and (**T3**).

**Remark 1.1.6** Sometimes in literature this is called a pretopology. Actually this was the name when it first appeared in Grothendieck's and Demazure's work([4]). Here topologies are defined by means of sieves and not of coverings: the reason in doing so is that it may happen that two different pretopologies generate the same category of sheaf, while a topology is uniquely determined by the associated sheaf category.
Nevertheless in our definition the geometric aspect is more evident, and it is enough for our scope.

Of course we need a good definition of functor between topologies.

**Definition 1.1.7** *Let $(\mathcal{C}, \mathcal{T})$ and $(\mathcal{C}', \mathcal{T}')$ be sites. We say that a functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is* **continuous** *if it satisfies the following propeties*

*i)* $\{U_i \to U\} \in \mathbf{Cov}(\mathcal{C}) \implies \{\mathcal{F}(U_i) \to \mathcal{F}(U)\} \in \mathbf{Cov}(\mathcal{C}')$.

*ii) For any covering $\{U_i \to U\} \in \mathbf{Cov}(\mathcal{C})$ and any arrow $V \to U$ the canonical morphism $\mathcal{F}(U_i \times_U V) \to \mathcal{F}(U_i) \times_{\mathcal{F}(U)} \mathcal{F}(V)$ is an isomorphism for all $i$.*

*We will say that $F$ yields a morphism of site $\mathcal{F} : (\mathcal{C}', \mathcal{T}') \to (\mathcal{C}, \mathcal{T})$.*

**Remark 1.1.8** Consider a continuous map between topological spaces, say $f : X \to Y$. Then the functor $[V \to f^{-1}(V)] : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a continuous functor.
In fact $i)$ means that if $\{V_i \subseteq V\}$ is an open covering of $V \subseteq Y$ then $\{f^{-1}(V_i) \subseteq f^{-1}(V)\}$ is an open covering of $f^{-1}(V) \subseteq X$ and $ii)$ means that $f^{-1}(U_i \cap V) = F(U_i) \cap F(V)$.

**Remark 1.1.9** Let $\mathcal{F} : (\mathcal{C}', \mathcal{T}') \to (\mathcal{C}, \mathcal{T})$ be a morphism of sites and assume that the underlying continuous functor $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories with continuous quasi inverse.
Then it is easy to see that $\mathcal{S}(\mathcal{T})$ is equivalent to $\mathcal{S}(\mathcal{T}')$ via the functor

$$[\mathscr{G} \to \mathcal{F}_*\mathscr{G} := \mathscr{G} \circ F] : \mathcal{S}(\mathcal{T}') \to \mathcal{S}(\mathcal{T}).$$

Finally we may use continuity to define the notion of finer topology:

**Definition 1.1.10** *Let $\mathcal{T}$, $\mathcal{T}'$ be sites on a category $\mathcal{C}$. We say that $\mathcal{T}$ is finer then $\mathcal{T}'$ if the identical functor $\mathcal{C} \to \mathcal{C}$ yields a morphism of sites $(\mathcal{C}, \mathcal{T}) \to (\mathcal{C}, \mathcal{T}')$.*
*This just means that any covering in $\mathcal{T}'$ is a covering in $\mathcal{T}$.*

The few notion introduced up to now are already enough to define cohomology groups.
Let $[\mathscr{F} \mapsto \Gamma_U(\mathscr{F}) := \mathscr{F}(U)] : \mathcal{S}(\mathcal{T}) \to \underline{\mathbf{Ab}}$ be the section functor on $U$.
Assume for a moment that the category of sheaves on a site is an abelian category with enough injective objects (all of these will be proven in the next sections).
Then we make the following definition:

**Definition 1.1.11** *Let $\mathscr{F} \in \mathcal{S}(\mathcal{T})$ and $U \in \mathcal{C}$. We call*

$$H^q(U, \mathscr{F}) := R^q\Gamma_U(\mathscr{F})$$

*the $q$-**th cohomology group of $U$ with value in $\mathscr{F}$***

## 1.2 Sheafification

The aim of this section is to describe sheafification, i.e. the left adjoint to the inclusion $i : \mathcal{S}(\mathcal{T}) \to \mathcal{P}(\mathcal{T})$. The existence of such an adjunction is of central importance for proving that categories of sheaves are abelian.
Throughout the section we fix a site $\mathcal{T}$ on a category $\mathcal{C}$. All covering will be in $\mathcal{T}$.

We sketch our strategy first.
Given an abelian presheaf $\mathscr{F}$ and a covering $\mathcal{U} = \{U_i \to U\}$ define

$$H(\mathcal{U}, \mathscr{F}) := \ker\left(\prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \times_U U_j)\right).$$

Consider the collection $\mathcal{J}_U$ of coverings of a fixed object $U \in \mathcal{C}$. We will make this in a filtered preordered category, and show that $H(-, \mathscr{F})$ can be seen as a functor $\mathcal{J}_U \to \underline{\mathbf{Ab}}$. Finally we will define the presheaf

$$\mathcal{F}^\dagger(U) := \varinjlim_{\mathcal{J}_U} H(-, \mathscr{F})$$

and prove that $(\mathcal{F}^\dagger)^\dagger$ is the sheafification of $\mathscr{F}$.

First we define $\mathcal{J}_U$. What we're missing is a good notion for arrows. Slightly more generally we can define maps between arbitrary coverings.

**Definition 1.2.1** *We call* **morphism of coverings** *from* $\mathcal{V} = \{V_j \to V\}_{j\in J}$ *to* $\mathcal{U} = \{U_i \to U\}_{i\in I}$ *an arrow* $V \to U$ *together with a pair* $(\epsilon, f)$ *consisting of :*

- *a map* $\epsilon : J \to I$
- *a collection of morphisms* $f = \{f_j : V_j \to U_{\epsilon(j)}\}_{j\in J}$ *such that*

$$\begin{tikzcd} V_j \arrow[r] \arrow[d] & U_{\epsilon(j)} \arrow[d] \\ V \arrow[r] & U \end{tikzcd}$$

  *commute for all* $j \in J$.

*We will denote this loosely with* $(\epsilon, f)$.
*In particular if* $V = U$ *and* $V \to U$ *is the identity map, we say that* $\mathcal{V}$ *is a* **refinement** *of* $\mathcal{U}$ *and we call* $(\epsilon, f)$ **refinement map**.

This makes $\mathbf{Cov}(\mathcal{C})$ into a category.
Moreover notice that a morphism of coverings induces maps

$$\begin{aligned} \prod_i \mathscr{F}(U_i) &\longrightarrow \prod_j \mathscr{F}(V_j) \\ (s_i) &\longmapsto \big(\mathscr{F} f_j(s_{\epsilon(j)})\big) \end{aligned}$$

$$\begin{aligned} \prod_{i,i'} \mathscr{F}(U_i \times_U U_{i'}) &\longrightarrow \prod_{j,j'} \mathscr{F}(V_j \times_U V_{j'}) \\ (s_{ii'}) &\longmapsto \big(\mathscr{F}(f_j \times f_{j'})(s_{\epsilon(j)\epsilon(j')})\big)_{jj'} \end{aligned}$$

making the following diagram commute

$$\begin{tikzcd} \prod_i \mathscr{F}(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,i'} \mathscr{F}(U_i \times_U U_{i'}) \arrow[d] \\ \prod_j \mathscr{F}(V_j) \arrow[r, shift left] \arrow[r, shift right] & \prod_{j,j'} \mathscr{F}(V_j \times_U V_{j'}) \end{tikzcd}$$

Hence we get a well defined group map $H(f, \mathscr{F}) : H(\mathcal{U}, \mathscr{F}) \to H(\mathcal{V}, \mathscr{F})$ and we see that $H(-, \mathscr{F})$ is a presheaf on $\mathbf{Cov}(\mathcal{C})$.

The following lemma is very important for many computation.

**Lemma 1.2.2** *Let $f, g : \mathcal{V} \to \mathcal{U}$ be morphisms of coverings inducing the same map $V \to U$. Then $H(f, \mathscr{F}) = H(g, \mathscr{F})$.*

**Proof**
It will suffice to exhibit a map $\Delta$ such that the following diagram commutes

$$\begin{tikzcd}
\prod \mathscr{F}(U_i) \arrow[rr, shift left, "H(f,\mathscr{F})"] \arrow[rr, shift right, "H(g,\mathscr{F})"'] \arrow[dr, shift left] \arrow[dr, shift right] & & \prod \mathscr{F}(V_j) \\
& \prod \mathscr{F}(U_i \times_U U_{i'}) \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

Consider the product map $f_j \times g_j : U_j \longrightarrow U_{\epsilon(j)} \times_U U_{\eta(j)}$. This makes the following diagram commute

$$\begin{tikzcd}
& V_j \arrow[dl, "f_j"'] \arrow[d, "f_j \times g_j"] \arrow[dr, "g_j"] & \\
U_{\epsilon(j)} & U_{\epsilon(j)} \times_U U_{\eta(j)} \arrow[l, "\pi_j^1"] \arrow[r, "\pi_j^2"'] & U_{\eta(j)}
\end{tikzcd}$$

Thus applying $\mathscr{F}$ we see that

$$\mathscr{F}(f_j \times g_j) \circ \mathscr{F}(\pi_j^1) = \mathscr{F}(f_j)$$
$$\mathscr{F}(f_j \times g_j) \circ \mathscr{F}(\pi_j^2) = \mathscr{F}(g_j)$$

for all $j$.
Then it suffice to define the $j$-th component of $\Delta$ to be the map $\mathscr{F}(f_j \times g_j)$ precomposed with the projection $\prod \mathscr{F}(U_i \times_U U_{i'}) \to \mathscr{F}(U_{\epsilon(j)} \times_U U_{\eta(j)})$ . $\square$

Now we focus on the category $\mathcal{J}_U$, the category $\mathbf{Cov}(U)$ with refinements as arrows.
We can order $\mathcal{J}_U$ by saying that $\mathcal{U} \leq \mathcal{U}'$ (we'll say that $\mathcal{U}'$ is finer then $\mathcal{U}$) if there exists a refinement map $(\epsilon, f) : \mathcal{U}' \to \mathcal{U}$.
$\mathcal{J}_U$ is a filtered partially ordered category, since for coverings $\mathcal{U} = \{U_i \to U\}$ and $\mathcal{U}' = \{U_i' \to U\}$ we have that $\mathcal{U}, \mathcal{U}' \leq \{U_i \times_U U_j'\}$.
Now, thanks to lemma 1.2.2, we can see $H(-, \mathscr{F})$ as a contravariant functor

$\mathcal{J}_U \to \underline{\mathbf{Ab}}$. For all $U \in \mathcal{C}$ we just define $\mathscr{F}^\dagger(U)$ to be the direct limit of groups

$$\mathcal{F}^\dagger(U) := \varinjlim_{\mathcal{J}_U} H(-, \mathscr{F})$$

This is indeed a presheaf:
for any map $V \to U$ in $\mathcal{C}$ and any covering $\mathcal{U} = \{U_i \to U\}$, there is a natural morphism of coverings $\mathcal{V} = \{U_i \times V \to V\} \to \mathcal{U}$, thus a group map $H(\mathcal{U}, \mathscr{F}) \to H(\mathcal{V}, \mathscr{F})$. These maps, for varying $\mathcal{U}$, give a morphism of direct system of groups, thus a group map $\mathscr{F}^\dagger(U) \to \mathscr{F}^\dagger(V)$. This is easily verified to be functorial in $U$ and the following diagram commute by construction.

$$\begin{tikzcd}
H(\mathcal{U}, \mathscr{F}) \arrow[r] \arrow[d] & \mathscr{F}^\dagger(U) \arrow[d] \\
H(\mathcal{V}, \mathscr{F}) \arrow[r] & \mathscr{F}^\dagger(V)
\end{tikzcd}$$

Moreover notice that $(-)^\dagger$ is a functor $\mathcal{P}(\mathcal{T}) \to \mathcal{P}(\mathcal{T})$. In fact for any presheaf morphism $\eta : \mathscr{F} \to \mathscr{G}$ and any covering $\mathcal{U}$ we get a commutative diagram

$$\begin{tikzcd}
\prod_i \mathscr{F}(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\eta(U_i)"'] & \prod_{i,j} \mathscr{F}(U_i \times_U U_j) \arrow[d, "\eta(U_i \times_U U_j)"] \\
\prod_i \mathscr{G}(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} \mathscr{G}(U_i \times_U U_j)
\end{tikzcd}$$

and thus group maps $H(\mathcal{U}, \eta) : H(\mathcal{U}, \mathscr{F}) \to H(\mathcal{U}, \mathscr{G})$. Again for varying $\mathcal{U}$ these give a morphism between direct system of groups, thus a group map $\mathscr{F}^\dagger(U) \to \mathscr{G}^\dagger(U)$.

**Remark 1.2.3** At this point it's definitely worth to spend a couple of word about Čech cohomology.
The groups $H(\mathcal{U}, \mathscr{F})$ are usually denoted by $H^0(\mathcal{U}, \mathscr{F})$.
It is easy to see that the functor $H^0(\mathcal{U}, -) : \mathcal{P}(\mathcal{T}) \to \underline{\mathbf{Ab}}$ is left exact, thus the right derived functor $H^0(\mathcal{U}, -)$ are defined.
The groups $H^q(\mathcal{U}, \mathscr{F})$ are referred to as **q-th Čech cohomology groups of $\mathscr{F}$ with respect to $\mathcal{U}$**.

There's an alternative way, more technical but more concrete, to define these groups.
For a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ define

$$C^q(\mathcal{U}, \mathscr{F}) := \prod_{(i_0,\dots,i_q) \in I^{q+1}} \mathscr{F}(U_{i_0} \times_U \cdots \times_U U_{i_q}).$$

Now define differentials $d^q : C^q(\mathcal{U}, \mathscr{F}) \to C^{q+1}(\mathcal{U}, \mathscr{F})$ by

$$(d^q s)_{i_0,\dots,i_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^\nu \mathscr{F}(\pi_\nu)(s_{i_0,\dots,\hat{i_\nu},\dots,i_{q+1}})$$

where $\pi_\nu$ is the projection $U_{i_0} \times_U \cdots \times_U U_{i_{q+1}} \to U_{i_0} \times_U \cdots \times_U \widehat{U_{i_\nu}} \times_U \times_U U_{i_{q+1}}$.
It can be shown that this yields a chain complex and that the groups $H^q(\mathcal{U}, \mathscr{F})$ are its $q$-th cohomology group (see [22] I 2.2 for details: in fact since $C^*(\mathcal{U}, -)$ coincide with $H^*(\mathcal{U}, \mathscr{F})$ in degree 0, it is enough to show that it is a universal $\partial$-functor).
The group $\mathscr{F}^\dagger(U)$ is usually denoted with $\check{H}^0(U, \mathscr{F})$.
We will see (proposition 1.2.7) that the functor $\check{H}^0(U, -) : \mathcal{P}(\mathcal{T}) \to \underline{\mathbf{Ab}}$ is left exact and thus its right derived functors $\check{H}^q(U, -)$ are defined.
The groups $\check{H}^q(U, \mathscr{F})$ are called **q-th Čech cohomology groups of $\mathscr{F}$ over $U$**.
Moreover one can actually prove (using the same technique we mentioned before) that $\check{H}^q(U, \mathscr{F}) \simeq \varinjlim_{\mathcal{U}} H^q(\mathcal{U}, \mathscr{F})$.

Lemma 1.2.2 implies another result useful for computations.

**Lemma 1.2.4** *Let $\mathcal{V}, \mathcal{V}' \in \mathbf{Cov}(V)$ and let $\mathcal{V} \to \mathcal{U}$ and $\mathcal{V}' \to \mathcal{U}$ be morphisms of coverings inducing the same arrow $V \to U$.*
*Then the images of $(s_i) \in H(\mathcal{U}, \mathscr{F})$ in $H(\mathcal{V}, \mathscr{F})$ and $H(\mathcal{V}', \mathscr{F})$ represent the same element in $\mathscr{F}^\dagger(V)$.*

**Proof**
Let $\mathcal{V} = \{V_j \to V\}$ and $\mathcal{V}' = \{V_{j'} \to V\}$ and let $(\epsilon, f)$ and $(\epsilon', f')$ be the refinements map.
Form the covering $\mathcal{W} = \{V_j \times_V V_{j'} \to V\}$. This is a common refinement for $\mathcal{V}$ and $\mathcal{V}'$, and we have naturally induced map of coverings $\mathcal{W} \to \mathcal{V} \to \mathcal{U}$ and $\mathcal{W} \to \mathcal{V}' \to \mathcal{U}$. These clearly induce the same arrow $V \to U$ then by lemma 1.2.2 they induce the same arrow $H(\mathcal{U}, \mathscr{F}) \to H(\mathcal{W}, \mathscr{F})$, i.e. there is a commutative diagram

$$\begin{tikzcd}
 & H(\mathcal{V},\mathscr{F}) \arrow[dr] & \\
H(\mathcal{U},\mathscr{F}) \arrow[ur] \arrow[dr] & & H(\mathcal{W},\mathscr{F}) \\
 & H(\mathcal{V}',\mathscr{F}) \arrow[ur] &
\end{tikzcd}$$

Then the images of $(s_i) \in H(\mathcal{U},\mathscr{F})$ in $H(\mathcal{V},\mathscr{F})$ and $H(\mathcal{V}',\mathscr{F})$ map to the same element in $H(\mathcal{W},\mathscr{F})$, which means they represent the same element in $\mathscr{F}^\dagger(V)$. $\square$

Now we will prove that applying $(-)^\dagger$ twice is actually left adjoint to the inclusion $\mathcal{S}(\mathcal{T}) \to \mathcal{P}(\mathcal{T})$.
By definition of adjunction this amount to prove that:

- $\left(\mathscr{F}^\dagger\right)^\dagger$ is a sheaf;
- for any $\mathscr{F} \in \mathcal{P}(\mathcal{T})$ we have a map $\theta : \mathscr{F} \to \left(\mathscr{F}^\dagger\right)^\dagger$;
- any morphism from $\mathscr{F}$ to a sheaf $\mathscr{G}$ factors as $\mathcal{F} \to \left(\mathscr{F}^\dagger\right)^\dagger \to \mathcal{G}$.

First of all notice that we have natural maps functorial in $U$

$$\mathscr{F}(U) = H(\{U \to U\},\mathscr{F}) \longrightarrow \varinjlim_{\mathcal{J}_U} H(-,\mathscr{F}) = \mathscr{F}^\dagger(U)$$

This gives a morphism $\theta : \mathscr{F} \to \mathscr{F}^\dagger$. Moreover if $\mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, this makes the following diagram commute

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}^\dagger \arrow[d] \\
\mathscr{G} \arrow[r] & \mathscr{G}^\dagger
\end{tikzcd}$$

If moreover $\mathcal{G}$ is a sheaf then $H(\mathcal{U},\mathcal{G}) = \mathcal{G}(U)$ for any $\mathcal{U} \in \mathcal{J}_U$, and we see at once that the map $\mathcal{G} \to \mathcal{G}^\dagger$ is an isomorphism.
Hence for any sheaf $\mathcal{G}$, any morphism $\mathcal{F} \to \mathcal{G}$ factors as $\mathcal{F} \to \mathcal{F}^\dagger \to G$.
This factorization is actually unique, since the 0 morphism $\mathcal{F} \to \mathcal{G}$ maps to the 0 morphism $\mathcal{F}^\dagger \to \mathcal{G}^\dagger$ just by definition.

Applying twice this reasoning, we see that any morphism form $\mathcal{F}$ to a sheaf

$\mathcal{G}$ factors in a unique way as as $\mathcal{F} \to \left(\mathscr{F}^\dagger\right)^\dagger \to \mathcal{G}$.
From now on we shall write $\mathscr{F}^\# := \left(\mathscr{F}^\dagger\right)^\dagger$.

So finally we just need to prove that $\mathscr{F}^\#$ is a sheaf.
The next proposition does the job

**Proposition 1.2.5**

*i) $\mathscr{F}^\dagger$ is a separated presheaf, i.e. the map $\mathscr{F}^\dagger(U) \to \prod_i \mathscr{F}^\dagger(U_i)$ is injective for any covering $\mathcal{U} = \{U_i \to U\}$.*

*ii) if $\mathscr{F}$ is separated then $\mathscr{F}^\dagger$ is a sheaf.*

**Proof**
**i**) Take $\bar{s} \in \ker\left(\mathscr{F}^\dagger(U) \to \prod \mathscr{F}^\dagger(U_i)\right)$. We have to show that $\bar{s} = 0$.
Let $\bar{s}$ be represented by an element $(s_j) \in H(\mathcal{V}, \mathscr{F})$ with $\mathcal{V} = \{V_j \to U\}$. We have to exhibit a refinement $g : \mathcal{W} \to \mathcal{V}$ such that $H(g, \mathscr{F})\big((s_j)\big) = 0$.
Applying lemma 1.2.4 (take $\mathcal{V} = \{V_{j'} \times_V V_j \to V_j\}_{j'}$ and $\mathcal{V}' = \{V_j \to V_j\}$) we see that $\bar{s}|_{V_j} = \theta(s_j)$. Now consider a common refinement of $\mathcal{U}$ and $\mathcal{V}$, say $\mathcal{W} = \{W_r \to U\}$, with refinement maps $(\epsilon, f) : \mathcal{W} \to \mathcal{U}$ and $(\epsilon', f') : \mathcal{W} \to \mathcal{V}$.
We have that for all $r$

$$\begin{cases} \bar{s}_{|W_r} & = (\bar{s}_{|U_{\epsilon(r)}})|_{W_r} = 0 \\ \bar{s}_{|W_r} & = \theta(s_{\epsilon'(r)})|_{W_r} = \theta(s_{\epsilon'(r)}|_{W_r}) \end{cases} \implies \theta(s_{\epsilon'(r)}|_{W_r}) = 0 \in \mathscr{F}^\dagger(W_r)$$

This means there are coverings $\mathcal{W}_r = \{W_{rt} \to W_r\}_r$ such that $s_{\epsilon'(r)}|_{W_r}$ maps to 0 via the map $\mathscr{F}(W_r) = H(\{W_r \to W_r\}, \mathscr{F}) \to H(\mathcal{W}_r, \mathscr{F})$, i.e. $(s_{\epsilon'(r)}|_{W_{rt}})_r = 0$.
Now consider the covering $\mathcal{W} = \{W_{rt} \to W_r \to U\}$. This is a refinement of $\mathcal{V}$ in a natural way and via the map $H(\mathcal{V}, \mathscr{F}) \to H(\mathcal{W}, \mathscr{F})$ we have that

$$(s_j) \mapsto (s_{\epsilon'(r)}|_{W_{rt}}) = 0.$$

We're done.

**ii**) Let $\mathcal{U} = \{U_i \to U\}$ be a covering and take

$$(\overline{s_i}) \in \ker\left(\prod \mathscr{F}^\dagger(U_i) \rightrightarrows \prod \mathscr{F}^\dagger(U_i \times_u U_j)\right).$$

Let $\overline{s_i} \in \mathscr{F}^\dagger(U_i)$ be represented by $(s_r^i)_r \in H(\mathcal{U}_i, \mathscr{F})$, where $\mathcal{U}_i = \{U_{ir} \to U_i\}_r$. Once again $(\overline{s_i})_{|U_{ir}} = \theta(s_r^i)$. Moreover we have the following commutative diagram

$$\begin{tikzcd}
 & U_{ir} \arrow[rr] & & U_i \\
U_{ir} \times_U U_{jt} \arrow[ur] \arrow[rr] \arrow[dr] & & U_i \times_U U_j \arrow[ur] \arrow[dr] & \\
 & U_{jt} \arrow[rr] & & U_j
\end{tikzcd}$$

Thus we find that

$$\begin{cases}
\overline{s_i}_{|U_{ir}\times_U U_{jt}} = \left(\overline{s_i}_{|U_i\times_U U_j}\right)\big|_{U_{ir}\times_U U_{jt}} = \left(\overline{s_j}_{|U_i\times_U U_j}\right)\big|_{U_{ir}\times_U U_{jt}} = \overline{s_j}_{|U_{ir}\times_U U_{jt}} \\
\overline{s_i}_{|U_{ir}\times_U U_{jt}} = \left(\overline{s_i}_{|U_{ir}}\right)\big|_{U_{ir}\times_U U_{jt}} = \theta(s_r^i)|_{U_{ir}\times_U U_{jt}} = \theta(s^i_{r|_{U_{ir}\times_U U_{jt}}}) \\
\overline{s_j}_{|U_{ir}\times_U U_{jt}} = \left(\overline{s_j}_{|U_{jt}}\right)\big|_{U_{ir}\times_U U_{jt}} = \theta(s_t^j)|_{U_{ir}\times_U U_{jt}} = \theta(s^j_{t|_{U_{jt}\times_U U_{jt}}})
\end{cases}$$

$$\implies \theta(s^i_{r|_{U_{ir}\times_U U_{jt}}}) = \theta(s^j_{t|_{U_{jt}\times_U U_{jt}}})$$

But $\theta$ is a monomorphism. In fact if $s \in \mathscr{F}(U)$ maps to $0 \in \mathscr{F}^\dagger(U)$ then there exists a covering $\{U_i \to U\}$ such that $s_{|_{U_i}} = 0$ for all $i$. But $\mathscr{F}$ is separated, thus $s = 0$.
Then we get $s^i_{r|_{U_{ir}\times_U U_{jt}}} = s^j_{t|_{U_{ir}\times_U U_{jt}}}$, which means that

$$(s_r^i) \in \ker\left(\prod \mathscr{F}(U_{ir}) \rightrightarrows \prod \mathscr{F}(U_{ir} \times_U U_{jt})\right) = H(\{U_{ir} \to U\}, \mathscr{F}).$$

Denote with $\bar{s}$ the image of $(s_r^i)$ in $\mathscr{F}^\dagger(U)$. We claim that $\bar{s}_{|_{U_i}} = \overline{s_i}$. But this is straight forward since we have a natural map of coverings $\{U_{ir} \to U_i\}_r \to \{U_{ir} \to U\}_{ir}$ which gives the commutative diagram

$$\begin{tikzcd}
H(\{U_{ir} \to U\}, \mathscr{F}) \arrow[r] \arrow[d] & H(\{U_{ir} \to U_i\}, \mathscr{F}) \arrow[d] \\
\mathscr{F}^\dagger(U) \arrow[r] & \mathscr{F}^\dagger(U_i)
\end{tikzcd}$$

Now we readily see that $(s_r^i) \mapsto \bar{s}_{|_{U_i}}$ via

$$H(\{U_{ir} \to U\}, \mathscr{F}) \to H(\{U_{ir} \to U_i\}, \mathscr{F}) \to \mathscr{F}^\dagger(U_i)$$

and $(s_r^i) \mapsto \overline{s_i}$ via

$$H(\{U_{ir} \to U\}, \mathscr{F}) \to \mathscr{F}^\dagger(U) \to \mathscr{F}^\dagger(U_i).$$

We're done. $\square$

**Corollary 1.2.6** *The functor* $\# : \mathscr{F} \mapsto \mathscr{F}^{\#}$ *is left adjoint to the inclusion* $\mathcal{S}(\mathcal{T}) \to \mathcal{P}(\mathcal{T})$. $\square$

It is useful to study the exactness property of sheafification.

**Proposition 1.2.7** $(-)^{\dagger}$ *is left exact.*

**Proof**
Let $0 \to \mathscr{F} \xrightarrow{\eta} \mathscr{G} \xrightarrow{\mu} \mathscr{C} \in \mathcal{P}(\mathcal{T})$ be an exact sequence of presheaves, i.e. $0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{C}(U)$ is exact for all $U \in \mathcal{C}$. We have to show that $0 \to \mathscr{F}^{\dagger}(U) \to \mathscr{G}^{\dagger}(U) \to \mathscr{C}^{\dagger}(U)$ is exact for all $U$.
Let $\bar{s} \in \ker\big(\mathscr{F}^{\dagger}(U) \to \mathscr{G}^{\dagger}(U)\big)$ be represented by $(s_i) \in H(\mathcal{U}, \mathscr{F}), \mathcal{U} = \{U_i \to U\}$.
Then there exists a refinement $f : \mathcal{U}' = \{U'_j \to U\} \to \mathcal{U}$ such that $H(f, \mathscr{G}) \circ H(\mathcal{U}, \eta)\big((s_i)\big) = 0$.
But the following diagram commute

$$\begin{tikzcd}
H(\mathcal{U}, \mathscr{F}) \arrow[r, "H(\mathcal{U},\eta)"] \arrow[d, "H(f,\mathscr{F})"'] & H(\mathcal{U}, \mathscr{G}) \arrow[d, "H(f,\mathscr{G})"] \\
H(\mathcal{V}, \mathscr{F}) \arrow[r, "H(\mathcal{V},\eta)"'] & H(\mathcal{V}, \mathscr{G})
\end{tikzcd}$$

so that $H(\mathcal{V}, \eta) \circ H(f, \mathscr{F})\big((s_i)\big) = H(f, \mathscr{G}) \circ H(\mathcal{U}, \eta)\big((s_i)\big) = 0$.
But recalling how $H(\mathcal{V}, \eta)$ is defined and using the fact that $\eta(U_i)$ is injective for all $i$ we get that $H(f, \mathscr{F})\big((s_i)\big) = 0$ which means that $\bar{s} = 0$.
It is quite clear that $\operatorname{im}\big(\mathscr{F}^{\dagger} \to \mathscr{G}^{\dagger}\big) \subseteq \ker\big(\mathscr{G}^{\dagger} \to \mathscr{C}^{\dagger}\big)$.
Finally let $\bar{s} \in \ker\big(\mathscr{G}^{\dagger} \to \mathscr{C}^{\dagger}\big)$ be represented by $(s_i) \in H(\mathcal{U}, \mathscr{G})$.
Once again we find a refinement $f : \mathcal{U}' \to \mathcal{U}$ such that $H(\mathcal{V}, \mu) \circ H(f, \mathscr{F})\big((s_i)\big) = 0$. Recalling how $H(\mathcal{V}, \eta)$ is defined and using exactness of $\mathscr{F}(U_i) \to \mathscr{G}(U_i) \to \mathscr{C}(U_i)$, this time we get an element $(t_j) \in H(\mathcal{U}', \mathscr{F})$ such that $H(\mathcal{U}', \eta)\big((t_j)\big) = H(f, \mathscr{F})\big((s_i)\big)$. Then the image of $(t_j)$ in $\mathscr{F}^{\dagger}(U)$ maps to $\bar{s}$.
We're done. $\square$

**Corollary 1.2.8** *Sheafification is an exact functor.*

**Proof**
Since it is a left adjoint, it is right exact.
Plus the composition $i \circ \# = \dagger \circ \dagger$ ($i$ being the inclusion $\mathcal{S}(\mathcal{T}) \to \mathcal{P}(\mathcal{T})$) is left exact by what we have just proved. But since $i$ is fully faithful $\#$ has to be left exact as well. $\square$

## 1.3 $\mathcal{S}(\mathcal{T})$ is abelian

It's easy to prove that $\mathcal{P}(\mathcal{T})$ is an abelian category.

Since $\mathcal{S}(\mathcal{T})$ is a full subcategory of $\mathcal{P}(\mathcal{T})$, it inherits from it the additive structure.
Let $\alpha : \mathscr{F} \to \mathscr{G}$ be a morphism in $\mathcal{S}(\mathcal{T})$ and denote with $\mathscr{K}$ and $\mathscr{C}$ its kernel and cokernel in the category of presheaves.
The following proposition shows that $\mathcal{S}(\mathcal{T})$ is abelian.

**Proposition 1.3.1** *The following are true in $\mathcal{S}(\mathcal{T})$:*

**i)** $\mathscr{K} = \ker(\alpha)$.

**ii)** $\mathscr{C}^{\#} = \operatorname{coker}(\alpha)$.

**iii)** $\ker\big(\operatorname{coker}(\alpha)\big) \simeq \operatorname{coker}\big(\ker(\alpha)\big)$.

**Proof**
**i)** First we show that $\mathscr{K}$ is in fact a sheaf.
Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathscr{K}(U) \arrow[r] \arrow[d] & \prod \mathscr{K}(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod \mathscr{K}(U_i \times_U U_j) \arrow[d] \\
\mathscr{F}(U) \arrow[r] \arrow[d] & \prod \mathscr{F}(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod \mathscr{F}(U_i \times_U U_j) \arrow[d] \\
\mathscr{G}(U) \arrow[r] & \prod \mathscr{G}(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod \mathscr{F}(U_i \times_U U_j)
\end{tikzcd}$$

Since all the columns and the two bottom rows are exact, the first row is exact as well, which means that $\mathscr{K}$ is a sheaf.
Next we show that $\mathscr{K} = \ker\big(\mathscr{F} \to \mathscr{G}\big)$, i.e. that $0 \to \mathscr{K} \to \mathscr{F}$ is exact in $\mathcal{S}(\mathcal{T})$. But this is exact in $\mathcal{P}(\mathcal{T})$ and since $\#$ is exact (Corollary 1.22) and $\mathscr{K}$ and $\mathscr{F}$ are sheaves, we get that $0 \to \mathscr{K} \to \mathscr{F}$ is exact in $\mathcal{S}(\mathcal{T})$.
We're done.

**ii)** This is true since sheafification is a left adjoint and thus it's right exact.

**iii)** Let $\mathscr{I} = \ker\big(\operatorname{coker}(\alpha)\big)$ and $\mathscr{J} = \operatorname{coker}\big(\ker(\alpha)\big)$ in the category of presheaves. Then there exist a unique isomorphism of presheaves $\bar{\alpha}$ such that $\alpha$ factorize as $\mathscr{F} \to \mathscr{J} \xrightarrow{\bar{\alpha}} \mathscr{I} \to \mathscr{G}$.

Applying sheafification we get a factorization $\mathscr{F} \to \mathscr{J}^{\#} \xrightarrow{\bar{\alpha}^{\#}} \mathscr{I}^{\#} \to \mathscr{G}$ where $\bar{\alpha}^{\#}$ is again an isomorphism.
Then we just need to show that $\mathscr{I}^{\#} = \ker\big(\mathrm{coker}(\alpha)\big)$ and $\mathscr{J}^{\#} = \mathrm{coker}\big(\ker(\alpha)\big)$ in the category of sheaves.
The assertion for $\mathscr{J}^{\#}$ follows from i) and ii).
For $\mathscr{I}^{\#}$ notice that we have an exact sequence of presheaves $0 \to \mathscr{I} \to \mathscr{G} \to \mathscr{C}$. Since sheafification is exact (Corollary 1.22) we get a sequence of sheaves $0 \to \mathscr{I}^{\#} \to \mathscr{G} \to \mathscr{C}^{\#}$ which is exact in the category of presheaves.
Thus $\mathscr{I}^{\#} = \ker\big(\mathscr{G} \to \mathscr{C}^{\#}\big) = \ker\big(\mathrm{coker}(\alpha)\big)$ in $\mathcal{P}(\mathcal{T})$. But kernel in the category of sheaves are just kernels in the category of presheaves.
We're done. □

Now that we proved that $\mathcal{S}(\mathcal{T})$ is abelian, we may wonder whether the section functor is exact.

**Lemma 1.3.2** *The section functor is left exact.*

**Proof**
Notice that $\Gamma_V$ factor as

$$\mathcal{S}(\mathcal{T}) \longrightarrow \mathcal{P}(\mathcal{T}) \xrightarrow{\Gamma_V} \underline{\mathbf{Ab}}.$$

Now the inclusion $\mathcal{S}(\mathcal{T}) \to \mathcal{P}(\mathcal{T})$ is a right adjoint and thus left exact and the section functor is clearly exact on $\mathcal{P}(\mathcal{T})$. Thus $\Gamma_V : \mathcal{S}(\mathcal{T}) \to \underline{\mathbf{Ab}}$ is left exact. □

In addition we show that $\mathcal{S}(\mathcal{T})$ satisfies axioms Ab3, Ab4 and Ab5 introduced by Grothendieck in its Tohoku paper ([8]). We recall them here.

**(Ab 1)** $\mathcal{C}$ admits arbitrary direct sum.

**(Ab 2)** Direct sum of monomorphisms is a monomorphism.

**(Ab 3)** If $\{A_i\}$ is a direct system of subobject of $A \in \mathcal{C}$ together with arrows $u_i : A_i \to B$ for each $i$, such that $u_j$ extend $u_i$ whenever $A_i \subseteq A_j$, then there exists a unique arrow $\sum_i A_i \to B$ extending all the $u_i$.

**Lemma 1.3.3** $\mathcal{S}(\mathcal{T})$ *satisfies* **Ab 3**.

**Proof**
Let $\{\mathscr{F}_i\}$ be a collection of sheaves. Clearly the presheaf defined by

$$\mathscr{F}(U) = \bigoplus_i \mathscr{F}_i(U)$$

together with the natural injections $\mathscr{F}_i \to \mathscr{F}$ is the coproduct of the $\mathscr{F}_i$ in the category of presheaves. But # is a left adjoint, thus it preserve colimit. It follows that $\mathscr{F}^{\#}$ is the direct sum of the $\mathscr{F}_i$ in the category of sheaves. □

**Lemma 1.3.4** *$\mathcal{S}(\mathcal{T})$ satisfies* **Ab 4**.

**Proof**
Let $\{u_i : \mathscr{F}_i \to \mathscr{G}_i\}$ be a collection of monomorphism in $\mathcal{S}(\mathcal{T})$. Then $\oplus u_i : \oplus \mathscr{F}_i \to \oplus \mathscr{G}_i$ is a monomorphism in $\mathcal{P}(\mathcal{T})$. Clearly the direct sum of the $u_i$ in $\mathcal{S}(\mathcal{T})$ is $(\oplus u_i)^{\#}$, which is a monomorphism since # is exact. □

**Lemma 1.3.5** *$\mathcal{S}(\mathcal{T})$ satisfies* **Ab 5**.

**Proof**
First notice that the property hold in $\mathcal{P}(\mathcal{T})$.
In fact consider a direct system $\{\mathscr{F}_i\}$ of subobject of a given presheaf $\mathscr{F}$, together with morphisms $u_i : \mathscr{F}_i \to \mathscr{G}$ into another presheaf. Let $\sum_i \mathscr{F}_i$ be the supremum of the $\mathscr{F}_i$ in the category of presheaves. By definition $\sum_i \mathscr{F}_i = \mathrm{im}(\bigoplus_i \mathscr{F}_i \to \mathscr{F})$ is the presheaf $U \mapsto \sum_i \mathscr{F}_i(U)$ (the subgroup of $\mathscr{F}(U)$ generated by the $\mathscr{F}_i(U)$). Obviously there exists a unique morphism of presheaves $u : \sum_i \mathscr{F}_i \to \mathscr{G}$ extending the $u_i$ (define $u(U) : \sum_i a_i \mapsto \sum u_i(a_i)$). Thus $\mathcal{P}(\mathcal{T})$ has property (**Ab5**).
Now consider a direct system $\{\mathscr{F}_i\}$ of subobject of a given sheaf $\mathscr{F}$, together with morphisms $u_i : \mathscr{F}_i \to \mathscr{G}$ into another sheaf. By proposition 1.3.1 $\left(\sum_i \mathscr{F}_i\right)^{\#}$ is the supremum of the $\mathscr{F}_i$ in the category of sheaves (as before $\sum_i \mathscr{F}_i$ is the supremum in the category of presheaves).
Now in $\mathcal{P}(\mathcal{T})$ there exists a morphism $u : \sum_i \mathscr{F}_i \to \mathscr{G}$ extending the $u_i$, and since $\mathscr{G}$ is a sheaf this factors as $\sum_i \mathscr{F}_i \to \left(\sum_i \mathscr{F}_i\right)^{\#} \xrightarrow{u^{\#}} \mathscr{G}$.
Clearly $u^{\#}$ is the unique morphism extending the $u_i$. □

## 1.4 Injective objects in $\mathcal{S}(\mathcal{T})$

The last task we need to accomplish is proving that the category of sheaves over a site has enough injective objects.
To do so we will use a theorem proven by Grothendieck in its Tohoku paper ([8]).

First we shall recall the definitions of subobjects and generators.
Consider two monomorphism $u : B \to A$ and $u' : B' \to A$. We say that $B \leq B'$ if there is a factorization

$$\begin{tikzcd} B \arrow[d, "w"'] \arrow[dr, "u"] & \\ B' \arrow[r, "u'"'] & A \end{tikzcd}$$

Define an equivalence relation $\sim$ on the collection of monomorphisms with codomain $A$ by saying that $u \sim u'$ if and only if $u \leq u'$ and $u' \leq u$.

**Definition 1.4.1** *We call* **subobject of** $A$ *an equivalence class of* $\sim$.

Now we recall what a family of generator is:

**Definition 1.4.2** *We say that a collection* $\{U_i\}$ *of object in* $\mathcal{C}$ *is a family of generators for* $\mathcal{C}$ *if for any subobject* $B \subseteq A$ *we have*

$$\mathrm{im}(U_i \to A) \subseteq B \; \forall \, U_i \to A \implies B \simeq A$$

We're ready to state the theorem:

**Theorem 1.4.3** *A Grothendieck category (i.e. an abelian category with generators satisfying axioms Ab 3, Ab 4 and Ab5) has enough injective objects.*

**Proof** (sketch)
We briefly review the steps of the proof. For details see the original source [8] or [17].

First of all is not difficult to see that if $\{U_i\}$ is a family of generators then $U = \bigoplus_i U_i$ is a generator.

Next one can show that the lifting property for injective objects just need to be checked on maps $V \to U$. Precisely an object $I$ is injective if for any subobject $V \to U$, any map $V \to A$ extends to a map $U \to A$. This is analogous to the Baer's criterion for injective modules (see [30] 2.3.1).

Now fix an object $A$. We want to prove it is a subobject of some injective object.
Define

$$I = \{V \to A | V \subseteq U\}$$

and consider the map

$$\bigoplus_{i \in I} V_i \to A \times \big(\bigoplus_{i \in I} U\big)$$

which $(V_i \to A)$-component is the the product of $V_i \to A$ and the inclusion $V_i \to \bigoplus_{i\in I} U$ in the $(V_i \to A)$-component.
Define $M_1(A)$ to be the quotient of this map.
We clearly have a canonical map $A \to M_1(A)$ and one can prove this is an injection.

Now we define by induction $M_{\alpha+1}(A) := M_1\big(M_\alpha(A)\big)$ for all $\alpha \geq 1$.
Further using transfinite induction we can define $M_\alpha$ for $\alpha$ an ordinal number.

Finally one choose the smallest infinite ordinal $\Omega$ which cardinality is strictly grater then the cardinality of the collection of subobject of $U$.
It can be proved that $M_\Omega(A)$ is injective and this conclude the proof. $\square$

To apply the theorem we still need to prove that $\mathcal{S}(\mathcal{T})$ admit a family of generators.
For doing so it suffices to show that the section functors are representable.
In fact assume that $\Gamma_V$ is represented by a sheaf $\mathscr{F}_V$ for all objects $V$. We claim that $\{\mathscr{F}_V\}_{V\in\mathcal{C}}$ is a family of generator for $\mathcal{P}(\mathcal{T})$.
In fact if $\mathscr{G} \subsetneq \mathscr{F}$ is a proper subobject then $\mathscr{G}(V) \subsetneq \mathscr{F}(V)$ for some $V \in \mathcal{C}$ and by representability we have a commutative diagram

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{P}(\mathcal{T})}(\mathscr{F}_V, \mathscr{F}) \\
\mathscr{G}(V) \arrow[r, "\sim"] \arrow[u, hook] & \mathrm{Hom}_{\mathcal{P}(\mathcal{T})}(\mathscr{F}_V, \mathscr{G}) \arrow[u, hook]
\end{tikzcd}$$

Thus the element in $\mathscr{F}(V)\backslash\mathscr{G}(V)$ correspond to a morphism $\mathscr{F}_V \to \mathscr{F}$ which does not factorize as $\mathscr{F}_V \to \mathscr{G} \to \mathscr{F}$.

The next lemma does the job.

**Proposition 1.4.4** *Every section functor is representable.*

**Proof**
Define $\mathscr{F}_V$ as follow. Let

$$\mathscr{F}_V(U) = \bigoplus_{\mathrm{Hom}_{\mathcal{C}}(U,V)} \mathbb{Z}.$$

and for any $u : U' \to U$ define

$$\mathscr{F}_V(u) : \bigoplus_{\mathrm{Hom}_{\mathcal{C}}(U,V)} \mathbb{Z} \longrightarrow \bigoplus_{\mathrm{Hom}_{\mathcal{C}}(U',V)} \mathbb{Z}$$

to be the map whose $h$ component ($h \in \mathrm{Hom}_{\mathcal{C}}(U,V)$) is the injection on the $h \circ u$ component (from now on if $h : U \to V$, we will denote with

$\mathbb{Z} \xrightarrow{h} \bigoplus_{\mathrm{Hom}_{\mathcal{C}}(U,V)} \mathbb{Z}$ the injection in the $h$ component).
In other words $\mathscr{F}_V(u)$ is the unique arrow such that the following diagram commute for all $h$

$$\begin{tikzcd}
 & \bigoplus\limits_{\mathrm{Hom}_{\mathcal{C}}(U,V)} \mathbb{Z} \arrow[d] \\
\mathbb{Z} \arrow[ur, "h"] \arrow[r, "h\circ u"'] & \bigoplus\limits_{\mathrm{Hom}_{\mathcal{C}}(U',V)} \mathbb{Z}
\end{tikzcd}$$

Clearly $\mathscr{G}(V) \simeq \mathrm{Hom}_{\underline{\mathbf{Ab}}}\big(\mathbb{Z}, \mathscr{G}(V)\big)$, thus it will suffice to construct an isomorphism

$$\mathrm{Hom}_{\underline{\mathbf{Ab}}}\big(\mathbb{Z}, \mathscr{G}(V)\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{P}(\mathcal{T})}\big(\mathscr{F}_V, \mathscr{G}\big)$$

functorial in $\mathscr{G}$.
To any arrow $v : \mathbb{Z} \to \mathscr{G}(V)$ we associate a natural transformation $\eta_v$ whose component $\eta_v(U)$ is the map $\bigoplus_{\mathrm{Hom}(U,V)} \mathbb{Z} \to \mathscr{G}(U)$ whose $h$ component is $\mathbb{Z} \to \mathscr{G}(V) \xrightarrow{\mathscr{G}(h)} \mathscr{G}(U)$.
In other words $\eta_v(U)$ is the unique arrow such that the following diagram commute for all $h$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "h"] \arrow[d, "v"'] & \bigoplus\limits_{\mathrm{Hom}(U,V)} \mathbb{Z} \arrow[d, "\eta_v(U)"] \\
\mathscr{G}(V) \arrow[r, "\mathscr{G}(h)"'] & \mathscr{G}(U)
\end{tikzcd}$$

Conversely to any natural transformation $\eta : \mathscr{F}_V \to \mathscr{G}$ we associate the $1_V$ component of the map $\eta(V) : \mathscr{F}_V(V) \to \mathscr{G}(V)$, i.e. the composition

$$v_\eta : \mathbb{Z} \xrightarrow{1_V} \bigoplus_{\mathrm{Hom}(V,V)} \mathbb{Z} \xrightarrow{\eta(V)} \mathscr{G}(V)$$

This is easily seen to be functorial.
We show now that these are mutually inverse.
$v_{\eta_v}$ is the $1_V$ component of the map $\eta_v(V) : \mathscr{F}_V(V) \to \mathscr{G}(V)$, i.e. the composition

$$\mathbb{Z} \xrightarrow{1_V} \bigoplus_{\mathrm{Hom}(V,V)} \mathbb{Z} \xrightarrow{\eta_v(V)} \mathscr{G}(V)$$

But by definition of $\eta_v$ the following diagram commute

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "1_V"] \arrow[d, "v"'] & \bigoplus_{\mathrm{Hom}(U,V)} \mathbb{Z} \arrow[d, "\eta_v(V)"] \\
\mathscr{G}(V) \arrow[r, equal, "\mathscr{G}(1_V)"'] & \mathscr{G}(V)
\end{tikzcd}$$

and thus the composition above it's just $v$.
On the other hand consider $\eta_{v_\eta}$.
By definition of $v_\eta$ we obtain for all $h : U \to V$ the commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "1_V"] \arrow[dr, "v_\eta"'] & \bigoplus_{\mathrm{Hom}(V,V)} \mathbb{Z} \arrow[r, "\mathscr{F}_V(h)"] \arrow[d, "\eta(V)"] & \bigoplus_{\mathrm{Hom}(U,V)} \mathbb{Z} \arrow[d, "\eta(U)"] \\
& \mathscr{G}(V) \arrow[r, "\mathscr{G}(h)"'] & \mathscr{G}(U)
\end{tikzcd}$$

But the composition on the first row is the $1_V$ component of $\mathscr{F}_V(h)$, which by definition is the injection $\mathbb{Z} \to \mathscr{F}_V(U)$ in the $h$ component.
By the uniqueness property in the definition of $\eta_{v_\eta}$ we see at once that $\eta_{v_\eta} = \eta$. $\square$

Then we obtain

**Corollary 1.4.5** *$\{\mathscr{F}_V\}_{V\in\mathcal{C}}$ is a family of generator for $\mathcal{P}(\mathcal{T})$.*

Furthermore we easily obtain

**Corollary 1.4.6** *$\{\mathscr{F}_V^{\#}\}_{V\in\mathcal{C}}$ is a family of generator for $\mathcal{S}(\mathcal{T})$.*

**Proof**
By the adjoint property of sheafification we see that for any sheaf $\mathscr{G}$

$$\mathrm{Hom}_{\mathcal{S}(\mathcal{T})}\big(\mathscr{F}_V^{\#}, \mathscr{G}\big) \simeq \mathrm{Hom}_{\mathcal{P}(\mathcal{T})}\big(\mathscr{F}_V, \mathscr{G}\big) \simeq \mathscr{G}(U).$$

The proof now runs as before. $\square$

## 1.5 The canonical topology

It is an interesting question whether representable presheaves are sheaves for a given site.
Actually there is a topology, called canonical, which is the finer such that all representable presheaves are sheaves.
We're going to construct this now.
From now on we write $Z(U)$ for $\mathrm{Hom}_{\mathcal{C}}(U, Z)$.

**Definition 1.5.1** *We say that a covering $\{U_i \to U\}_{i \in I}$ is* **surjective** *if*

$$Z(V) \longrightarrow \prod_i Z(U_i)$$

*is injective for all $Z \in \mathcal{C}$.*
*We say that it is universally surjective if for any arrow $V \to U$ the covering $\{U_i \times_U V \to V\}$ is surjective.*

**Remark 1.5.2** This generalizes the notion of (universal) epimorphism: in fact a (universal) epimorphism is just a (universally) surjective covering consisting of a single arrow.

**Remark 1.5.3** If $\mathcal{C}$ has arbitrary coproduct this condition can be stated in a nicer way. In fact $\prod_i Z(U_i) \simeq Z(\amalg_i U_i)$ so that $\{U_i \to U\}_{i \in I}$ is surjective if and only if $\amalg_i U_i \to U$ is an epimorphism.
In many concrete category this means that $U = \bigcup_i \mathrm{im}(U_i \to U)$.

**Definition 1.5.4** *We say that a covering $\{U_i \to U\}_{i \in I}$ is* **effectively surjective** *(or strictly surjective) if*

$$Z(U) \longrightarrow \prod_i Z(U_i) \rightrightarrows \prod_{i,j} Z(U_i \times_U U_j)$$

*is exact for all $Z \in \mathcal{C}$.*

**Remark 1.5.5** When we consider coverings consisting of a single arrow, this time we get the notion of **strict epimorphism**.
Precisely this is an arrow $U' \to U$ such that the sequence

$$U' \times_U U' \rightrightarrows U' \to U$$

is exact.

**Example 1.5.6** We claim that $\{\varphi_i : U_i \to U\}_{i\in I}$ is effectively surjective in <u>**Set**</u> if and only if $U = \cup_i \varphi_i(U_i)$.
We already proved the only if implication.
Conversely assume that $U = \bigcup_i \varphi_i(U_i)$.
We already know that $Z(U) \to \prod_i Z(U_i)$ is injective and it's quite clear that $\mathrm{im}\big(Z(U) \to \prod_i Z(U_i)\big) \subseteq \ker\big(\prod_i Z(U_i) \rightrightarrows \prod_{i,j} Z(U_i \times_U U_j)\big)$.
We're just left to prove that a map $h : \amalg_i U_i \to Z$ on which the last two arrow agree come from a map $h' : U \to Z$.
But for any $x \in U$, $x = \varphi_i(x_i)$ for some $x_i \in U_i$, so we may define $h'(x) := h(x_i)$. It is straight forward to see that this doesn't depend from the choice of $x_i$, thus everything work.

Clearly every covering in the canonical topology must satisfy these conditions. Thus the set of coverings of an object $U$ must be a subset of its effectively surjective coverings.
Since in addition we need the condition (**T1**) to be satisfied, we make the following definition:

**Definition 1.5.7** *We say that an effectively surjective covering $\{U_i \to U\}_{i\in I}$ is* **universal** *if for any arrow $V \to U$ the covering $\{U_i \times_U V \to V\}_{i\in I}$ is again effectively surjective.*

**Example 1.5.8** Again universal effective surjective coverings in <u>**Set**</u> are the one such that $U = \cup_i \varphi_i(U_i)$.
In fact let $\varphi : V \to U$. For any $x \in V$ we have that $\varphi(x) = \varphi_i(x_i) \in U$ for some $x_i \in U_i$. Thus $(x_i, x) \in U_i \times_U V$ maps to $x \in V$, which means that $V = \bigcup_i \mathrm{im}(U_i \times_U V \to V)$.

We just define $\mathbf{Cov}(U)$ to be the collection of all universal effectively surjective covering of $U$.
It is quite clear, provided this is a topology, that it is the finest such that all representable presheaves are sheaves.
We just need to prove that this defines indeed a topology.

**Proposition 1.5.9** *Taking coverings as above define a topology.*

**Proof**
(**T1**) is clearly satisfied.
(**T3**) is fulfilled as well, since if $U' \to U$ is an isomorphism then $U' \times_U U' \simeq U'$ and for any arrow $V \to U$ we have that $U' \times_U V \simeq V$.
Finally we need to prove that (**T2**) holds true.
Let $\{U_i \to U\}_{i\in I}$ and $\{U_{ij} \to U_i\}_{j\in J_i}$ be universal effectively surjective coverings.

First of all we easily see that $\{U_{ij} \to U_i \to U\}$ is universally surjective. In fact for any $V \to U$ we have a factorization

$$\begin{tikzcd}
Z(U) \arrow[rr] \arrow[dr] & & \prod_i \prod_j Z(U_{ij}) \\
& \prod_i Z(U_i) \arrow[ur] &
\end{tikzcd}$$

the diagonal arrows being injective by hypothesis, and for any arrow $V \to U$ we have that $U_{ij} \times_U V \simeq U_{ij} \times_{U_i} (U_i \times_U V)$.
Secondly we show that $\{U_{ij} \to U_i \to U\}$ is effectively surjective.
Form the following commutative diagram:

$$\begin{tikzcd}
U & \prod_i U_i \arrow[l] & \prod_{i,j} U_i \times_U U_j \arrow[l, shift right] \arrow[l, shift left] \\
& \prod_i \prod_s U_{is} \arrow[u] & \prod_{i,j} \prod_{s,t} U_{is} \times_U U_{jt} \arrow[l, shift right] \arrow[l, shift left] \arrow[u] \\
& \prod_i \prod_{s,t} U_{is} \times_{U_i} U_{it} \arrow[u, shift left] \arrow[u, shift right] \arrow[ur] &
\end{tikzcd}$$

(the arrows to be understood).
In it the first row is exact by hypothesis and the first column is exact since it is product of sequences exact by hypothesis.
Then we get the following commutative diagram

$$\begin{tikzcd}
X(U) \arrow[r] & \prod_i X(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,j} X(U_i \times_U U_j) \arrow[d] \\
& \prod_i \prod_s X(U_{is}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_{i,j} \prod_{s,t} X(U_{is} \times_U U_{jt}) \arrow[dl] \\
& \prod_i \prod_{s,t} X(U_{is} \times_{U_i} U_{it}) &
\end{tikzcd}$$

in which the first row and column are exact.
The proof now follows by diagram chasing.
Take an element $(\alpha_{ij}) \in \ker\Big(\prod_i \prod_s X(U_{is}) \rightrightarrows \prod_{i,j}\prod_{s,t} X(U_{ij} \times_U U_{st})\Big)$.
Since the triangle commute, $(\alpha_{ij}) \in \ker\Big(\prod_i \prod_s X(U_{is}) \rightrightarrows \prod_i \prod_{s,t} X(U_{is} \times_U U_{it})\Big)$ as well.
Thus there is an element $(\alpha_i) \in \prod_i X(U_i)$ mapping to $(\alpha_{ij})$.
But since the square commute, $(\alpha_i)$ is in the kernel of

$$\prod_i X(U_i) \rightrightarrows \prod_{i,j} X(U_i \times_U U_j) \longrightarrow \prod_{i,j}\prod_{s,t} X(U_{is} \times_U U_{jt}).$$

If we prove that the second arrow is injective, it will follow that $(\alpha_i) \in \ker\Big(\prod_i X(U_i) \rightrightarrows \prod_{i,j} X(U_i \times_U U_j)\Big)$.
Then we would find an element $\alpha \in X(U)$ mapping to $(\alpha_i)$, and thus to $(\alpha_{ij})$, and we would be done.
Since the map $\prod_{i,j} X(U_i \times_U U_j) \to \prod_{i,j}\prod_{s,t} X(U_{is} \times_U U_{it})$ is the product of the maps $X(U_i \times_U U_j) \to \prod_{s,t} X(U_{is} \times_U U_{it})$, the following lemma does the job.

**Lemma 1.5.10** *$\{U_{is} \times_U U_{jt} \to U_i \times_U U_j\}$ is a surjective covering.*

**Proof**
It is easy to see that the following squares are cartesian

$$\begin{tikzcd}
U_{is} \arrow[r] & U_i & \\
U_{is} \times_U U_{jt} \arrow[u] \arrow[r] & U_i \times_U U_{jt} \arrow[u] \arrow[r] \arrow[d] & U_i \times_U U_j \arrow[d] \\
 & U_{jt} \arrow[r] & U_j
\end{tikzcd}$$

Thus the coverings $\{U_i \times_U U_{jt} \to U_i \times_U U_j\}_t$ and $\{U_{is} \times_U U_{jt} \to U_i \times_U U_{jt}\}_s$ for all $t$ are surjective by hypothesis.
Finally $\{U_{is} \times_U U_{jt} \to U_i \times_U U_j\}$ is surjective (being "composition" of surjective coverings). □

We're just left to show that $\{U_{ij} \to U_i \to U\}$ is a universal effective surjective covering, i.e. that for any arrow $V \to U$ the covering $\{U_{ij} \times_U V \to U_i \times_U V \to V\}$ is effectively surjective.

But $U_{ij} \times_U V \simeq U_{ij} \times_{U_i} (U_i \times_U V)$ and by hypothesis $\{U_{ij} \times_{U_i} (U_i \times_U V) \to U_i \times_U V\}_j$ and $\{U_i \times_U V \to V\}$ are effectively surjective.
We're done. $\square$

## 1.6 Profinite groups

Profinite groups will play a central role in what follows and this seems a suitable place to introduce them.
They are a special kind of topological groups, and the category $G$-**Set** of finite sets (or modules) equipped with a continuous action of a profinite group $G$ has nice topological properties.
In particular the main result we shall prove is that every sheaf for the canonical topology on $G$-**Set** is representable. This means that $G$-**Set** is equivalent to the category of sheaves on its canonical topology.

Consider an inverse system of finite groups $\{G_i\}_{i \in I}$, each group endowed with the discrete topology. Then we can consider the inverse limit $G = \varprojlim G_i$ as a topological group, with the topology induced by the product topology.

**Definition 1.6.1** *We say that a topological group is profinite if it is obtained by the above construction.*

Let's state some topological property of profinite groups.

- $G$ is a closed subset of the product $\prod_i G_i$.
  In fact it is easy to see that

$$G = \bigcap_{i \in I} \left( \bigcup_{g \in G_i} \left( \bigcap_{j \geq i} \pi_j^{-1} \mu_{ji}^{-1}(g) \right) \right)$$

  ($\pi_j$ being the projections and $\mu_{ji}$ the maps in the inverse system). Thus $G$ is an intersection of closed subsets.

- $G$ is Hausdorff and compact.
  This is true since both properties are stable under product and remains true for closed subsets.

- It is totally disconnected, i.e. its only connected components are its points.
  First note that an arbitrary product of finite discrete groups is totally disconnected.
  Indeed consider $U \subseteq G$ which is not a singleton. Then there exists

$(g_i) \neq (g'_i) \in U$. So it will be $g_j \neq g'_j$ for some $j$. Thus for varying $g \in G_j$ the sets $U_g := U \cap \pi_i^{-1}(g)$ are open disjoint subset of $U$ such that $U = \bigcup_{g \in G_i} U_g$.
Now just note that closed subsets of totally disconnected spaces are totally disconnected.

- The open normal subgroup of $G$ form a fundamental system of neighborhood of 1.
  To see this notice that the subgroups $H_i := \ker(\pi_i)$ are normal and open, being the kernel of continuous maps, and intersect trivially.
  As a consequence we see that $G \simeq \varprojlim G/H$, where the limit is taken over all open normal subgroups of $G$.

- The open subgroups of $G$ are exactly the closed subgroups of finite index.
  In fact if $U \subseteq G$ is open, then the corresponding cosets are open (for the map $[u \mapsto gu] : U \to gU$ is a homeomorphism) and the complement of $U$ is the union of such cosets. Plus these are in finite number since they cover $G$ and $G$ is compact.
  Conversely the complement of a closed subgroup of finite index is the finite union of the corresponding cosets (once again being homeomorphic to the subgroup itself).

Actually some of this properties characterize profinite groups. Namely:

**Proposition 1.6.2** *A topological group is profinite if and only if it is compact, Hausdorff and totally disconnected*

**Proof**
See [19] I 1.1.3 □.

Next define a category which objects are finite sets endowed with the discrete topology on which $G$ act continuously from the left, and which arrows are $G$-equivariant maps. We shall call it the category of continuous left $G$-set and denote it with $G$-**Set** .

As we mentioned before, an interesting result holds for the category of sheaves over the canonical topology on $G$-**Set** , namely that all sheaves are representable.
Precisely:

**Proposition 1.6.3** *Denote with $\mathcal{T}_G$ the canonical topology on $G$-**Set** .*
*The functor $Z \mapsto Z(-)$ is an equivalence $G$-**Set** $\to \mathcal{S}(\mathcal{T}_G)$ (sheaves of set).*

**Proof**
We define the quasi-inverse.
Let $\mathscr{F} \in \mathcal{S}(T_G)$ be a sheaf for the canonical topology. Notice that for any $H \leq G$ open and normal the group $G/H$ is naturally a left $G$-set. Define a $G$-action on $\mathscr{F}(G/H)$ as follows:

$$gs = \mathscr{F}(\cdot g)(s) \qquad \forall g \in G, s \in \mathscr{F}(G/H)$$

($\cdot g : G/H \to G/H$ denotes multiplication by $g$).
The collection $\mathscr{F}(G/H)$ form a direct system of left $G$-set, the maps being $\mathscr{F}(G/H \to G/H')$ for all $H \subseteq H'$. We may then consider the direct limit $\varinjlim \mathscr{F}(G/H)$. This have a natural structure of left $G$-set.
We claim that the functor $\mathscr{F} \mapsto \varinjlim \mathscr{F}(G/H)$ is the quasi-inverse.
One composition is the functor

$$Z \mapsto \varinjlim \mathrm{Hom}_G(G/H, Z).$$

It is easy to see that $\mathrm{Hom}_G(G/H, Z) \simeq Z^H$ (send a map $\varphi \in \mathrm{Hom}_G(G/H, Z)$ to $\varphi(1_{G/H})$ and conversely an element $z \in Z^H$ to the map $\varphi_z : gH \mapsto gz$.)
Thus we have

$$\varinjlim \mathrm{Hom}_G(G/H, Z) = \varinjlim Z^H = \bigcup Z^H.$$

We claim this is $Z$, i.e. that any element $z \in Z$ is fixed by some open normal subgroup of $G$. But this is true since these subgroup form a fundamental system of neighborhood of 1, thus $z$ is fixed by a normal subgroup contained in the stabilizer $G_z$.
Verifying that the other composition is isomorphic to the identity is more involved.
We need to exhibit a natural isomorphism $\mathscr{F} \longrightarrow \mathrm{Hom}_G\left(-, \varinjlim \mathscr{F}(G/H)\right)$ functorial in $\mathscr{F}$.
First we show that it suffices to exhibit such an isomorphism for transitive $G$-sets.
In fact consider a $G$-set $U$ and write it as the disjoint union of its orbits, say $U = \amalg U_i$. This is the coproduct in the category of $G$-set, thus we have

$$\mathrm{Hom}_G\left(\amalg U_i, \varinjlim \mathscr{F}(G/H)\right) \simeq \prod \mathrm{Hom}_G\left(U_i, \varinjlim \mathscr{F}(G/H)\right).$$

Moreover $\{U_i \hookrightarrow U\}$ is a covering for the canonical topology. In fact $\{U_i \to U\}$ is a covering for the canonical topology if and only if $\amalg U_i \to U$ is surjective (compare example 1.5.8).
Hence we have an exact sequence

$$\mathscr{F}(U) \to \prod \mathscr{F}(U_i) \rightrightarrows \mathscr{F}(U_i \times_U U_j).$$

But $U_i \times_U U_j = U_i \cap U_j$, hence we find $\mathscr{F}(U) = \prod \mathscr{F}(U_i)$.
Thus consider a transitive $G$-set $U$.
We are going to prove the following chain of isomorphisms:

$$\begin{aligned}
\mathscr{F}(U) &\overset{(1)}{\simeq} \varprojlim \mathscr{F}(U^H) \\
&\overset{(2)}{\simeq} \varprojlim \mathrm{Hom}_{G/H}\left(U^H, \mathscr{F}(G/H)\right) \\
&\overset{(3)}{\simeq} \varprojlim \mathrm{Hom}_G\left(U^H, \varinjlim \mathscr{F}(G/H')\right) \\
&\overset{(4)}{\simeq} \mathrm{Hom}_G\left(\varinjlim U^H, \varinjlim \mathscr{F}(G/H')\right) \\
&\overset{(5)}{\simeq} \mathrm{Hom}_G\left(U, \varinjlim \mathscr{F}(G/H')\right)
\end{aligned}$$

(here $H$ runs through all open subgroup of $G$ and $H'$ runs through all open subgroup of $H$).
**(1)** Since $U = \bigcup U^H$, $\{U^H \hookrightarrow U\}$ is a covering for the canonical topology.
Thus we have an exact sequence

$$\mathscr{F}(U) \longrightarrow \prod_H \mathscr{F}(U^H) \rightrightarrows \prod_{H,H'} \mathscr{F}(U^H \cap U^{H'})$$

(again $U^H \times_U U^{H'} = U^H \cap U^{H'}$).
As a consequence we obtain

$$\mathscr{F}(U) = \varprojlim \mathscr{F}(U^H).$$

**(2)** Since $H \leq G$ is an open subgroup, $G/H$ is finite and thus discrete. This implies that all action are continuous.
Hence we just need to show that for a generic group $G$, every sheaf $\mathscr{F}$ for the canonical topology on $G$-**Set** is represented by $\mathscr{F}(G)$.
For this consider a transitive $G$-sets $U$ and fix an element $u \in U$. Then the map $[g \to gu] : G \to U$ form a covering itself, and we have an exact sequence

$$\mathscr{F}(U) \to \mathscr{F}(G) \rightrightarrows \mathscr{F}(G \times_U G).$$

But

$$G \times_U G = \{(g, g') \in G \times G | gu = g'u\} = \{(g, gh) | g \in G, h \in S_u\} = \coprod_{h \in S_u} G$$

(here $S_u$ is the stabilizer of $u$ in $G$).
Hence the exact sequence above becomes:

$$\mathscr{F}(U) \to \mathscr{F}(G) \rightrightarrows \prod_{h \in S_u} \mathscr{F}(G),$$

where the rightmost arrows send an element $s \in \mathscr{F}(G)$ respectively to $(s)_{h \in S_u}$ and to $(\mathscr{F}(\cdot h)s)_{h \in S_u}$.
Then we finally find

$$\mathscr{F}(U) = \ker\left(\mathscr{F}(G) \rightrightarrows \prod_{h \in S_u} \mathscr{F}(G)\right) = \mathscr{F}(G)^{S_u} = \mathrm{Hom}_G\big(U, \mathscr{F}(G)\big).$$

**(3)** It suffices to show that $\mathscr{F}(G/H) \simeq \big(\varinjlim \mathscr{F}(G/H')\big)^H$.
Notice that for $H' \subseteq H$ normal subgroup the quotient map $G/H' \to G/H$ form a covering itself, thus e have an exact sequence

$$\mathscr{F}(G/H) \to \mathscr{F}(G/H') \rightrightarrows \mathscr{F}(G/H' \times_{G/H} G/H').$$

But just as before $G/H' \times_{G/H} G/H' = \amalg_{h \in H}(G/H')$. Hence we see that $\mathscr{F}(G/H) \simeq \mathscr{F}(G/H')^{H/H'}$ and the natural injective map $\mathscr{F}(G/H) \to \varinjlim \mathscr{F}(G/H')$ identifies $\mathscr{F}(G/H)$ with $\big(\varinjlim \mathscr{F}(G/H')\big)^H$.
**(4)** This is true by a formal categorical argument.
**(5)** $\varinjlim U^H = \cup U^H = U$, again because open normal subgroups form a fundamental system of neighborhood of 1. □

**Remark 1.6.4** Similarly one may define the category $G$-**mod** of discrete $G$-modules, i.e. the category of finite abelian groups endowed with the discrete topology on which $G$-act continuously.
Repeating step by step the proof of the proposition above we find that every abelian sheaf for the canonical topology on $G$-mod is representable.

To conclude this interlude, we want to evidence how cohomology theory on $\mathcal{T}_G$ gives an alternative point of view on classical profinite groups cohomology theory.

We recall some definition.
Let $G$ be a profinite group and consider the functor

$$[A \to A^G] : G\text{–}\mathbf{mod} \to G\text{–}\mathbf{mod}$$

where $A^G := \{a \in A | ga = a \, \forall g \in G\}$ is the subgroup of $G$-invariant. This is easily verified to be a left exact functor.
Moreover the category of $G$-modules has enough injective objects (this actually follows from the remark above and theorem 1.4.3).

Thus we can give the following definition:

**Definition 1.6.5** *We call* $q$**-th cohomology group of** $G$ **with values in** $A \in G$**-mod** *the* $G$*-module*

$$H^q(G, A) := R^q\big((-)^G\big)(A).$$

We will see (proposition 3.3.3) that Galois groups of fields are profinite. When this is the case we speak about Galois cohomology groups of the base field. The study of these cohomology groups gives information about the arithmetic of the field, and it is the central tool in class field theory.
For a comprehensive treatment about cohomology of profinite groups see [20].

Now fix a one-element $G$-set $e$ with the unique possible $G$-action defined on it (clearly this is a $G$-module too). Since every sheaf $\mathscr{F}$ on $\mathcal{T}_G$ is represented by some $G$ module $A$, for the section functor on $e$ we get

$$\Gamma_e : \mathscr{F} \to \mathscr{F}(e) \simeq \mathrm{Hom}_G(e, A) = A^G.$$

Hence we have $\partial$-functorial isomorphisms

$$H^q(e, \mathscr{F}) \simeq H^q(G, A),$$

which is to say that the cohomology groups of $e$ with value in $\mathscr{F} \simeq \mathrm{Hom}(-, A)$ for the canonical topology on $G$-**mod** are just the cohomology groups of $G$ with values in $A$.

Conversely we can study the cohomology groups of an arbitrary $G$-module $U$ with value in $\mathscr{F} \simeq \mathrm{Hom}(-, A)$ by using classical group cohomology theory. First write $U \simeq \amalg U_i$ as the finite disjoint union of its orbits (this is the coproduct in the category of $G$-modules).
Then we have $\mathscr{F}(U) \simeq \mathrm{Hom}_G(\amalg U_i, A) \simeq \prod \mathrm{Hom}_G(U_i, G)$ and thus we obtain

$$H^q(U, \mathscr{F}) \simeq H^q\big(\amalg U_i, \mathrm{Hom}_G(-, A)\big) \simeq \prod H^q\big(U_i, \mathrm{Hom}_G(-, A)\big).$$

For being a transitive $G$-module, each $U_i$ is isomorphic to some left coset space $G/H_i$, where $H_i$ is an open subgroup of $G$.
Moreover for the section functor on $G/H_i$ we find

$$\Gamma_{G/H_i}\big(\mathrm{Hom}_G(-, A)\big) = \mathrm{Hom}_G(G/H_i, A) = A^{H_i}.$$

Hence we obtain

$$H^q(U, \mathscr{F}) \simeq \prod H^q\big(U_i, \mathrm{Hom}_G(-, A)\big) \simeq \prod H^q\big(G/H_i, \mathrm{Hom}_G(-, A)\big) \simeq$$

$$\simeq \prod H^q(H_i, A).$$

# 2 GALOIS CATEGORIES

Fundamental groups arise in many different fields of mathematics.
We want to give two meaningful example.

**Example 2.0.6** Let $S$ be a topological space. Define a cover of $S$ to be a topological space $X$ together with a continuous map $X \to S$ that locally looks like the projection $S \times I \to S$ for some discrete set $I$. We say that the cover is finite if $I$ is finite.
We define a morphism of coverings $X \to X'$ to be an homeomorphism making the following diagram commute

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X' \arrow[d] \\ & S \end{tikzcd}$$

This makes the collection of (finite) covers of $S$ in a category.
Now define the universal cover $\bar{X}$ of $S$ (if it exists) to be a cover satisfying the following universal property:
for all connected cover $X \to S$ there exists a unique covering $\bar{X} \to X$ such that the following diagram commute

$$\begin{tikzcd} \bar{X} \arrow[r] \arrow[dr] & X \arrow[d] \\ & S \end{tikzcd}$$

It can be proved, assuming sufficiently nice properties for $S$, that the universal cover exists and that its group of automorphism is isomorphic to the usual fundamental group of $S$.
Moreover this group classifies covers of $S$, meaning that there is a biunivocal correspondence between covers and sets equipped with a continuous action of the group.
See for example [12] for a comprehensive treatment about topological fundamental group.

**Example 2.0.7** Consider a field $k$ and fix a separable closure $k_s/k$.
For any separable extension $K/k$ there exists an inclusion $K \hookrightarrow k_s$ making the following diagram commute

$$\begin{tikzcd} K \arrow[r, hook] & k_s \\ k \arrow[u, hook] \arrow[ur, hook] & \end{tikzcd}$$

Moreover the group $G_k$ of $k$-automorphisms of $k_s$ (the absolute Galois group of $k$) classifies finite separable extensions of $k$, meaning that there is a biunivocal correspondence $K \mapsto \mathrm{Aut}_k(K)$ between finite separable extensions of $k$ and transitive sets on which $G_k$ acts continuously.
We will deal with this particular case in chapter three.

These two situations are manifestly analogous: covers correspond to finite separable extensions; the universal cover corresponds to the separable closure; the fundamental group corresponds to the absolute Galois group.
The theory of Galois category succeeds to a great degree in unifying them.

Strictly speaking Galois categories axiomatize the categories of finite discrete sets on which a fixed profinite group acts continuously.
A Galois category is given by the datum of a category $\mathcal{C}$ together with a functor $\mathcal{F} : \mathcal{C} \to \underline{\mathbf{FSet}}$ satisfying some axioms.
To any such category we can associate (in a non-canonical way) a fundamental group, which is defined to be $\pi_1(\mathcal{C}, \mathcal{F}) = \mathrm{Aut}_{\mathbf{Fct}}(\mathcal{F})$. The main theorem of Galois categories, to which proof we devote this section, states that $\pi_1(\mathcal{C}, \mathcal{F})$ is a profinite group and that $\mathcal{F}$ factors through an equivalence of categories $\mathcal{F} : \mathcal{C} \to \pi_1(\mathcal{C}, \mathcal{F})$-set.

The original source for these topics is SGAI ([11]).
The sources I followed mostly are the course notes [3] and in some case [28].
A really nice treatment can be found in [15] too.

## 2.1 Definition and first properties

In order to define Galois categories we need the notion of group action on an object in a category.

**Definition 2.1.1** *We say that a group $G$ acts on an object $X \in \mathcal{C}$ if there exists a group map*

$$\alpha : G \to \mathrm{Aut}_{\mathcal{C}}(X, X).$$

*We will say that $X$ is a $G$-***object***.*

Then we can define the notion of quotient by a group action.

**Definition 2.1.2** *Let $X$ be a $G$-object. We say that an object $X^G$ together with an arrow $X \to X^G$ is the* **quotient of $X$ by the action of $G$**, *if any arrow $u : X \to Y$ fixed by $G$ (meaning that $u \circ \alpha(g) = u \; \forall g \in G$) factors as $X \to X/G \to Y$.*

It is easy to see that in the category of sets, these correspond to the usual notions of group action and quotient by a group action.

We may give now the definition of Galois category.

**Definition 2.1.3** *We call* **Galois category** *a category $\mathcal{C}$ together with a functor $\mathcal{F} : \mathcal{C} \to \underline{\mathbf{FSet}}$ (called fiber functor or fundamental functor) satisfying the following conditions*

*(G1) $\mathcal{C}$ has a final object and finite fibered products.*

*(G2) $\mathcal{C}$ has finite coproducts and quotients by finite group actions.*

*(G3) Any morphism $X \to Y$ in $\mathcal{C}$ facors as $X \to Y' \to Y \simeq Y' \amalg Y''$ where $X \to Y'$ is a strict epimorphism and $Y' \to Y$ is a monomorphism inducing an isomorphism into a direct summand of $Y$.*

*(G4) $\mathcal{F}$ preserves the final object and commute with fibered product.*

*(G5) $\mathcal{F}$ commutes with finite coproducts and quotients by finite group action.*

*(G6) $\mathcal{F}$ sends strict epimorphisms (*see remark 1.5.5*) to surjective maps.*

*(G7) $\mathcal{F}$ reflects isomorphisms.*

**Remark 2.1.4** We want to stress the fact that the fundamental functor is not unique in general.
It can be proved, but it is beyond the scope of this work, that different fundamental functors on the same Galois category are isomorphic. See [15] 3.19 for details.

It's worth to be precise on what "commutes with quotients" means.
Just notice that if $G$ act on an object $X$, then it act naturally on $\mathcal{F}(X)$. Moreover if $X \to X/G$ is the quotient map then we have an induced map $\mathcal{F}(f) : F(X) \to \mathcal{F}(X/G)$.
(G5) just states that $\mathcal{F}(X/G) \simeq \mathcal{F}(X)/G$ and that $F(f)$ is the quotient map $\mathcal{F}(X) \to \mathcal{F}(X)/G$.

**Example 2.1.5 (example 2.0.6 revised)**
It can be shown that the category of finite covers of a "sufficiently nice" topological space is a Galois category.
In fact for each point $s \in S$ we can define a fundamental functor by assigning to each finite cover the fiber over $s$.
It can be shown (see [15] theorem 1.15) that it suffices that $S$ is connected for this machinery to work. Nevertheless in order to talk about homotopy classes of closed loops and universal cover we need to make stronger assumption. Moreover this has the advantage that infinite covers can be considered.

**Example 2.1.6 (example 2.0.7 revised)**
Consider the full subcategory of $\mathfrak{Sch}_k$ which objects are spectrum of finite separable field extension of $k$.
This fails to be a Galois category because it does not have finite coproduct. Still we can bypass this problem by considering the category which objects are disjoint union of spectrum of finite separable field extensions of $k$.
It can be shown that this is a Galois category, the fundamental functor being

$$\amalg \operatorname{Spec} k_i \mapsto \operatorname{Spec} k_s \times_{\operatorname{Spec} k} (\amalg \operatorname{Spec} k_i).$$

We will study this case in details in chapter 3.

Some immediate consequences follow at once from the definition:

**Remark 2.1.7**

- $\mathcal{C}$ has initial object, since the coproduct over the empty set is initial.
- $\mathcal{F}$ preserves monomorphisms. This follows from (G4) and the fact that an arrow $X \to Y$ is a monomorphism if and only if $X \times_Y X \simeq X$ with the identity maps as projections.

With few more effort we see that the fiber functor has many nice preservation properties.

**Lemma 2.1.8**

*i)* $\mathcal{F}(X)$ *is initial if and only if* $X$ *is initial.*

*ii)* $\mathcal{F}(X)$ *is final if and only if* $X$ *is final.*

*iii)* $u : X \to Y$ *is a strict epimorphism if and only if* $\mathcal{F}(u)$ *is surjective.*

*iv)* $u : X \to Y$ *is a monomorphism if and only if* $\mathcal{F}(u)$ *is injective.*

*v) $u : X \to Y$ is an isomorphism if and only if it is a monomorphism and a strict epimorphism.*

**Proof**

**i)** If $X$ is initial then $X \amalg Y \simeq Y$ for any $Y \in \mathcal{C}$, then $\mathcal{F}(X \amalg Y) = \mathcal{F}(X) \amalg \mathcal{F}(Y) \simeq \mathcal{F}(Y)$ and then $\mathcal{F}(X) = \emptyset$.
Conversely let $\mathcal{F}(X) = \emptyset$ and consider the unique arrow $u : \emptyset \to X$. Then $\mathcal{F}(u) : \emptyset \to \emptyset$ is the identity map and thus $u$ is an isomorphism by axiom (G7).

**ii)** The "if" implication is just axiom (G4).
Conversely assume that $\mathcal{F}(X) = \{*\}$. Denote with $e$ the final object in $\mathcal{C}$ and consider the unique arrow $u : X \to e$. Then again $\mathcal{F}(u)$ is an isomorphism and $u$ is an isomorphism by (G7).

**iii)** The "only if" implication is axiom (G6).

Conversely consider a factorization of $u$ as in (G3), say $X \xrightarrow{u'} Y' \xrightarrow{u''} Y = Y' \amalg Y''$.
$\mathcal{F}(u'')$ is injective by the remark above and surjective since $\mathcal{F}(u)$ is surjective. Then $\mathcal{F}(u'')$ is an isomorphism and by (G7) $u''$ is an isomorphism too.
But then $u$ is a strict epimorphism, being the composition of a strict epimorphism and an isomorphism.

**iv)** The "only if" implication is the remark above.

Conversely consider a decomposition $X \xrightarrow{u'} Y' \xrightarrow{u''} Y = Y' \amalg Y''$ as before.
$\mathcal{F}(u')$ is surjective by part iii) and injective since $\mathcal{F}(u)$ is injective. Then $\mathcal{F}(u')$ is an isomorphism and by (G7) $u'$ is an isomorphism too.
But then $u'$ is a monomorphism, being the composition of an isomorphism and a monomorphism.

**v)** This follows immediately by i) and ii). □

Another important result is the uniqueness of the decomposition in (G3).

**Lemma 2.1.9** *Let $X \to Y' \to Y$ be a decomposition as in (G3). Then $Y'$ is unique up to a unique isomorphism.*

**Proof**

Consider two such decompositions $X \xrightarrow{u_1'} Y_1 \xrightarrow{u_1''} Y = Y_1 \amalg Y_1'$ and $X \xrightarrow{u_2'} Y_2 \xrightarrow{u_2''} Y = Y_2 \amalg Y_2'$ and denote with $p_i$, $i = 1, 2$ the projections $X \times_{Y_1} X \rightrightarrows X$.
Since $u_2'$ is a strict epimorphism there is a coequalizer diagram

$$Y_2(Y_1) \to Y_2(X) \rightrightarrows Y_2(X \times_{Y_1} X).$$

First we claim that $u_2' \in \ker\big(Y_2(X) \rightrightarrows Y_2(X \times_{Y_1} X)\big)$.
In fact

$$u_1' \circ p_1 = u_1' \circ p_2 \implies (u_2'' \circ u_2') \circ p_1 = (u_1'' \circ u_1') \circ p_1 = (u_1'' \circ u_1') \circ p_2 = (u_2'' \circ u_2') \circ p_2$$

$$\Longrightarrow \; u_2' \circ p_1 = u_2' \circ p_2 \qquad \text{(since } u_2'' \text{ is a monomorphism).}$$

Thus there exists a unique arrow $\phi : Y_1 \to Y_2$ such that $\phi \circ u_1' = u_2'$. This is an isomorphism by (G7): indeed $F(\phi)$ is surjective (since $\mathcal{F}(u_2')$ is) and thus bijective (since $\#\mathcal{F}(Y_1) = \#\mathcal{F}(Y_2) = \#\mathcal{F}(u)(X)$). □

Finally Galois categories satisfy the following important property:

**Lemma 2.1.10** *Any Galois category is Artinian, i.e. any chain of monomorphism* $\cdots X_2 \overset{u_2}{\hookrightarrow} X_1 \overset{u_1}{\hookrightarrow} X$ *stabilizes.*

**Proof**
From remark 2.1.7 we know that the maps $\mathcal{F}(u_i) : \mathcal{F}(X_i) \to \mathcal{F}(X_{i-1})$ are injective, and so $\#\mathcal{F}(X_i) \leq \#\mathcal{F}(X_{i-1})$. But $\#\mathcal{F}(X)$ is finite, thus it must be $\#\mathcal{F}(X_n) = \#\mathcal{F}(X_{n-1})$ for $n$ big enough. Then $\mathcal{F}(u_n)$ is bijective and $u_n$ is an isomorphism by (G7). □

## 2.2 The main theorem

Roughly speaking the main theorem states that all Galois category reduce to the following example:

**Example 2.2.1** The category $G$-**Set** (see section 1.6) is a Galois category with the forgetful functor $G$-**Set** $\to$ **<u>FSet</u>** as fiber functor.

The first natural question arising is which profinite group we should choose for a given Galois category $(\mathcal{C}, \mathcal{F})$.
This definition answers the question:

**Definition 2.2.2** *We call* $\pi_1(\mathcal{C}, \mathcal{F}) := \mathrm{Aut}_{\mathfrak{Fct}}(\mathcal{F})$ *the* **fundamental group of $\mathcal{C}$ relative to $\mathcal{F}$.**

**Remark 2.2.3** It can be shown that fundamental groups relative to different fundamental functors differ by a canonical inner automorphism.
See [15] 3.19 for a reference.

**Remark 2.2.4** This was the point of the new insight into fundamental groups.
These are not seen anymore as automorphism groups of some object (the universal cover in example 2.0.6 and a separable closure of a field in example 2.0.7), but rather as the automorphism groups of some functor.

The fundamental group obviously act on the left on any $\mathcal{F}(X)$. Actually the following proposition holds:

**Proposition 2.2.5** *$\pi_1(\mathcal{C},\mathcal{F})$ is a profinite group acting continuously on any $\mathcal{F}(X)$.*

**Proof**
First we prove that $\pi_1(\mathcal{C},\mathcal{F})$ is profinite.
It will suffice to show that

$$G = \prod_{X\in\mathcal{C}} \mathrm{Aut}_{\underline{\mathbf{FSet}}}\big(\mathcal{F}(X)\big)$$

is a profinite group and then show that $\pi_1(\mathcal{C},\mathcal{F}) \subseteq G$ is a closed subgroup (since being compact, Hausdorff and totally disconnected are properties stable under taking closed subsets).
In fact if $G = \prod_i G_i$ is an arbitrary product of finite group endowed with the discrete topology, it is a profinite group if endowed with the product topology.
It is compact since arbitrary product of compact spaces is compact (Tychonoff's theorem).
It is Hausdorff since arbitrary product of Hausdorff spaces is Hausdorff.
It is totally disconnected (this was already proven in section 1.6).

Now we prove that $\pi_1(\mathcal{C},\mathcal{F}) \subseteq G$ is closed.
In fact it is naturally injected into $G$ by the map $\theta \mapsto \big(\theta(X)\big)_{X\in\mathcal{C}}$. Moreover define for any $u : Y' \to Y$ the set $U_u := \big\{(\theta_X)_{X\in\mathcal{C}} | F(u)\circ\theta_{Y'} = \theta_Y \circ F(u)\big\}$.
It is easy to see that

$$\pi_1(\mathcal{C},\mathcal{F}) = \bigcap_{\substack{Y',Y\in\mathcal{C}\\ u:Y'\to Y}} U_u.$$

But the $U_u$ are closed. In fact we have that

$$U_u = \bigcup_{(\theta',\theta)} \big(\pi_{Y'}^{-1}(\theta') \cap \pi_Y^{-1}(\theta)\big)$$

where the union is taken over all $(\theta',\theta) \in \mathrm{Aut}_{\underline{\mathbf{FSet}}}\big(F(Y')\big) \times \mathrm{Aut}_{\underline{\mathbf{FSet}}}\big(\mathcal{F}(Y)\big)$ such that $\mathcal{F}(u)\circ\theta' = \theta\circ\mathcal{F}(u)$ (these are clearly finitely many).
This proves that $\pi_1(\mathcal{C},\mathcal{F}) \subseteq G$ is closed and thus profinite.

It remains to prove that the action is continuous.
Since we consider discrete topology on $\mathcal{F}(X)$ we just need to prove that the stabilizer of any element $\zeta \in \mathcal{F}(X)$ is open. But $G$ acts on $\mathcal{F}(X)$ too and $\mathrm{Stab}_{\pi_1(\mathcal{C},\mathcal{F})}(\zeta) = \mathrm{Stab}_G(\zeta) \cap \pi_1(\mathcal{C},\mathcal{F})$. Finally

$$\mathrm{Stab}_G(\zeta) = \pi_X^{-1}\big(\mathrm{Stab}_{\mathrm{Aut}_{\underline{\mathbf{FSet}}}(\mathcal{F}(X))}(\zeta)\big)$$

is open in $G$.
We're done. $\square$

Thus the fiber functor factors through a functor $\mathcal{F} : \mathcal{C} \to \pi_1(\mathcal{C}, \mathcal{F})$-$\underline{\mathbf{FSet}}$. The main theorem states that this is an equivalence.

**Theorem 2.2.6** (Main theorem of Galois Category) *Let $(\mathcal{C}, \mathcal{F})$ be a Galois category. Then $\mathcal{F} : \mathcal{C} \to \pi_1(\mathcal{C}, \mathcal{F})$-set is an equivalence of category.*

**Remark 2.2.7** This clarifies the concept that the fundamental groups classifies "covers": any cover (i.e. object in the category) correspond biunivocally to a continuous set together with an action of the fundamental group.

We begin by sketching the proof.
The main point is that the fundamental functor and the fundamental group may be described as limits.
To understand how fix an object $X \in \mathcal{C}$ and an element $\zeta \in \mathcal{F}(X)$ and define a natural transformation as follows:

**Definition 2.2.8** *We call* **evaluation on** $\zeta$ *the natural transformation $v_\zeta$ defined on component as*

$$\begin{aligned} v_\zeta(Y) : \mathrm{Hom}_{\mathcal{C}}(X, Y) &\longrightarrow \mathcal{F}(Y) \\ u &\longmapsto \mathcal{F}(u)(\zeta) \end{aligned}$$

We will see that there are certain object, the Galois object, for which $v_\zeta$ is an isomorphism (if restricted to some full subcategory of $\mathcal{C}$).
Moreover we will make the full subcategory of Galois object into a filtered inverse system of object and arrows. This will allows us in turn to describe $\mathcal{F}$ as a direct limit of the representable functors $\mathrm{Hom}_{\mathcal{C}}(X, -)$ and $\pi_1(\mathcal{C}, \mathcal{F})$ as the inverse limit of the groups $\mathrm{Aut}_{\mathcal{C}}(X)$.
These descriptions are actually the central part of the proof.

Finally we introduce a terminology we shall use from now on.

**Definition 2.2.9** *Let $\mathcal{C}$ be a category and $\mathcal{F} : \mathcal{C} \to \underline{\mathbf{Set}}$ a functor.*
*We call* **pointed category** *associated to $\mathcal{C}$ the category $\mathcal{C}^{pt}$ which objects are pairs $(X, \zeta)$ with $X \in \mathcal{C}$, $\zeta \in \mathcal{F}(X)$ and which arrows $(X, \zeta) \to (X', \zeta')$ are morphisms $u : X \to X'$ in $\mathcal{C}$ such that $\mathcal{F}(u)(\zeta) = \zeta'$.*
*Moreover for two object $Y, X$ in a category we will write $Y \geq X$ if there exists an arrow $Y \to X$.*

## 2.3 Connected objects

**Definition 2.3.1** *We say that an object $X \in \mathcal{C}$ is* **connected** *if*

$$X = X' \amalg X'' \implies X' = \emptyset \ \vee \ X'' = \emptyset.$$

*We write $\mathcal{C}_0 \subseteq \mathcal{C}$ for the full subcategory of connected object.*

First of all we notice a couple of fact:

**Remark 2.3.2**

- An arrow $X \to Y$ is a monomorphism if and only if $Y \simeq X \amalg X'$.
  In fact if $X \to Y$ is a monomorphism then by uniqueness of the factorization in (G3) it must be $Y \simeq X \amalg X'$.
  Conversely if $Y \simeq X \amalg X'$ then $\mathcal{F}(Y) = \mathcal{F}(X) \amalg \mathcal{F}(X')$ by (G5), thus $\mathcal{F}(X) \to \mathcal{F}(Y)$ is injective and $X \to Y$ is a monomorphism by lemma 2.1.8.

- Morphism with connected codomain and non-initial domain are strict epimorphism.
  In fact let $\emptyset \neq X_0 \in \mathcal{C}_0$, $X \to X_0$ be an arrow and $X \xrightarrow{u'} X_0' \xrightarrow{u''} X_0$ be the factorization of $u$ as in (G3). By the remark above and the fact that $X_0$ is connected we see that $u''$ is an isomorphism and thus $u$ is a strict epimorphism.

- Epimorphism with connected domain have connected codomain.
  In fact let $X_0 \in \mathcal{C}_0$ and let $u : X_0 \to X$ be a strict epimorphism. Write $X = X' \amalg X''$ with $X' \neq \emptyset$ and form the cartesian square

$$\begin{tikzcd} X_0' \arrow[r, hook, "p"] \arrow[d, two heads] & X_0 \arrow[d, two heads, "u"] \\ X' \arrow[r, hook] & X \end{tikzcd}$$

  $p$ is both a monomorphism (since monomorphisms are stable under pullback) and a strict epimorphism (by the previous remark). Thus it is an isomorphism (lemma 2.1.8) and $X_0 \to X' \to X$ is a factorization of $u$ as in (G3).
  But since $u$ is an epimorphism, $X_0 \xrightarrow{u} X \xrightarrow{\mathrm{id_X}} X$ is another factorization as in (G3).
  Hence, by uniqueness of such factorization, it must be $X \simeq X'$ and thus $X'' = \emptyset$.

The first remarkable result we shall prove is that any object can be written uniquely as a coproduct of finitely many connected object.
The proof uses the Artinian property together with the characterization of monomorphisms we gave above.

**Proposition 2.3.3** *For any object $X \in \mathcal{C}$ there is a unique (up to isomorphisms and reordering) decomposition*

$$X = X_1 \amalg \cdots \amalg X_n$$

*with the $X_i$ connected.*

**Proof**
First we prove the existence.
If $X$ is connected we're done.
Otherwise we claim that $X \simeq X_1 \amalg X'$ for some $X_1$ connected. In fact write $X = X_1' \amalg X_1''$ with $X_1', X_2' \neq \emptyset$. If $X_1'$ is connected we're done. If not we repeat the argument and write $X_1' = X_2' \amalg X_2''$. We then get a chain of monomorphism $\cdots X_2' \hookrightarrow X_1' \hookrightarrow X$. Since $\mathcal{C}$ is Artinian this must stabilize, thus there exist a connected object $X_1$ such that $X \simeq X_1 \amalg Y$ (this could eventually be the final object).
Now we repeat the argument for $Y$ and at each step we obtain
$X \simeq X_1 \amalg \cdots \amalg X_n \amalg Y$, where the $X_i$ are connected and non-empty. Moreover $\sum |\mathcal{F}(X_i)| \leq |\mathcal{F}(X)|$, thus this process must terminate in a finite number of step and we find

$$X \simeq X_1 \amalg \cdots \amalg X_n.$$

It remains to prove uniqueness.
Assume that $Y \neq \emptyset$ is a connected component of $\amalg X_i$.
By the previous remark there is a monomorphism $Y \to \amalg X_i$ and thus $\mathcal{F}(Y)$ may be identified with a subset of $\mathcal{F}(\amalg X_i) = \amalg \mathcal{F}(X_i)$. Since $\mathcal{F}(Y) \neq \emptyset$ (lemma 2.1.8), there exists some $j$ such that $\mathcal{F}(Y) \cap \mathcal{F}(X_j) \neq \emptyset$.
Now form the cartesian square

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & X_j \arrow[d] \\ Y \arrow[r] & \amalg X_i \end{tikzcd}$$

Since $\mathcal{F}$ preserve fibered product, we have that $\mathcal{F}(Y') = \mathcal{F}(X) \cap \mathcal{F}(X_j) \neq \emptyset$ and thus $\mathcal{F}(Y') \neq \emptyset$. Hence $Y' \neq \emptyset$ and by the remark above we have that both $Y' \to X_j$ and $Y' \to Y$ are strict epimorphisms. But these are monomorphism too since monomorphisms are stable under pull back. Thus they are isomorphisms and we get that $Y \simeq Y' \simeq X_j$. $\square$

As a consequence we obtain an important lemma, saying that representable covariant functors preserve finite coproducts:

**Lemma 2.3.4** *Let $X, X_1, ..., X_n \in \mathcal{C}_0$. Then*

$$\mathrm{Hom}_{\mathcal{C}}(X, \amalg_i X_i) \simeq \amalg_i \mathrm{Hom}_{\mathcal{C}}(X, X_i).$$

**Proof**
We clearly have a map $\amalg_i \mathrm{Hom}_{\mathcal{C}}(X, X_i) \to \mathrm{Hom}_{\mathcal{C}}(X, \amalg_i X_i)$ sending $X \to X_i$ to $X \to X_i \to \amalg_i X_i$.
Conversely take a map $X \to \amalg_i X_i$. By (G4) this factorizes as $X \to X' \to \amalg_i X_i = X' \amalg X''$. But $X'$ is connected by remark 2.3.2, thus by uniqueness of the decomposition it must be $X' \simeq X_i$ for some $i$. This means that any map $X \to \amalg_i X_i$ factors through some $X_i$, thus we obtain a map $\mathrm{Hom}_{\mathcal{C}}(X, \amalg_i X_i) \to \amalg_i \mathrm{Hom}_{\mathcal{C}}(X, X_i)$.
It is easy to see that these maps are mutually inverse. □

The evaluation maps relative to connected objects has nice properties.

**Lemma 2.3.5** *Let $X_0 \in \mathcal{C}_0$, $X_0 \neq \emptyset$, $\zeta_0 \in \mathcal{F}(X_0)$. Then $v_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, -) \to \mathcal{F}$ is a monomorphism.*

**Proof**
This just means that $v_{\zeta_0}(X)$ is injective for all $X \in \mathcal{C}$.
This is true if $X = \emptyset$. In fact $\mathrm{Hom}_{\mathcal{C}}(X_0, \emptyset) = \emptyset$, since any arrow $X_0 \to \emptyset$ give a map $\mathcal{F}(X_0) \to \mathcal{F}(\emptyset) = \emptyset$, which forces $X_0$ to be the initial object.
Thus let $X \neq \emptyset$ and take $u, u' : X_0 \to X$ such that $F(u)(\zeta_0) = F(u')(\zeta_0)$. This imply that $\zeta_0 \in \ker\big(\mathcal{F}(u), \mathcal{F}(u')\big) = \mathcal{F}\big(\ker(u, u')\big)$ and thus that $\ker(u, u') \neq \emptyset$. But then, since $\ker(u, u') \to X_0$ is a monomorphism and $X_0$ is connected, by remark 2.3.2 we get that $\ker(u, u') \to X_0$ is an isomorphism, i.e. that $u = u'$. □

**Lemma 2.3.6** *For any $(X_1, \zeta_1), ..., (X_n, \zeta_n) \in \mathcal{C}^{pt}$ there exists $(X_0, \zeta_0) \in \mathcal{C}_0^{pt}$ such that $(X_0, \zeta_0) \geq (X_i, \zeta_i)$ $\forall i = 1, ..., n$.*

**Proof**
Define $X := X_1 \times \cdots \times X_n$ and let $\zeta := (\zeta_1, ..., \zeta_n) \in F(X) = F(X_1) \times \cdots \times F(X_n)$.
Clearly $(X, \zeta) \geq (X_i, \zeta_i)$ $\forall i = 1, ..., n$ via the projections $X \to X_i$. Now let $X_0$ be the connected component of $X$ such that $\zeta \in \mathcal{F}(X_0)$. Clearly $(X_0, \zeta) \geq (X_i, \zeta_i)$ via $\pi_i \circ \iota$ ($\iota$ being the monomorphism $X_0 \to X$). □

**Corollary 2.3.7** *For any $X \in \mathcal{C}$ there exists $X_0 \in \mathcal{C}_0$ such that $v_{\zeta_0}(X) : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \to \mathcal{F}(X)$ is bijective.*

**Proof**
Let $\mathcal{F}(X) = \{\zeta_1, ..., \zeta_n\}$ and chose $X_i = X$ for $i = 1, ..., n$. $\square$

Finally a really important property is that endomorphisms of connected object are automatically automorphisms.

**Lemma 2.3.8** *If $X_0 \in \mathcal{C}$ is connected then* $\mathrm{Hom}_{\mathcal{C}}(X_0, X_0) = \mathrm{Aut}_{\mathcal{C}}(X_0)$.

**Proof**
We need to prove that any arrow $u : X_0 \to X_0$ is an isomorphism, i.e. that $\mathcal{F}(u)$ is an isomorphism, i.e. (since $\mathcal{F}(X_0)$ is finite) that $\mathcal{F}(u)$ is surjective. But this is true since any arrow $X_0 \to X_0$ is a strict epimorphism by remark 2.3.2. $\square$

## 2.4 Galois objects

We have seen in the last section that for any connected object $X_0$ and any $\zeta_0 \in \mathcal{F}(X_0)$ there is an injective map

$$v_{\zeta_0}(X_0) : \mathrm{Aut}_{\mathcal{C}}(X_0) \longrightarrow \mathcal{F}(X_0).$$

We make the following definition.

**Definition 2.4.1** *We say that a connected object $X_0$ is a* **Galois object** *if $v_{\zeta_0}(X_0)$ is bijective for some (and thus for all) $\zeta_0 \in \mathcal{F}(X_0)$.*
*We write $\mathcal{G} \subseteq \mathcal{C}_0$ for the full subcategory of Galois object.*

We can rephrase this definition in different ways.
Notice that $X_0$ is naturally an $\mathrm{Aut}_{\mathcal{C}}(X_0)$-object and $\mathcal{F}(X_0)$ in an $\mathrm{Aut}_{\mathcal{C}}(X_0)$-set via the map $\mathrm{Aut}_{\mathcal{C}}(X_0) \to \mathrm{Aut}_{\underline{\mathbf{FSet}}}\big(\mathcal{F}(X_0)\big)$.

We have:

**Lemma 2.4.2** *The following are equivalent*

*i)* *$X_0$ is a Galois object.*

*ii)* $\mathrm{Aut}_{\mathcal{C}}(X_0)$ *act simply transitively on $\mathcal{F}(X_0)$.*

*iii)* $\#\mathcal{F}(X_0) = \#\mathrm{Aut}_{\mathcal{C}}(X_0)$.

*iv)* $X_0/\mathrm{Aut}_{\mathcal{C}}(X_0)$ *is final in $\mathcal{C}$.*

**Proof**
The only non immediate part is the equivalence of the last condition.
But $X_0/\mathrm{Aut}_{\mathcal{C}}(X_0)$ is final if and only if $\mathcal{F}(X_0/\mathrm{Aut}_{\mathcal{C}}(X_0)) = \mathcal{F}(X_0)/\mathrm{Aut}_{\mathcal{C}}(X_0) = \{*\}$ and this is clearly equivalent to ii).

Galois object are important because their associated evaluation maps are isomorphisms under certain condition:

**Proposition 2.4.3** *Let $X$ be a Galois object and $Y \leq X$ connected. Then $v_\zeta(Y) : \mathrm{Hom}\mathcal{C}(X, Y) \to \mathcal{F}(Y)$ is a bijection for all $\zeta \in \mathcal{F}(X)$.*

**Proof**
We already know by lemma 2.3.5 that $v_\zeta$ is injective.
It just remain to prove surjectivity. For any $u : X \to Y$ we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Aut}_{\mathcal{C}}(X) \arrow[r, "v_\zeta(X)"] \arrow[d, "u\circ(-)"'] & \mathcal{F}(X) \arrow[d, "\mathcal{F}(u)"] \\
\mathrm{Hom}_{\mathcal{C}}(X, Y) \arrow[r, "v_\zeta(Y)"'] & \mathcal{F}(Y)
\end{tikzcd}$$

But since $X$ is connected, $u$ is a strict epimorphism by remark 2.3.2, thus $\mathcal{F}(u)$ is surjective by (G6). Moreover $v_\zeta(X)$ is an isomorphism since $X$ is Galois. Surjectivity of $v_\zeta(Y)$ follows at once. □

**Corollary 2.4.4** *Let $X$ be a Galois object and let $\mathcal{C}^X$ be the full subcategory of $\mathcal{C}$ whose family of object is $\{\amalg X_i | X \geq X_i \text{ connected}\}$.*
*Then $\mathcal{F}|_{\mathcal{C}^X} \simeq \mathrm{Hom}_{\mathcal{C}^X}(X, -)$, i.e. $\mathcal{F}|_{\mathcal{C}^X}$ is represented by $X$.*

**Proof**
This follow at once by the previous proposition and lemma 2.3.4. □

For this reason, given an object $X \in \mathcal{C}$, is important to ask whether there exists a Galois object dominating it.

**Proposition 2.4.5** (Galois closure) *For every $X \in \mathcal{C}$ there exists a Galois object $\bar{X}$ dominating $X$.*

**Proof**
Choose $X_0 \in \mathcal{C}_0$ such that $v_{\zeta_0}(X)$ is bijective.
Furthermore let $\pi : X_0 \to X^n$ be the arrow induced by $u_1, ..., u_n \in \mathrm{Hom}_{\mathcal{C}}(X_0, X)$ and let $X_0 \xrightarrow{\pi'} \bar{X} \xrightarrow{\pi''} X^n = \bar{X} \amalg \bar{X}'$ be the factorization of $\pi$ as in (G4).
We claim that $\bar{X}$ is a Galois object, which is to say that $v_{\bar{\zeta}}(\bar{X})$ is surjective for some $\bar{\zeta} \in \mathcal{F}(\bar{X})$.
Let $\bar{\zeta} := \mathcal{F}(\pi')(\zeta_0)$ and let $\zeta \in \mathcal{F}(\bar{X})$.
First of all $\bar{X}$ is connected by remark 2.3.2.
By proposition 2.3.6 we may assume the existence of an arrow $\rho : (X_0, \zeta_0) \to (\bar{X}, \zeta)$. Thus if we prove that there exists $\omega : \bar{X} \to \bar{X}$ such that $\omega \circ \pi' = \rho$,

we would have $\zeta = \mathcal{F}(\rho)(\zeta_0) = \mathcal{F}(\omega) \circ \mathcal{F}(\pi')(\zeta_0) = \mathcal{F}(\omega)(\bar{\zeta})$ and thus we would be done.
To prove the existence of such a $\omega$ it will suffice to prove that the $\pi_i \circ \pi' \circ \rho,\ i = 1, ..., n$ are up to a permutation the $u_i$.
In fact this would imply (by universal property of product) the existence of an isomorphism $\sigma : X^n \to X^n$ such that $\pi_i \circ \pi' \circ \rho = \pi_i \circ \sigma \circ \pi' \circ \pi''$ for all $i$. Thus we would get that $\pi' \circ \rho = \sigma \circ \pi' \circ \pi''$ and by lemma 2.1.9 we would obtain the $\omega$ we are looking for.
So finally we're left to prove that $\{\pi_i \circ \pi' \circ \rho, ..., \pi_i \circ \pi' \circ \rho\} = \{u_1, ..., u_n\}$.
The $\subseteq$ inclusion is obvious.
To prove the converse inclusion it suffices to prove that the $\pi_i \circ \pi' \circ \rho$ are all distinct. But $\rho$ is a strict epimorphism by remark 2.3.2, thus $\pi_i \circ \pi' \circ \rho = \pi_j \circ \pi' \circ \rho \implies \pi_i \circ \pi' = \pi_j \circ \pi' \implies \pi_i \circ \pi' \circ \pi'' = \pi_j \circ \pi' \circ \pi'' \implies u_i = u_j \implies i = j$.
We're done. □

## 2.5 Fundamental functor and fundamental group

In this section we will give descriptions of $\mathcal{F}$ and $\pi_1(\mathcal{C}, \mathcal{F})$ as limits. This is the central point in the proof of the main theorem.
First fix $\zeta = (\zeta_X) \in \prod_{X \in \mathcal{G}} \mathcal{F}(X)$ and let $\mathcal{G}^\zeta$ be the collection

$$\mathcal{G}^\zeta := \big\{(X, \zeta_X) | X \in \mathcal{G}\big\}.$$

By lemma 2.3.5 for any $Y \geq X$ we have a unique morphism $u_{Y,X} : (Y, \zeta_Y) \to (X, \zeta_X)$. This make $\mathcal{G}^\zeta$ in a filtered preordered category. Moreover the morphisms $u_{Y,X}$ induce naturally the structure of a direct system of object on $\{\mathrm{Hom}_{\mathcal{C}}(X, -) | X \in \mathcal{G}^{\bar{\zeta}}\}$. The following result holds.

**Proposition 2.5.1**

$$\mathcal{F} \simeq \varinjlim \mathrm{Hom}_{\mathcal{C}}(X, -).$$

**Proof**
We clearly have natural transformations $v_{\zeta_X} : \mathrm{Hom}_{\mathcal{C}}(X, -) \to \mathcal{F}$. Compatibility with the direct system means that if $(X, \zeta_X) \leq (Y, \zeta_Y)$, i.e. there exists $u : (Y, \zeta_Y) \to (X, \zeta_X)$, then for any $Z \in \mathcal{C}$ the following diagram commute

$$\begin{tikzcd}
 & \mathcal{F}(Z) & \\
\mathrm{Hom}_{\mathcal{C}}(X, Z) \arrow[ur, "v_{\zeta_X}(Z)"] \arrow[rr, "(-)\circ u"'] & & \mathrm{Hom}_{\mathcal{C}}(Y, Z) \arrow[ul, "v_{\zeta_Y}(Z)"']
\end{tikzcd}$$

But if $h : X \to Z$ then

$$v_{\zeta_Y}(Z)(h \circ u) = \mathcal{F}(h \circ u)(\zeta_Y) = \mathcal{F}(h)(\zeta_X) = v_{\zeta_X}(Z)(h).$$

It remains to prove universal property.
So suppose we have a functor $G : \mathcal{C} \to \underline{\mathbf{FSet}}$, together with natural transformations $\mathrm{Hom}_{\mathcal{C}}(X, -) \to G$ for all $X \in \mathcal{G}$, compatible with the morphisms in the direct system. We need to define a natural transformation $\eta : \mathcal{F} \to G$ making the following diagram commute for all $X \in \mathcal{G}$ and for all $Z \in \mathcal{C}$

$$\begin{tikzcd}
\mathcal{F}(Z) \arrow[rr, "\eta(Z)"] & & G(Z) \\
& \mathrm{Hom}_{\mathcal{C}}(X, Z) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Using lemma lemma 2.3.4 we may assume that $Z \in \mathcal{C}_0$ is connected . By proposition 2.4.5 there exists a Galois object $X$ dominating $Z$ and by lemma 2.4.3 $\mathcal{F}(Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Z)$. Thus we clearly have a unique arrow $\mathcal{F}(Z) \to G(Z)$ making the diagram above commute.
We're done. □

Next we will make the collection $\{\mathrm{Aut}_{\mathcal{C}}(X)\}_{X \in \mathcal{G}}$ in an inverse system of groups and prove that the fundamental group is its limit.

**Lemma 2.5.2** *For any $u_{Y,X} : (Y, \zeta_Y) \geq (X, \zeta_X)$ there exists a surjective group map $\phi_{Y,X} : \mathrm{Aut}_{\mathcal{C}}(Y) \to \mathrm{Aut}_{\mathcal{C}}(X)$.*
*These maps make $\{\mathrm{Aut}_{\mathcal{C}}(X)\}_{X \in \mathcal{G}}$ in an inverse system of groups.*

**Proof**
This follows at once since $\mathrm{Aut}_{\mathcal{C}}(Y) \simeq \mathcal{F}(Y)$, $\mathrm{Aut}_{\mathcal{C}}(X) \simeq \mathcal{F}(X)$ and $\mathcal{F}(Y) \to \mathcal{F}(X)$ is surjective (since $Y \to X$ is a strict epimorphism by remark 2.3.2).
Then we can define $\phi_{Y,X}$ to be the unique map making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Aut}_{\mathcal{C}}(Y) \arrow[r, "v_{\zeta_Y}"] \arrow[d, dashed, "\phi_{Y,X}"'] & \mathcal{F}(Y) \arrow[d, "\mathcal{F}(u_{Y,X})"] \\
\mathrm{Aut}_{\mathcal{C}}(X) \arrow[r, "v_{\zeta_X}"'] & \mathcal{F}(X)
\end{tikzcd}$$

□

**Remark 2.5.3** Let $\omega \in \mathrm{Aut}_{\mathcal{C}}(Y)$. We claim that $\phi_{Y,X}(\omega)$ is the unique arrow such that $u_{Y,X} \circ \omega = \phi_{Y,X}(\omega) \circ u_{Y,X}$.

Write $u = u_{Y,X}$. First notice that $(-) \circ u : \mathrm{Aut}_{\mathcal{C}}(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Y, X)$ is an isomorphism. In fact it is injective since $u$ is a strict epimorphism (again remark 2.3.2) and thus it is an isomorphism since $|\mathrm{Aut}_{\mathcal{C}}(X)| = |\mathcal{F}(X)| = |\mathrm{Hom}_{\mathcal{C}}(Y, X)|$.
Hence there exists a unique $\sigma_\omega \in \mathrm{Aut}_{\mathcal{C}}(X)$ such that $\sigma_\omega \circ u = u \circ \omega$.
But the function $[\omega \mapsto \sigma_\omega] : \mathrm{Aut}_{\mathcal{C}}(Y) \to \mathrm{Aut}_{\mathcal{C}}(X)$ makes the above diagram commute, thus it must be $\phi_{Y,X}(\omega) = \sigma_\omega$.

We can give now the promised description of the fundamental group.

**Proposition 2.5.4** *There is an isomorphism of profinite groups*

$$\pi_1(\mathcal{C}, \mathcal{F}) \simeq \varprojlim_{\mathcal{G}^\zeta} \mathrm{Aut}_{\mathcal{C}}(X).$$

**Proof**
First we need to define for all $X \in \mathcal{G}$ group maps $\mathrm{Aut}(\mathcal{F}) \to \mathrm{Aut}_{\mathcal{C}}(X)$ compatible with the maps in the inverse system.
Take $\theta \in \mathrm{Aut}_{\mathcal{C}}(\mathcal{F})$. Since $\mathrm{Aut}_{\mathcal{C}}(X) \simeq \mathcal{F}(X)$, $\theta(X)(\zeta_X)$ correspond to a unique automorphism $\omega_{\theta,X} : X \to X$, precisely the one such that $\mathcal{F}(\omega_{\theta,X})(\zeta_X) = \theta(X)(\zeta_X)$. We define $\omega_{\theta,X}$ to be the image of $\theta$ in $\mathrm{Aut}_C(X)$.
In one formula

$$\theta \mapsto \left(v_{\zeta_X}^{-1}\big(\theta(X)(\zeta_X)\big)\right)_{(x,\zeta_X)\in\mathcal{G}}.$$

We need to prove compatibility with the inverse system, i.e. that for $u : (Y, \zeta_Y) \to (X, \zeta_X)$ the following diagram commute

$$\begin{tikzcd}
 & \mathrm{Aut}(\mathcal{F}) \arrow[dl] \arrow[dr] & \\
\mathrm{Aut}_{\mathcal{C}}(Y) \arrow[rr] & & \mathrm{Aut}_{\mathcal{C}}(X)
\end{tikzcd}$$

Take $\theta \in \mathrm{Aut}(\mathcal{F})$. By remark 2.5.3 the image of $\omega_{\theta,Y}$ in $\mathrm{Aut}_{\mathcal{C}}(X)$ is the unique automorphism $\omega : X \to X$ making the following diagram commute

$$\begin{tikzcd}
Y \arrow[r, "\omega_{\theta,Y}"] \arrow[d, "u"'] & Y \arrow[d, "u"] \\
X \arrow[r, "\omega"'] & X
\end{tikzcd}$$

We need to prove that $\omega = \omega_{\theta,X}$ and to do so it will suffice to prove that $\mathcal{F}(\omega)(\zeta_X) = \theta(X)(\zeta_X)$. From the diagram above

$$\mathcal{F}(\omega)(\zeta_X) = \mathcal{F}(\omega)\mathcal{F}(u)(\zeta_Y) = \mathcal{F}(u)\mathcal{F}(\omega_{\theta,Y})(\zeta_Y) = \mathcal{F}(u)\theta(Y)(\zeta_Y).$$

Moreover we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(Y) \arrow[r, "\theta(Y)"] \arrow[d, "\mathcal{F}(u)"'] & \mathcal{F}(Y) \arrow[d, "F(u)"] \\ \mathcal{F}(X) \arrow[r, "\theta(X)"'] & \mathcal{F}(X) \end{tikzcd}$$

thus we obtain $\mathcal{F}(u)\theta(Y)(\zeta_Y) = \theta(X)\mathcal{F}(u)(\zeta_Y) = \theta(X)(\zeta_X)$.
This proves compatibility with the inverse system of groups.
Thus we obtain a group map $\phi : \mathrm{Aut}(\mathcal{F}) \to \varprojlim \mathrm{Aut}_{\mathcal{C}}(X)$. We just need to prove it is an isomorphism, i.e. that for any $\omega = (\omega_X) \in \varprojlim \mathrm{Aut}_{\mathcal{C}}(X)$ there exists a unique $\theta \in \mathrm{Aut}(\mathcal{F})$ mapping to $\omega$.
By proposition 2.5.1 to give an automorphism $\theta : \mathcal{F} \to \mathcal{F}$ it is enough to give natural transformations $\mathrm{Hom}_{\mathcal{C}}(X, -) \to \mathrm{Hom}_{\mathcal{C}}(X, -)$ for all $X \in \mathcal{G}$.
Moreover take $Y \in \mathcal{C}$ and let $X$ be a Galois object dominating all the components of $Y$.
By corollary 2.4.4 and Yoneda lemma we have the commutative diagram

$$\begin{tikzcd} \mathcal{F}(Y) \arrow[r, "\theta(Y)"] \arrow[d, "v_{\zeta_X}^{-1}"'] & \mathcal{F}(Y) \\ \mathrm{Hom}_{\mathcal{C}}(X, Y) \arrow[r, "(-)\circ\omega"'] & \mathrm{Hom}_{\mathcal{C}}(X, Y) \arrow[u, "v_{\zeta_X}"'] \end{tikzcd}$$

for some $\omega : X \to X$.
Writing down things explicitly one sees that for any $\zeta \in F(Y)$ we have $\theta(Y)(\zeta) = F(\omega_\zeta \circ \omega)(\zeta_X)$, where $\omega_\zeta$ is the unique morphism $X \to Y$ such that $F(\omega_\zeta)(\zeta_X) = \zeta$.
In particular if $Y$ is itself Galois, so that we may choose $X = Y$ and $\zeta = \zeta_X$, we see that $\omega_\zeta = id_Y$. Thus $\theta(X)(\zeta_X) = F(\omega)(\zeta_X)$ and $\omega_{\theta,X}$ is the unique arrow $X \to X$ such that $F(\omega_{\theta,X})(\zeta_X) = \theta(X)(\zeta_X) = F(\omega)(\zeta_X)$.
Then we readily see that choosing $\omega = \omega_X$ yields the unique automorphism $\theta$ mapping to $\omega = (\omega_X) \in \varprojlim \mathrm{Aut}_{\mathcal{C}}(X)$.
This proves that

$$\pi_1(\mathcal{C}, \mathcal{F}) \simeq \varprojlim_{\mathcal{G}^\varsigma} \mathrm{Aut}_{\mathcal{C}}(X)$$

as groups. Finally we need to prove that $\phi$ is an homeomorphism.
Denote with $\pi_Y : \varprojlim \mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Aut}_{\mathcal{C}}(Y)$ the projection on the $Y$ component. Since the open subsets $U_{Y,\omega} := \pi_Y^{-1}(\omega)$ for varying $Y$ and $\omega \in \mathrm{Aut}_{\mathcal{C}}(Y)$

form a basis for the topology on $\varprojlim \mathrm{Aut}_{\mathcal{C}}(X)$, it will suffice to prove that the $\phi^{-1}(U_{Y,\omega})$ are open. But

$$\begin{aligned}\phi^{-1}(U_{Y,\omega}) &= \left\{\theta \in \mathrm{Aut}_{\mathbf{Fct}}(\mathcal{F}) | \omega_{\theta,Y} = \omega\right\} = \\ &= \left\{\theta \in \mathrm{Aut}_{\mathbf{Fct}}(\mathcal{F}) | \theta(Y)(\zeta_Y) = F(\omega)(\zeta_Y)\right\} = \\ &= p_Y^{-1}\left(\left\{\sigma \in \mathrm{Aut}_{\underline{\mathbf{FSet}}}(\mathcal{F}(Y)) | \sigma(\zeta_Y) = F(\omega)(\zeta_Y)\right\}\right)\end{aligned}$$

which is clearly open in $\pi_1(\mathcal{C}, \mathcal{F})$ (we wrote $p_Y : \pi_1(\mathcal{C}, \mathcal{F}) \to \mathrm{Aut}_{\underline{\mathbf{FSet}}}(\mathcal{F}(Y))$ for the projection on the $\mathrm{Aut}_{\underline{\mathbf{FSet}}}(\mathcal{F}(Y))$ factor).
Finally $\phi^{-1}$ is continuous as well by the closed map lemma. □

We have now all the ingredients to prove the main theorem.

## 2.6 Proof of the main theorem

We are ready to prove theorem 2.2.6.

**Essential surjectivity**.

Consider a finite continuous $\pi_1(\mathcal{C}, \mathcal{F})$-set $E$.
It is easy to see that in the category of $G$-**Set** connected means transitive. Then, since $\mathcal{F}$ commutes with coproducts, we may assume that $E$ is transitive.
Denote with $S_e$ the stabilizer of an element $e \in E$. This is open in $\pi_1(\mathcal{C}, \mathcal{F})$ since the action is continuous. Thus the finite intersection $\cap_{e \in E} S_e$ is open too and thus of finite index. This implies it is non trivial, and using the description of $\pi_1(\mathcal{C}, \mathcal{F})$ given in proposition 2.5.4, we see that there exists a normal open subgroup $H \subseteq \cap_{e \in E} S_e$ of $\pi_1(\mathcal{C}, \mathcal{F})$ such that $\pi_1(\mathcal{C}, \mathcal{F})/H \simeq \mathrm{Aut}_C(X)$. Moreover, by definition of $H$, the action of $\pi_1(\mathcal{C}, \mathcal{F})$ on $E$ factors through an action of $\mathrm{Aut}_C(X)$ on $E$.
We claim that $\mathcal{F}(X/S_e) \simeq E$ as $\mathrm{Aut}_{\mathcal{C}}(X)$-set (here we consider $S_e$ as a subgroup of $\mathrm{Aut}_C(X)$).
But being transitive $E$ is isomorphic to the left coset space $\mathrm{Aut}_{\mathcal{C}}(X)/S_e$. Moreover for $X$ being Galois we have that $\mathrm{Aut}_{\mathcal{C}}(X) \simeq F(X)$ as $\mathrm{Aut}_{\mathcal{C}}(X)$-sets. Then

$$E \simeq \mathrm{Aut}_{\mathcal{C}}(X)/S_e \simeq F(X)/S_e \simeq F(X/S_e)$$

(the last isomorphism follows by (G5)).
Now this isomorphism of $\mathrm{Aut}_{\mathcal{C}}(X)$-set clearly lift to an isomorphism of $\pi_1(\mathcal{C}, \mathcal{F})$-set. Thus $E \simeq \mathcal{F}(X/S_e)$ and $\mathcal{F}$ is essentially surjective. □

**Faithfulness**.

Consider two object $X, Y \in \mathcal{C}$, which we may again suppose to be connected

by lemma 2.3.4.
First of all if $u, u' : X \to Y$ are such that $\mathcal{F}(u) = \mathcal{F}(u')$, then $v_\zeta(u) = v_\zeta(u')$ for any $\zeta \in \mathcal{F}(X)$. But $v_\zeta$ is injective by lemma 2.3.5 , thus $u = u'$ and this prove faithfulness. $\square$

**Fullness**.

Take a $\pi_1(\mathcal{C}, \mathcal{F})$-equivariant map $u' : \mathcal{F}(X) \to \mathcal{F}(Y)$ and consider a Galois object $Z$ dominating both $X$ and $Y$. Moreover let $f : Z \to X$ be an arrow.
By bijectivity of $v_{\zeta_Z}(Y)$ there exists a unique $f' : Z \to Y$ such that $\mathcal{F}(f')(\zeta_Z) = u' \circ \mathcal{F}(f)(\zeta_Z)$. Moreover $\mathcal{F}(f')$ and $u' \circ \mathcal{F}(f)$ are equivariant maps between transitive sets. Thus they are uniquely determined by the image of a single element and we get that $\mathcal{F}(f') = u' \circ \mathcal{F}(f)$.
If we show that $f'$ factors as $Z \xrightarrow{f} X \xrightarrow{u} Y$ it will follow that $\mathcal{F}(u) \circ \mathcal{F}(f)(\zeta_Z) = \mathcal{F}(f')(\zeta_Z) = u' \circ \mathcal{F}(f)(\zeta_Z)$. But then it would be $\mathcal{F}(u) = u'$ (once again because they are equivariant maps between transitive sets) and we would be done.
We need a lemma.

**Lemma 2.6.1** *Let $Z$ be Galois and $X$ connected. Consider an arrow $f : Z \to X$ and let $H_f = \{h \in \mathrm{Aut}(Z) | f \circ h = f\}$.*
*Then $X \simeq Z/H_f$.*

**Proof**

By the definition of $H_f$ we see that $f$ factors as $Z \to Z/H \xrightarrow{f'} X$.
By remark 2.3.2 we see that $f$ is a strict epimorphism, thus $f'$ is a strict epimorphism too.
We just need to prove that $f'$ is a monomorphism, i.e. that $\mathcal{F}(f')$ is injective.
By (G5) $\mathcal{F}(f')$ is the quotient map $\mathcal{F}(Z)/H \to \mathcal{F}(X)$, so this amount to prove that $\mathcal{F}(f)(\zeta) = \mathcal{F}(f)(\zeta') \implies \mathcal{F}(h)(\zeta) = \zeta'$ for some $h \in H$.
But $\mathrm{Aut}_{\mathcal{C}}(Z)$ act transitively on $\mathcal{F}(Z)$, thus there exists $h \in \mathrm{Aut}_{\mathcal{C}}(Z)$ such that $\mathcal{F}(h)(\zeta) = \zeta'$. Moreover $\mathcal{F}(f) \circ \mathcal{F}(h)(\zeta) = \mathcal{F}(f)(\zeta') = \mathcal{F}(f)(\zeta)$ and by injectivity of $v_\zeta(Z)$ this implies that $f \circ h = f$.
Thus $h \in H$ and this conclude the proof. $\square$

Thus to show that $f'$ factors as $Z \xrightarrow{f} X \xrightarrow{u} Y$ it suffices to prove that $f'$ is fixed by the action of $H_f$.
But by (G5) $F(f)$ is the quotient map $F(Z) \to F(Z)/H$, thus we get that for all $h \in H$

$$F(f') \circ F(h) = u' \circ F(f) \circ F(h) = u' \circ F(f) = F(f').$$

By faithfulness we get $f' \circ h = f'$.
This conclude the proof. $\square$

# 3 ÉTALE TOPOLOGY AND ÉTALE FUNDAMENTAL GROUP

The aim of this final chapter is to apply the general theory we studied so far to the category of schemes.

In the first part of the chapter we deal with Gothendieck topologies in such a category.
This brought outstanding result. The reason is that the Zariski topology is too coarse for many purposes: for example it does not usually give useful informations when computing cohomology groups. The general theory of sites is a smart generalization, which allows in some sense to refine the Zariski topology.
To begin with, we describe a general procedure to construct Grothendieck topologies in the category of schemes. In few words we equip with a site structure a full subcategory $\mathcal{C} \subseteq \mathfrak{Sch}_S$, defining coverings to be surjective families of morphisms belonging to some class $\mathcal{E}$.
The most important example is étale topology, in which $\mathcal{C}$ is the category of étale $S$-schemes of finite type, and $\mathcal{E}$ is the class of étale morphisms of finite type. Étale is a French word used to describe the condition of the sea when it's completely calm. In fact the conditions defining étale morphisms, flatness and unramifiedness, imply a certain regularity of their fibers.

The second part of the chapter deals with the étale fundamental group of a scheme.
We will show that the category of schemes finite and étale over a connected base scheme is a Galois category. Then the theory developed in the second chapter yields a powerful dictionary between geometry and algebra.

To conclude the thesis we give two examples of how this geometric theory has deep implication in purely arithmetic problems.
The first one deal with the case in which the base scheme is the spectrum of a field.
This is an easy but already interesting situation. In fact we will see that in this case the étale fundamental group is just the absolute Galois group of the field and that étale cohomology is just a geometric reformulation of classical Galois cohomology.
In the second example we revise and generalize a classical arithmetic theorem, Hilbert Theorem 90.

The notion of étale cohomology first appeared in [9]; the subject was treated again in deeper details in SGA IV ([5]).
There are mainly three modern books on this topic: [6], [18] and [22]. Another reliable source is the Stacks Project (see [24], [25] and [26]). A really nice treatment about Galois theory for schemes, with emphasis on arithmetic implications is [14].

## 3.1 Étale topology

### Topologies in the category of scheme

We're going to describe a general procedure to construct topologies on the category of schemes.

Consider a class $\mathcal{E}$ of morphisms satisfying the following properties (*):

- All isomorphism are in $\mathcal{E}$.
- Composition of morphisms in $\mathcal{E}$ is again in $\mathcal{E}$.
- Base change of morphisms in $\mathcal{E}$ is again in $\mathcal{E}$.

We call $\mathcal{E}$-morphism a morphism belonging to $\mathcal{E}$.

Now fix a subcategory $\mathcal{C}_S \subseteq \mathfrak{Sch}_S$ satisfying the following property (**):

- for any object $X \to S$ in $\mathcal{C}_S$ and any $\mathcal{E}$-morphism $Y \to X$, the composite $Y \to X \to S$ belongs to $\mathcal{C}_S$.

For example we can consider the categories $\mathcal{E}_S$ and $\mathbf{LFT}_S$ respectively of $\mathcal{E}$-morphisms and of morphisms locally of finite type over $S$.

This set up allows some definition.

**Definition 3.1.1** *We call* **$\mathcal{E}$-covering** *of an object $X \in \mathfrak{C}_S$ a surjective covering $\{X_i \to X\}$ consisting of $\mathcal{E}$-morphisms.*
*We call* **$\mathcal{E}$-topology** *(or* **$\mathcal{E}$-site***) on $\mathcal{C}_S$ the topology given by the $\mathcal{E}$ covering.*
*We call* **small $\mathcal{E}$-site** *the $\mathcal{E}$-topology on $\mathcal{E}_S$.*
*We call* **big $\mathcal{E}$-site** *the $\mathcal{E}$-topology over* $\mathbf{LFT}_S$.

**Remark 3.1.2** All of these are well defined because of the conditions (*) and (**) and the fact that surjectivity is stable under base change.

In the following sections we will introduce some possible choices for $\mathcal{E}$. We anticipate them here.

Choosing $\mathcal{E}$ to be the class of open immersions, we get the usual **Zariski topology**. In fact all the other topologies we will consider are finer then this, meaning that the class of open immersion is contained in $\mathcal{E}$.

The most important case we will deal with is the **étale topology**, which is the small $\mathcal{E}$-site obtained choosing $\mathcal{E}$ to be the class of étale morphism (definition 3.1.24) of finite type.
This is the most used Grothendieck topology on the category of schemes. In fact étale morphisms are the algebraic analogous of local isomorphisms in differential geometry and étale topology is the natural substitute of the euclidean topology on complex varieties.

For some purpose one may need an even finer topology. A commonly used one is the **flat topology**, which is the big $\mathcal{E}$-site, where $\mathcal{E}$ is the collection of flat morphism (definition 3.1.10) locally of finite type.

These are just some examples, but many other topologies may be defined using the same technique.

## Flat morphisms

For simplicity all rings are Noetherian and all schemes are locally Noetherian.

We start by recalling the definition of flat modules and algebras.

**Definition 3.1.3** *An $A$-module $M$ is said to be* **flat** *if (-)$\otimes_A M$ is an exact functor $A$-mod$\to$ $A$-mod.*
*An $A$-algebra $f : A \to B$ is said to be flat (or $f$ is said to be flat) if $B$ is flat as an $A$-module.*

We want to give a couple examples to clarify the geometric intuition about flatness.
This in some sense correspond to the idea that fibers of a morphism should vary continuously.

**Example 3.1.4** Let $k$ be an algebraically closed field and consider the ring map $k[y] \to A := k[x, y]/(y - x^2)$.
We readily see that this is a flat ring map since $A$ is a free $k[y]$-module, with generators $1, x$ (see proposition 3.1.9).
In fact the fibers of the projection $\operatorname{Spec} A \to \operatorname{Spec} k[y]$ over the closed points

$(y-a) \in \operatorname{Spec} k[y]$ are either two different point (when $a \neq 0$) or one point with multiplicity 2 (when $a = 0$).
There is no pathological variation of the fibers.
Now consider the map $k[x] \to A := k[x,y]/(xy)$.
This is not flat: multiplication by $x$ is an injective map $k[x] \to k[x]$, but it is not after tensoring with $A$.
In fact the fiber of the projection $\operatorname{Spec} A \to \operatorname{Spec} k[x]$ over a closed point $(x-a) \in \operatorname{Spec} k[x]$ is a single point whenever $a \neq 0$, but becomes a whole line when $x = 0$.
The fibers does not vary in a continuous way.

Another really important notion we need to introduce is faithful flatness.

**Definition 3.1.5** *We say that a flat $A$-module $M$ is* **faithfully flat** *if*

$$N \neq 0 \implies M \otimes_A N \neq 0.$$

*We say that an $A$-algebra $B$ is faithfully flat if it is so as an $A$-module.*

**Remark 3.1.6** A faithfully flat ring map is injective.
In fact taking $M = (a) \subseteq A$ for some $0 \neq a \in A$ we find $B \otimes_A (a) \simeq aB \neq 0 \implies a \cdot 1 = f(a) \neq 0$.

We give a useful characterization of faithfully flat algebras.

**Proposition 3.1.7** *Let $A \neq 0$ and let $\varphi : A \to B$ be flat.*
*Then the following are equivalent:*

*i) $\varphi$ is faithfully flat.*

*ii) $B \otimes_A M' \to B \otimes_A M \to B \otimes_A M''$ exact $\implies M' \to M \to M''$ exact.*

*iii) The induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.*

**Proof**
**i)** $\iff$ **ii)**
If ii) holds then $B \otimes_A M = 0 \implies B \otimes_A M \xrightarrow{\cdot 0} B \otimes_A M \to 0$ exact $\implies M \xrightarrow{\cdot 0} M \to 0$ exact $\implies M = 0$.
Conversely assume that $M' \xrightarrow{g_1} M \xrightarrow{g_2} M''$ becomes exact after tensoring with $B$.
Clearly $\operatorname{im} g_1 \subseteq \ker g_2$. Moreover it is easy to see that $B \otimes_A (\ker g_2 / \operatorname{im} g_1) \simeq \ker(1 \otimes g_2)/\operatorname{im}(1 \otimes g_1) = 0$. Thus $\ker g_2 / \operatorname{im} g_1 = 0$ since $B$ is faithfully flat.
**i)** $\iff$ **iii)**

Assume that $A \to B$ is faithfully flat.
For any prime $\mathfrak{p} \subseteq A$ the residue field $k(\mathfrak{p})$ is a non-zero $A$-module. Thus $B \otimes_A k(\mathfrak{p})$ is non-zero too and $\operatorname{Spec}\big(B \otimes_A k(p)\big)$ is non-empty.
Conversely let $0 \neq M \in A$-mod and let $0 \neq x \in M$.
Since $B$ is $A$-flat and $Ax \subseteq M$, we have $B \otimes_A Ax \subseteq B \otimes_A M$. Thus it suffices to show that $B \otimes_A Ax \neq 0$ for all $x \in M \setminus \{0\}$. But $Ax = A/I$ where $I = \ker\big([a \to ax] : A \to M\big)$, so $B \otimes_A Ax \simeq B/IB$ and we're just left to show that $IB \neq B$. Finally $IB \subseteq \mathfrak{m}B$ where $\mathfrak{m}$ is a maximal ideal of $A$ containing $I$, and $\mathfrak{m}B \neq B$, otherwise there would be no primes in $B$ mapping to $\mathfrak{m}$ via $\operatorname{Spec} B \to \operatorname{Spec} A$. □

An important feature of faithful flatness is that many properties can be verified after a faithfully flat base change.
Here is a first example:

**Lemma 3.1.8** *If $A \to B$ is a ring map and $M$ is $A$-flat, then $B \otimes_A M$ is $B$-flat.*
*Conversely an $A$-module $M$ is flat if it is flat after a faithfully flat base change.*

**Proof**
Let $N$ be an $A$-module.
The thesis follows at once by the isomorphism

$$B \otimes_A (M \otimes_A N) \simeq (B \otimes_A M) \otimes_A N$$

and point ii) of the previous proposition. □

Next we're going to investigate the local properties of flatness in the special case of finitely generated modules.

**Proposition 3.1.9** *Let $M$ be a finitely generated $A$-module. The following are equivalent:*

*i) $M$ is $A$-flat.*

*ii) $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-free for all $\mathfrak{m} \subseteq A$ prime (actually $\mathfrak{m} \subseteq A$ maximal will do).*

*iii) $\widetilde{M}$ is a locally free $\mathcal{O}_{\operatorname{Spec} A}$-module.*

**Proof**
**i)** $\implies$ **ii)**
First we prove that $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-flat, then we prove that flat module over local rings are free.

Let $N$ be an $A_{\mathfrak{p}}$-module. Local flatness follows from the isomorphism $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N \simeq \big(M \otimes_A N\big)_{\mathfrak{p}}$ and the fact that injectivity is a local property (see [2], proposition 3.9).
Now let $M$ be a non-zero flat $A$-module, with $(A, \mathfrak{m})$ local ring. We claim that $M$ is free.
First note that $M/\mathfrak{m}M$ is an $A/\mathfrak{m}$-vector space. Take an $A/\mathfrak{m}$-basis $\bar{x}_1, ..., \bar{x}_n$ for $M/\mathfrak{m}M$, and lift it to get elements $x_1, ..., x_n \in M$. By Nakayama lemma this is a system of generator for $M$, which means we have a surjective map $\big[(a_1, ..., a_n) \mapsto \sum a_i x_i\big] : A^n \to M$.
Now let $K$ be its kernel, so that we have the exact sequence

$$0 \to K \to A^n \to M \to 0.$$

Tensoring this with $A/\mathfrak{m}$ preserve exactness (look at the long exact sequence of Tor, keeping in mind that $\mathrm{Tor}_A^1(M, N) = 0$ for all $N \in A$-mod), thus we get the exact sequence

$$0 \to K/\mathfrak{m}K \to \big(A/\mathfrak{m}\big)^n \to M/\mathfrak{m}M \to 0.$$

But the rightmost map is an isomorphism, then $K/\mathfrak{m}K = 0$ and by Nakayama lemma $K = 0$ and $A^n \simeq M$.

**ii)** $\implies$ **iii)**

Take $\mathfrak{p} \in \operatorname{Spec} A$. We need to show there exists $a \notin \mathfrak{p}$ such that $M_a$ is a free $A_a$-module.
Let $x_1, ..., x_n \in M$ be elements whose images in $M_{\mathfrak{p}}$ form an $A_{\mathfrak{p}}$-basis and consider the map

$$\big[\varphi : (a_1, ..., a_n) \mapsto \sum a_i x_i\big] : A^n \to M.$$

We claim that this map induces an isomorphism localizing over some element $a \in A$.
Recall that given an $A$-module $N$, the support of $N$ is defined to be

$$\operatorname{Supp} N := \{\mathfrak{p} \in \operatorname{Spec} A | M_{\mathfrak{p}} \neq 0\}.$$

Now denote with $K$ and $C$ respectively the kernel and cokernel of $\varphi$.
Clearly $\varphi_a : A_a^n \to M_a$ is an isomorphism if and only if $K$ and $C$ are zero after localizing at any $\mathfrak{q} \in D(a)$, if and only if $D(a) \subseteq \operatorname{Spec} A \setminus \big(\operatorname{Supp} K \cup \operatorname{Supp} C\big)$.
Since $\mathfrak{p} \in \operatorname{Spec} A \setminus \big(\operatorname{Supp} K \cup \operatorname{Supp} C\big)$, if we show that $\operatorname{Supp} K$ and $\operatorname{Supp} C$ are closed we will be done. Actually the following result holds: if $M$ is a finitely generated $A$-module then $\operatorname{Supp} M = V(\operatorname{Ann}(M))$.
In fact if $\mathfrak{q} \not\supseteq \operatorname{Ann} M$ there is an element $a \in A \setminus \mathfrak{q}$ annihilating $M$, thus

$M_{\mathfrak{q}} = 0$.
Conversely assume that $M_{\mathfrak{q}} = 0$. Then for all $x_i$ there exists $a_i \in A \setminus \mathfrak{q}$ such that $a_i x_i = 0$. Thus $a := \prod a_i \in A \setminus \mathfrak{q}$ and $a \in \operatorname{Ann} M$.
**iii)** $\implies$ **i)**
By hypothesis there exists a covering $\operatorname{Spec} A = \bigcup_{i=1}^n D(a_i)$ (finite since $A$ is Noetherian) such that $M_{a_i}$ is a free $A_{a_i}$-module.
Define $B = \prod A_{a_i}$. First we claim this is a faithfully flat $A$-algebra, i.e. that $\amalg \operatorname{Spec} A_{a_i} \to \operatorname{Spec} A$ is surjective.
In fact since the $D(a_i)$ cover $\operatorname{Spec} A$, we have that $a_1, ..., a_n$ generate $A$ and a prime $\mathfrak{p} \subseteq A$ cannot contain all of the $a_i$ i.e. it is contained in some $D(a_i) \simeq \operatorname{Spec} A_{a_i}$.
Now since $B$ is a faithfully flat $A$-algebra, by lemma 3.1.8 it suffices to show that $M' := B \otimes_A M$ is $B$-flat. But

$$B \otimes_A M = \left(\prod A_{a_i}\right) \otimes_A M \simeq \prod \left(A_{a_i} \otimes_A M\right) \simeq \prod M_{a_i}$$

as $B$-modules and for any $B$-module $N$ we have $M' \otimes_B N \simeq \prod \left(M_{a_i} \otimes_{A_{a_i}} N\right)$ (with the obvious $B$-module structure on $M_{a_i} \otimes_{A_{a_i}} N$).
This shows that $M'$ is $B$-flat. $\square$

We're ready to define flatness for scheme morphisms.

**Definition 3.1.10** *We say that a scheme morphism $f : Y \to X$ is* **flat** *if for all $y \in Y$ the induced map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat.*

We already mentioned in the previous section the flat topology, and we need to show that everything actually works.
In its definition we chose $\mathcal{E}$ to be the the class of flat morphism locally of finite type. We didn't introduce yet the second condition:

**Definition 3.1.11** *We say that a ring map $A \to B$ is of* **finite type** *if $B$ is a finitely generated $A$-algebra.*
*We say that a scheme morphism $f : Y \to X$ is* **locally of finite type** *if for any $U \subseteq X$ and $V \subseteq f^{-1}(U)$ open affine, the induced map $\mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ is of finite type.*

We shall prove now that flat morphisms locally of finite type indeed satisfies conditions (*) (clearly isomorphisms are flat).
These are quite clear for the property "local of finite type".
For flatness we have:

**Proposition 3.1.12**

*i) Composition of flat morphisms is flat.*

*ii) Base change of flat morphism is flat.*
*Conversely a morphism is flat if it becomes flat after a faithfully flat base change.*

**Proof**
We may clearly reduce to the affine case.
**i**) If $A \to B$ and $B \to C$ are flat and $M$ is an $A$-module, the thesis follows from the canonical isomorphism of $A$-modules $M \otimes_A C \simeq (M \otimes_A B) \otimes_B C$ together with the fact that composition of exact functors is exact.
**ii**) This is just lemma 3.1.8. □

## Étale morphisms

First of all we need to introduce the notion of unramified morphism.

**Definition 3.1.13** *Let $f : A \to B$ be of finite type.*
*We say that $f$ (or eqivalently $B$) is* **unramified** *at $\mathfrak{q} \in \operatorname{Spec} B$ if $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$ is a field, finite and separable over $k(\mathfrak{p})$.*
*We say that $f$ is unramified if it is unramified at any $\mathfrak{q} \in \operatorname{Spec} B$.*

**Remark 3.1.14** The fact that $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$ is a field means that $\mathfrak{p}$ extends via $A \to B \to B_\mathfrak{q}$ to the maximal ideal of $B_\mathfrak{q}$.

Once again we would like to suggest a geometric intuition for unramifiedness.
This correspond to the idea that fibers does contain just simple points (with multiplicity equal to 1).

**Example 3.1.15** Let $k$ be an algebraically closed field with $\operatorname{char} k \neq 2$.
Then the ring map $k \to k[x]/(x^2 - a)$ is unramified if and only if $a \neq 0$ (see proposition 3.1.17).
In fact the fiber over the point of $\operatorname{Spec} k$ is given by the points $(x \pm a) \in \operatorname{Spec} k[x]/(x^2 - a)$ if $a \neq 0$ and by the single point $(x) \in \operatorname{Spec} k[x]/(x^2)$ "counted twice" if $a = 0$.

The next lemma shows that unramifiedness can be checked on fibers.

**Lemma 3.1.16** *Let $f : A \to B$ be of finite type. Then $f$ is unramified if and only if $k(\mathfrak{p}) \to k(\mathfrak{p}) \otimes_A B$ is unramified for all $\mathfrak{p} \in \operatorname{Spec} A$.*

**Proof**
We know that $\operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$ is the fiber over $\mathfrak{p}$ of the natural map $\operatorname{Spec} B \to \operatorname{Spec} A$. Thus a prime in $k(\mathfrak{p}) \otimes_A B$ is just a prime $\mathfrak{q} \subseteq B$ such that $f^{-1}(\mathfrak{q}) = \mathfrak{p}$.
Moreover $k(\mathfrak{p}) \otimes_A B \simeq B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$, where $B_{\mathfrak{p}}$ is the localization of $B$ over $f(A \backslash \mathfrak{p})$. The thesis now follows from the canonical isomorphism of $k(\mathfrak{p})$-algebras $(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})_{\mathfrak{q}} \simeq B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$. □

For this reason unramified rings over fields are of central importance.

We rephrase the definition in this case for sake of clarity:
$B$ is unramified at $\mathfrak{q}$ over a field $k$ if $B_{\mathfrak{q}}$ is a field, finite and separable over $k$.

Unramified algebras over fields can be characterized in many ways:

**Proposition 3.1.17** *Let $B$ be a finite algebra over a field $k$, and fix an algebraic closure $\bar{k}/k$.*
*Then the following are equivalent:*

*i) $B$ is unramified over $k$.*

*ii) $B$ is a finite product of finite separable field extension of $k$.*

*iii) $\bar{B} := B \otimes_k \bar{k}$ is unramified over $\bar{k}$ (i.e. a finite product of copies of $\bar{k}$).*

**Proof**
Before we start is important to mention that $B$ is an Artinian ring, being a finite dimensional $k$-vector space.
In particular it has finitely many prime ideal which are all maximal and

$$B \simeq B/\mathfrak{q}_1^r \oplus \cdots \oplus B/\mathfrak{q}_n^r$$

for some $r$.
**i)** $\iff$ **ii)**
If $B = \prod k_i$, the $k_i$ finite and separable over $k$, then the primes in $B$ are all of the form $\mathfrak{m}_i := \pi_i^{-1}(0)$, $\pi_i : B \to k_i$ being the projections. One easily compute that $B_{\mathfrak{m}_i} \simeq k_i$ and this shows that $B$ is unramified over $k$.
Conversely we know by hypothesis that

$$B_{\mathfrak{q}_i} \simeq \left(B/\mathfrak{q}_i^r\right)_{\mathfrak{q}_i} \simeq B_{\mathfrak{q}_i}/\mathfrak{q}_i^r B_{\mathfrak{q}_i}$$

is a field for all $i$. This can be the case only if $r = 1$, hence we find

$$B \simeq B/\mathfrak{q}_1 \oplus \cdots B/\mathfrak{q}_n.$$

Moreover $B_{\mathfrak{q}_i} = B_{\mathfrak{q}_i}/\mathfrak{q}_i B_{\mathfrak{q}_i}$ is the fraction field of $B/\mathfrak{q}_i$, i.e. $B/\mathfrak{q}_i$ itself.
**ii)** $\iff$ **iii)**
Assume that $B \simeq \prod k_i$ is a finite product of finite separable field extension of $k$.
By the primitive element theorem we have that $k_i \simeq k[x]/f_i(x)$. Thus

$$B \otimes_k \bar{k} = \left(\prod k[x]/f_i(x)\right) \otimes_k \bar{k} \simeq \prod \bar{k}[x]/f_i(x) \simeq \prod \bar{k}$$

(the last isomorphism since the $f_i$ splits completely in $\bar{k}[x]$).
Conversely assume that $\bar{B} \simeq \prod \bar{k}$ is a finite product of copies of $\bar{k}$.
We'll need the following lemma

**Lemma 3.1.18** *A finite algebra $B$ over a field $k$ is reduced if and only if it is a product of finite field extension of $k$*

**Proof**
The "if" part is obvious.
Conversely assume that $B$ is reduced. Since the nilradical of a direct sum is the direct sum of the nilradicals we may assume that $B$ is indecomposable, i.e. that it does not have idempotents different from 0 and 1.
Now take an element $b \in B \setminus \{0\}$. For $B$ being Artinian, the decreasing chain of ideal $(b) \supset (b^2) \supset (b^3) \supset \cdots$ must stabilize and for $n$ sufficiently large there exists an element $a \in B$ such that $b^n = b^{n+1}a$.
Then for all $i > 0$ we have $b^n = b^{n+i}a^i$ and in particular $b^n = b^{2n}a^n$.
Multiplying each side by $a^n$ we find that $(ba)^n = (ba)^{2n}$. i.e. that $a^n b^n$ is idempotent. Thus by assumption it must be equals to 0 or 1.
But $b^n = b^n(ab)^n$ and the first case is impossible. Hence $1 = b^n a^n = b(b^{n-1}a^n)$ and $b$ is invertible. □

The lemma implies that $B_{\text{red}} \simeq \prod k_i$ is a finite product of finite field extension of $k$.
Now we claim that

$$\mathrm{Hom}_k(B, \bar{k}) \simeq \mathrm{Hom}_k(\prod k_i, \bar{k}) \simeq \amalg \mathrm{Hom}_k(k_i, \bar{k}).$$

The first isomorphism follows because any $k$-algebra map $B \to \bar{k}$ factors through $B_{\text{red}}$.
To prove the second take $\psi \in \mathrm{Hom}_k(\prod k_i, \bar{k})$. Precomposing this with the inclusions $k_i \to B$, we get operation-preserving maps $\psi_i : k_i \to \bar{k}$. Assume that $\psi_i \neq 0$. Then for any $b \in k_j$ we have $0 = \psi_i(1)\psi_j(b) = 0 \implies \psi_j(b) = 0$.
This proves that $\mathrm{Hom}_k(\prod k_i, \bar{k}) \simeq \amalg \mathrm{Hom}_k(k_i, \bar{k})$.
Hence we find that

$$\left|\mathrm{Hom}_k(B, \bar{k})\right| = \sum \left|\mathrm{Hom}_k(k_i, \bar{k})\right|.$$

But it's easy to see that $\mathrm{Hom}_k(B,\bar{k}) \simeq \mathrm{Hom}_{\bar{k}}(\bar{B},\bar{k})$ and by hypothesis

$$\left|\mathrm{Hom}_{\bar{k}}(\bar{B},\bar{k})\right| = \dim_{\bar{k}}(\bar{B}) = \sum [k_i : k].$$

The thesis now follows observing that $\left|\mathrm{Hom}_k(k_i,\bar{k})\right| \leq [k_i : k]$, and equality holds if and only if $k_i$ is separable over $k$ (this is a well known result, see for example [21] lemma 1.1.6). $\square$

**Remark 3.1.19** Actually *i)* $\implies$ *ii)* holds with even weaker hypothesis, namely when $B$ is of finite type over $k$.
In fact given a prime $\mathfrak{q} \subseteq B$ we have canonical inclusion $k \subseteq B/\mathfrak{q} \subseteq B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} = B_{\mathfrak{q}}$. These imply in turn that $B/\mathfrak{q}$ is integral over $k$ (since $B_{\mathfrak{q}}$ is finite over $k$) and that $B/\mathfrak{q}$ is a field (see [2], proposition 5.7). Hence $B$ is a Noetherian ring of dimension 0, i.e. an Artin ring, and the proof now runs as in the previous case.

We're interested in giving two other characterization of unramifiedness, which have a more geometric nature.

Recall that given an $A$ algebra $B$, the module of Kahler differentials, denoted with $\Omega_{B/A}$, is defined to be the $B$-module representing the functor

$$\mathrm{Der}_A(B,-) : B-\mathrm{mod} \to B-\mathrm{mod}$$

(an $A$-derivation $B \to M$ is an $A$-module map satisfying the Liebniz rule).
It can be proved that $\Omega_{B/A} \simeq J/J^2$ where $J$ is the kernel of the multiplication map $B \otimes_A B \to B$.
For a comprehensive treatment about differentials see [23], section 125.

The following proposition holds:

**Proposition 3.1.20** *Consider an $A$ algebra $A \to B$. The following are equivalent:*

*i)* $A \to B$ *is unramified.*

*ii)* $\Omega_{B/A} = 0$.

*iii) the diagonal morphism* $\mathrm{Spec}\, B \to \mathrm{Spec}\,(B \otimes_A B)$ *is an open immersion.*

**Proof**
**i)** $\implies$ **ii)**
Clearly $\Omega_{B/A} = 0 \iff (\Omega_{B/A})_{\mathfrak{q}} = 0\ \forall \mathfrak{q} \in \mathrm{Spec}\, B$.
Moreover by Nakayama lemma $(\Omega_{B/A})_{\mathfrak{q}} = 0 \iff (\Omega_{B/A})_{\mathfrak{q}}/\mathfrak{q}(\Omega_{B/A})_{\mathfrak{q}} = 0$.

Now by elementary properties of tensor product and Kahler differentials we have the following isomorphisms of $B_{\mathfrak{q}}$-module

$$(\Omega_{B/A})_{\mathfrak{q}}/\mathfrak{q}(\Omega_{B/A})_{\mathfrak{q}} \simeq \Omega_{B/A} \otimes_B k(\mathfrak{q}) \simeq \big(\Omega_{B/A} \otimes_B (B \otimes_A k(\mathfrak{p}))\big) \otimes_{B\otimes_A k(\mathfrak{p})} k(\mathfrak{q}) \simeq$$
$$\simeq \Omega_{B\otimes_A k(\mathfrak{p})/k(\mathfrak{p})} \otimes_{B\otimes_A k(\mathfrak{p})} k(\mathfrak{q})$$

Thus we can reduce to the case where $A = k$ is a field.
Furthermore since $k \hookrightarrow \bar{k}$ is faithfully flat (lemma 3.1.7) and $\Omega_{\bar{B}/\bar{k}} \simeq \Omega_{B/k} \otimes_k \bar{k}$, it is sufficient to prove that $\Omega_{\bar{B}/\bar{k}} = 0$.
But $\Omega_{\bar{B}/\bar{k}}$ is zero after localizing at any maximal ideals. In fact these are of the form $\mathfrak{m} = \pi^{-1}(0)$, $\pi$ being a projection $\prod \bar{k} \to \bar{k}$, thus we have

$$\big(\Omega_{\bar{B}/\bar{k}}\big)_{\mathfrak{m}} \simeq \Omega_{\bar{B}_{\mathfrak{m}}/\bar{k}} \simeq \Omega_{\bar{k}/\bar{k}} = 0.$$

**ii)** $\implies$ **iii)**
Recall that $\Omega_{B/A} \simeq J/J^2$ where $J = \ker(B \otimes_k B \to B)$.
In general if $J \subseteq A$ is a finitely generated ideal in a ring and $J/J^2 = 0$, by Nakayama lemma there exists an element $a \in 1 + J$ annihilating $J$.
We claim that $A/J \simeq A_a$.
First of all we notice that $a$ is an idempotent. In fact $a = 1 + x$ for some $x \in J$, so $a^2 = a(1 + x) = a + ax = a$.
Thus $A \to A_a$ is surjective, since $1/a^n = a/a^{n+1} = a/a = 1$.
Moreover $b \in \ker(A \to A_a) \iff a^m b = ab = (1 + x)b = 0 \iff b \in J$.
Hence $\ker(A \to A_a) = J$ and $A/J \simeq A_a$.
This proves that the closed subscheme $V(J)$ is isomorphic to the open subscheme $D(a)$. In particular $\operatorname{Spec} B \simeq V(J) \to \operatorname{Spec}(B \otimes_k B)$ is an open immersion.

**iii)** $\implies$ **i)**
By lemma 3.1.16 and proposition 3.1.17 it is sufficient to prove that the geometric fibers of $A \to B$ are unramified.
In other words given an algebraic closure $k(\mathfrak{p}) \subseteq \bar{k}$ we need to prove that $\bar{k} \to \bar{k} \otimes_A B := \bar{B}$ is unramified.
Moreover since open immersions are stable under base change and the following square is cartesian

$$\begin{tikzcd}
\operatorname{Spec} \bar{B} \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec}\big(\bar{B} \otimes_{\bar{k}} \bar{B}\big) \arrow[r] & \operatorname{Spec}\big(B \otimes_A B\big)
\end{tikzcd}$$

we may just assume that $B$ is an algebra over an algebraically closed field $k$.
Furthermore, by definition of unramifiedness and the fact that $k$ is algebraically closed, it is sufficient to prove that $B_{\mathfrak{q}} \simeq k$ $\forall \mathfrak{q} \in \operatorname{Spec} B$. In fact,

since the diagonal morphism $\operatorname{Spec} B_{\mathfrak{q}} \to \operatorname{Spec}\big(B_{\mathfrak{q}} \otimes_k B_{\mathfrak{q}}\big)$ is again an open immersion (by an argument similar to the one we used above), we may just assume that $B$ is local.
Let $y$ be the closed point in $\operatorname{Spec} B$.
By Zariski lemma $B/\mathfrak{m}_y \simeq k$ (since $k$ is algebraically closed), which means there exists a section $g : \operatorname{Spec} k \to \operatorname{Spec} B$ for $\operatorname{Spec} B \to \operatorname{Spec} k$. Hence the following square is cartesian

$$\begin{tikzcd}
\{y\} \arrow[r, "g"] \arrow[d, "g"'] & \operatorname{Spec} B \arrow[d, "\Delta"] \\
\operatorname{Spec} B \arrow[r, "\langle gf,1\rangle"'] & \operatorname{Spec}(B \otimes_k B)
\end{tikzcd}$$

proving that $y$ is open in $\operatorname{Spec} B$.
This implies that $B$ is Artinian, i.e. $\mathfrak{m}_y$ is the unique prime ideal in $B$ (otherwise the set of primes strictly contained in $\mathfrak{m}_y$ couldn't be closed).
But then $\operatorname{Spec}(B \otimes_k B) = \operatorname{Spec} B \times_k \operatorname{Spec} B$ has only one point and the open immersion $\operatorname{Spec} B \to \operatorname{Spec}(B \otimes_k B)$ must be an isomorphism. Hence $B \simeq B \otimes_k B$ and since $\dim_k(B \otimes_k B) = (\dim_k B)^2$, we get $\dim_k B = 1$, i.e. $B \simeq k$. □

As a result we can study how unramifiedness behave with respect to base change:

**Lemma 3.1.21** *Base change of unramified morphism is unramified.*
*Conversely $A \to B$ is unramified if it is unramified after a faithfully flat base change.*

**Proof**
Let $A \to B$ and $A \to A'$ be ring maps and write $B' := A' \otimes_A B$.
The claim follows from the isomorphism $\Omega_{B'/A'} \simeq \Omega_{B/A} \otimes_A A'$. □

As usual we can adapt these definition to schemes.

**Definition 3.1.22** *We say that a scheme morphism $f : Y \to X$ locally of finite type is* **unramified** *at $y \in Y$ if the map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ induced on the stalk is unramified.*
*We say that $f$ is unramified if it is unramified at any $y \in Y$.*

Once again unramifiedness is a property local on the target.

We'd like to carry over to the case of schemes the results of lemma 3.1.16 and proposition 3.1.17.

First given a scheme morphism $f : Y \to X$ we need to define $\Omega_{Y/X}$, the $\mathcal{O}_X$-module of Kahler differentials of $f$.
For this consider open affine coverings $X = \cup_i U_i$ and $f^{-1}(U_i) = \cup_j V_{ij}$ for all $i$. Now $f|_{V_{ij}} : V_{ij} \to U_i$ is a morphism between affine schemes and thus it is induced by a ring map $\varphi : A_i \to B_{ij}$. Hence we can define for all $i, j$ the $\mathcal{O}_{\operatorname{Spec} B_{ij}}$-module $\widetilde{\Omega}_{B_{ij}/A_i}$. These modules glue together yielding an $\mathcal{O}_Y$-module which we define to be $\Omega_{Y/X}$.

**Proposition 3.1.23** *Let $f : Y \to X$ be locally of finite type. The following are equivalent:*

*i) $f$ is unramified.*

*ii) For any $x \in X$ the fiber $Y_x := Y \times_X \operatorname{Spec} k(x) \to \operatorname{Spec} k(x)$ is unramified.*

*iii) For any $x \in X$ the fiber $Y_x$ is a disjoint union of spectrum of finite separable field extension of $k(x)$.*

*iv) Any geometric fiber of $f$ is unramified (i.e. is a disjoint union of an spectra of algebraically closed field).*

*v) $\Omega_{Y/X} = 0$.*

*vi) The diagonal morphism $Y \to Y \times_X Y$ is an open immersion.*

**Proof**
**i)** $\iff$ **ii)**
Let $x$ be a point in $X$ and denote with $Y_x$ the fiber over $x$. The claim follows immediately from the isomorphism $\mathcal{O}_{Y_x,y} \simeq \mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$.
**ii)** $\iff$ **iii)**
This follows covering $Y_x$ with affine schemes and using remark 3.1.19.
**iii)** $\iff$ **iv)**
Let $k$ be a field and fix an algebraic closure $\bar{k}/k$. We need to prove that a morphism $f : X \to \operatorname{Spec} k$ is unramified if and only if $f : X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k} \to \operatorname{Spec} \bar{k}$ is unramified. But if we cover $X = \bigcup X_i$ with open affines, we see that $f$ is obtained by gluing the affine morphism $f_i := f|_{X_i}$, and $f'$ is obtained by gluing their base change along $\operatorname{Spec} k \to \operatorname{Spec} \bar{k}$.
Hence we can reduce to the affine case, which is proved in proposition 3.1.17.
**i)** $\implies$ **v)**
This follows directly by the definition of $\Omega_{Y/X}$ and proposition 3.1.20.
**v)** $\implies$ **vi)**
Consider open affine coverings $X = \cup_i U_i$ and $f^{-1}(U_i) = \cup_j V_{ij}$ for all $i$, where

$U_i \simeq \operatorname{Spec} A_i$ and $V_{ij} = \operatorname{Spec} B_{ij}$.
Then $Y \times_X Y$ is obtained by gluing the affine schemes $\operatorname{Spec}(B_{ij} \otimes_{A_i} B_{ij'})$ and the diagonal morphism $\Delta_{Y/X}$ is obtained by gluing the diagonal morphisms $\operatorname{Spec} B_{ij} \to \operatorname{Spec}(B_{ij} \otimes_{A_i} B_{ij})$. But these are open immersions by hypothesis and proposition 3.1.20.

**vi**) $\implies$ **i**)
Since the diagonal morphism of a base change is the base change of the diagonal morphism and open immersions are stable under base change, we may reduce to the case of a scheme over an algebraically closed field.
Now unraimfiedness may be checked locally on the source and we reduce again to proposition 3.1.20. □

We are finally ready to give the definition of étale morphism:

**Definition 3.1.24** *We call a morphism $f : Y \to X$ étale if it is flat and unramified.*

Recall we have mentioned étale topology at the beginning of the chapter, choosing $\mathcal{E}$ to be the class of étale morphisms of finite type.
Still we haven't met the second notion.

**Definition 3.1.25** *We say that a morphism $f : Y \to X$ is of* **finite type** *if it is locally of finite type and $f^{-1}(U)$ is quasi compact for any $UX$ open affine.*

Once again we need to prove that the axioms defining Grothendieck topologies are satisfied.
Finite typeness is easily seen to be stable under composition and base change.
Moreover we have already proved that the same is true for flatness.
Hence we're just left to prove the following:

**Proposition 3.1.26**

*i) Composition of unramified morphisms is unramified.*

*ii) Base change of unramified morphism is unramified.*
*Conversely a morphism is unramified if it is unramified after a faithfully flat base change.*

**Proof**
Since unramifiedness is local on the target we reduce to the affine case.
**i**) Given ring maps $A \to B \to C$ there is an exact sequence of $C$-modules

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

and the claim follows.
**ii**) This is lemma 3.1.21. □

## 3.2 Étale fundamental group

### Finite morphisms

The category $\mathbf{Fét}_S$ of étale covers is obtained by asking the following extra condition for the structure morphism.

**Definition 3.2.1** *We say that a scheme morphism $f : Y \to X$ is affine if for any $U \subseteq X$ open* **affine** *$V := f^{-1}(U)$ is affine.*
*If furthermore $\mathcal{O}_Y\big(V\big)$ is a finite $\mathcal{O}_X(U)$-module we say that $f$ is* **finite**.

**Definition 3.2.2** *We say that $f : Y \to X$ is an étale cover if it is finite, étale and surjective.*
*We'll sometime say that $Y$ is an étale cover of $X$.*
*We define $\mathbf{Fét}_S$ to be the full subcategory of $\mathfrak{Sch}_S$ whose object are étale covers of $S$.*

As usual we investigate the behavior of finiteness with respect to base change:

**Lemma 3.2.3** *The base change of a finite morphism is finite.*
*Conversely a morphism is finite if it is finite after a faithfully flat base change.*

**Proof**
Clearly it suffices to verify this in the affine case.
But base changing a finite ring map $A \to B$ along $A \to A'$, the claim follows at once tensoring with $A'$ the exact sequence $A^n \to B \to 0$. □

Finiteness is a really strong condition. To give an example it implies good topological properties.

**Proposition 3.2.4** *Finite morphisms are proper, i.e. separated, of finite type and universally closed.*

**Proof**
Let $f : Y \to X$ be a finite morphism of schemes.
By definition it is obtained by gluing affine scheme morphisms

$$f_i : f^{-1}(\operatorname{Spec} A_i) = \operatorname{Spec} B_i \to \operatorname{Spec} A_i,$$

hence the diagonal morphism $\Delta_{Y/X} : Y \to Y \times_X Y$ is obtained by gluing affine scheme morphisms $\operatorname{Spec} B_i \to \operatorname{Spec}(B_i \otimes_{A_i} B_i)$. Since these are closed immersion, $\Delta_{Y/X}$ is a closed immersion too, i.e. $f$ is separated.
$f$ is clearly of finite type.
Finally we prove it is universally closed.

Since finiteness is stable under base change we just need to prove it is closed. Moreover closeness is a property local on the target. Thus we can reduce to the affine case, and we will be done if we prove that $\varphi^{-1}(V(J)) = V(\varphi^{-1}(J))$. Since $V(J) = V(\sqrt{J})$ we reduce to the case of $J$ radical. Moreover if $J$ is radical then $J = \bigcap \mathfrak{p}$, the finite (since the rings are Noetherian) intersection taken on the minimal primes containing $J$. Thus, since $V(J \cap J') = V(J) \cup V(J')$ we can reduce to the case of $J$ prime.
The only non trivial part is $V(\varphi^{-1}(J)) \subseteq \varphi^{-1}(V(J))$. But this is just going-up theorem. $\square$

**Proposition 3.2.5** *A finite flat morphism is open.*

**Proof**
Let $f : Y \to X$ be finite and flat.
Since being open is a property local on the target, we may assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine and $f$ is induced by a ring map $\varphi : A \to B$. Moreover since $f$ is flat, by proposition 3.1.9 we may assume that $B$ is a free $A$-module.
To prove that $f$ is open it suffices to prove that the image of a standard affine open $D(b) \subseteq \operatorname{Spec} B$ is open.
First we claim that $\mathfrak{p} \in f(D(b)) \iff \mathfrak{p}B_b \neq B_b$.
If $\mathfrak{p} \in f(D(b))$ then there exists a prime $\mathfrak{q} \in D(b)$ lying over $\mathfrak{p}$. Thus $\mathfrak{p}B_b \subseteq \mathfrak{q}B_b \neq B_b$. Conversely $f^{-1}(\mathfrak{p}) \subseteq \operatorname{Spec}\big(B_b/\mathfrak{p}B_b\big)$. Hence if $\mathfrak{p}B_b \neq B_b$ we just need to prove that the fiber over $\mathfrak{p}$ is non empty. But this is just going up theorem.
Thus we find that $\mathfrak{p} \in f(D(f)) \iff B_b/\mathfrak{p}B_b \simeq \big(B/\mathfrak{p}B\big)_{\bar{b}} \neq 0 \iff \bar{b} \in B/\mathfrak{p}B$ is not nilpotent.
Now let $T^r + a_1T^{r-1} + \cdots + a_r$ be the characteristic polynomial of $b$ over $A$, i.e. the characteristic polynomial of the matrix associated to the $A$-linear map $B \to B$ given by multiplication by $b$.
Now $B$ is finite thus integral over $A$. Clearly $\bar{b}$ is nilpotent if and only if $a_i = 0 \pmod{\mathfrak{p}} \forall i$.
In conclusion $f(D(b)) = \bigcup_i D(a_i)$ is open. $\square$

**Remark 3.2.6** In fact a more general result holds: any flat finitely presented morphism is open.
For a proof of this fact see [18] I 2.12.

**Remark 3.2.7** The previous propositions shows that the structure morphism of a finite étale morphism is open and closed (at the level of topological spaces).
In particular if the base scheme is connected, the structure morphism must be surjective, i.e. an étale cover.

Another nice property is that morphisms in $\mathbf{Fét}_S$ are themselves finite étale:

**Proposition 3.2.8** *If $u : X \to Y$ and $v : Y \to S$ are scheme morphisms such that $v$ and $v \circ u$ are finite étale, then $u$ is finite étale.*

**Proof**
Using Yoneda lemma we see that $u = \Gamma_u \circ p$ where $\Gamma_u : X \to X \times_S Y$ is the graph of $u$ and $p : X \times_S Y \to Y$ is the projection (cause $\Gamma_u \circ p$ maps to $u \circ (-)$ via the Yoneda embedding).
$p$ is clearly finite étale, by stability under pull back.
Moreover using again Yoneda lemma we see that the following square is cartesian

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\Gamma_u"'] & Y \arrow[d, "\Delta_{Y/S}"] \\ X \times_S Y \arrow[r, "u \times_S id_Y"'] & Y \times_S Y \end{tikzcd}$$

But the diagonal morphism is an open immersion (proposition 3.1.23), thus $\Gamma_u$ is is an open immersion too, since open immersions are stable under base change.
Finally open immersions are clearly finite étale, hence $u = \Gamma_u \circ p$ is finite étale too, being the composition of two finite étale morphisms. $\square$

Finally we study the notion of degree of an étale cover.
The key fact is that if $f : X \to S$ is finite and flat (in particular if it is finite étale) then the $\mathcal{O}_{S,s}$-module $(f_*\mathcal{O}_X)_s$ has finite locally constant rank.
This is because the function

$$[s \mapsto \mathrm{rk}_{X/S}(s) := \mathrm{rk}_{\mathcal{O}_{S,s}}\big((f_*\mathcal{O}_X)_s\big)] : X \to \mathbb{Z}^+$$

is continuous when we endow $\mathbb{Z}^+$ with the discrete topology (since $f_*\mathcal{O}_B$ is a locally free $\mathcal{O}_A$-module by proposition 3.1.9).
We make the following definition:

**Definition 3.2.9** *If $\mathrm{rk}_{X/S}$ is a constant function we call its image the degree of $f$.*
*We'll sometimes denote it with $[X : S]$.*

This happen for example if $S$ is connected.

Given a finite flat morphism $X \to S$ the following properties are easy to prove:

- For any base change $X' \to S'$ of $X \to S$ we have $[X' : S'] = [X : S]$.
- If $X = X' \amalg X''$ then $[X : S] = [X' : S] + [X'' : S]$.

Clearly an isomorphism has constant degree equal to one. The converse holds too:

**Lemma 3.2.10** *If $u : Y \to X$ is finite and flat and $[X : S] = 1$, then $u$ is an isomorphism.*

**Proof**
We may assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. By proposition 3.1.7 all we need to show is that a finite faithfully flat ring map $A \to B$ such that $\operatorname{rk}(\mathfrak{p}) = 1 \, \forall \, \mathfrak{p} \in \operatorname{Spec} A$ is an isomorphism.
By faithful flatness we know it is injective (remark 3.1.6).
It remains to prove surjectivity. Since it is enough to prove it after localizing at all primes and $B$ is a locally free $A$ module of rank 1 by hypothesis, we may assume that $B$ is generated by a single element as $A$-module. Thus write $B = Ab$ for some $b \in B$ and let

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0$$

be an integral dependence relation for $b$ over $A$.
By hypothesis $ab = 1$ for some $a \in A$, thus multiplying the relation by $a^{n-1}$ we find $b = -(a_1 + a_2 a + \cdots + a_n a^{n-1}) \in A$. $\square$

Finally we prove an important technical result, which will reduce many proof to a trivial case:

**Proposition 3.2.11** *Consider an étale cover $f : X \to S$ of constant degree. Then there exists a finite étale base change $g : S' \to S$ such that $X \times_S S' \simeq S'^n$ is the disjoint union of $[X : S]$ copies of $S'$.*

**Proof**
Induction on $\deg(f) = r$.
If $r = 1$ the claim follows by the lemma we have just proved.
Now let $r \geq 2$.
Since the diagonal morphism $\Delta_{X/S} : X \to X \times_S X$ is both an open and closed immersion (proposition 3.1.23 and 3.2.4) we have $X \times_S X = X \amalg X'$ and thus

$$[X \times_S X : X] = [X : X] + [X' : X].$$

But $[X \times_S X : X] = [X : S] = r$. Hence $[X' : X] = r - 1$ and we can apply the induction hypothesis on $X' \to X$ to obtain a finite étale morphism

$S' \to X$ such that $S' \times_X X' \simeq S'^{r-1}$.
We claim that $S' \to X \to S$ does the job.
It is finite étale, since it is the composition of two finite étale morphism.
Moreover we have

$$X \times_S S' \simeq X \times_S (X \times_X S') \simeq (X \times_S X) \times_X S' \simeq (X \amalg X') \times_X S' \simeq$$

$$\simeq (X \times_X S') \amalg (X' \times_X S') \simeq S' \amalg S'^{r-1}.$$

We're done. □

## Galois structure on $\mathbf{Fét}_S$

We devote the remaining of the section to prove that $\mathbf{Fét}_S$ is a Galois category if $S$ is connected.

First we shall define a fundamental functor.
Let $\bar{s}$ be a geometric point of $S$ (i.e. a morphism $\bar{s} : \operatorname{Spec} \Omega_s \to S$ where $\Omega_s$ is an algebraically closed field).
We define the functor

$$\begin{aligned} \mathcal{F}_{\bar{s}} : \mathbf{Fét}_S &\longrightarrow \underline{\mathbf{Set}} \\ X &\longrightarrow X(\bar{s}) := \operatorname{Hom}_{\mathfrak{Sch}_S}(\operatorname{Spec} \Omega_s, X) \end{aligned}$$

**Theorem 3.2.12** *Let $S$ be a connected scheme. Then the pair $(\mathbf{Fét}_S, \mathcal{F}_{\bar{s}})$ is a Galois category.*

**Definition 3.2.13** *We denote the fundamental group relative to $\mathcal{F}_{\bar{s}}$ by $\pi_1(S, \bar{s})$ and we call it fundamental group of $S$ in $\bar{s}$.*

**Proof**
We proceed by proving the axioms defining Galois categories.

**(G1)**

<u>$\mathbf{Fét}_S$ has a final object.</u>

This is easily seen to be $S$. □

<u>$\mathbf{Fét}_S$ has fibered product.</u>

Given a diagram $X_1 \xrightarrow{f} Y \xleftarrow{g} X_2$ in $\mathbf{Fét}_S$ we know that $X_1 \times_Y X_2$ exists in $\mathfrak{Sch}_S$.

We just need to prove that $X_1 \times_Y X_2 \to S$ is finite étale.
Since $Y \to S$ is finite étale, it suffices to prove that $u : X_1 \times_Y X_2 \to Y$ is an étale cover.
It is étale by stability under base change.
Moreover if $U \simeq \operatorname{Spec} A \subseteq Y$, then $u^{-1}(U) \simeq \operatorname{Spec}(B \otimes_A C)$ where $B$ and $C$ are finite $A$-modules (proposition 3.2.8). Thus $B \otimes_A C$ is clearly a finite $A$-module too. $\square$

**(G2)**

<u>$\mathbf{Fét}_S$ has finite coproducts.</u>

We know that disjoint union is the coproduct in the category of $S$-schemes, so we just need to check that given étale covers $X_1, ..., X_n$ of $S$, $\amalg_i X_i \to S$ is still an étale cover. We may assume $S = \operatorname{Spec} A$. Then $X_i = \operatorname{Spec} B_i$ and the structural maps correspond to ring maps $\varphi_i : A \to B_i$.
Moreover

$$\amalg_i X_i = \amalg_i \operatorname{Spec} B_i = \operatorname{Spec}\big(\bigoplus_i B_i\big)$$

and $\amalg_i X_i \to S$ is the map induced by

$$\big[\varphi : a \mapsto \big(\varphi_1(a), ..., \varphi_n(a)\big)\big] : A \to \bigoplus_i B_i.$$

This is clearly flat and finite.
It is easily seen to be unramified too:
If $\mathfrak{p} \subseteq \bigoplus_i B_i$ is a prime ideal, it is of the form $\pi_i^{-1}(\mathfrak{p}_i)$ for some $i$, $\pi_i$ being the projection on $B_i$ and $\mathfrak{p}_i \subseteq B_i$ being a prime. Finally $\big(\bigoplus_i B_i\big)_{\mathfrak{p}} \simeq (B_i)_{\mathfrak{p}_i}$ and thus $\varphi$ is unramified.

<u>$\mathbf{Fét}_S$ has quotients by action of finite groups.</u>

Consider an étale cover $X \to S$ together with a finite group $G$ acting on $X$. First assume that $S = \operatorname{Spec} A$ is affine, so that $X = \operatorname{Spec} B$ and $X \to S$ is induced by a map $\varphi : A \to B$.
The (left) action of $G$ on $X$ is by definition a group map $G \to \operatorname{Aut}_S(X)$. This induces a (right) action on $B$, via the isomorphism $\operatorname{Aut}_S(X) \simeq \operatorname{Aut}_A(B)^{op}$. For any $g \in G$ we will denote with $g$ both its image in $\operatorname{Aut}_S(\operatorname{Spec} B)$ and in $\operatorname{Aut}_A(B)^{op}$.
Now let $B^G \subseteq B$ be the subring of elements fixed by $G$, i.e.

$$B^G := \big\{b \in B | gb = b \, \forall g \in G\big\}.$$

We claim that $\operatorname{Spec} B \to \operatorname{Spec} B^G$ is the categorical quotient of $X$ by the action of $G$.

To prove this consider an $S$-morphism $f : \operatorname{Spec} B \to \operatorname{Spec} C$ fixed by $G$ (i.e. $fg = f \, \forall g \in G$).
This corresponds uniquely to an $A$-algebra map $\phi : C \to B$ fixed by $G$, i.e. such that $gf = f \, \forall g \in G$. But this means that $\operatorname{im}(\phi) \subseteq B^G$ and thus $\phi$ factors uniquely as $C \to B^G \hookrightarrow B$, yielding the unique factorization $\operatorname{Spec} B \to \operatorname{Spec} B^G \to \operatorname{Spec} C$ we were seeking for.
Still we need to prove that $\operatorname{Spec} B^G \to \operatorname{Spec} A$ is étale (it is finite, being a submodule of a finitely generated module over a Noetherian ring), and we know by lemmas 3.1.8 and 3.1.21 that it is sufficient to do this after a faithfully flat base change.
We will need the following lemma:

**Lemma 3.2.14** *Let $X \to S \in \mathbf{Fét}_S$ and let $S' \to S$ be flat and finite. Then any $G$-action on $X$ in $\mathbf{Fét}_S$ induces a $G$-action on $X \times_S S'$ in $\mathbf{Fét}'_S$ and $(X \times_S S')^G \simeq X^G \times_S S'$.*

**Proof**
The $G$-action on $X \times_S S'$ is induced by the map $[\sigma \mapsto \langle \sigma, 1 \rangle] : \operatorname{Aut}_S(X) \to \operatorname{Aut}'_S(X \times_S S')$. Moreover the natural morphism $X \times_S S' \to X^G \times_S S'$ is clearly fixed by $G$, thus we have a map $(X \times_S S')^G \to X^G \times_S S'$
We want to prove it is an isomorphism and to do so we may clearly reduce to the affine case, i.e. we need to prove that $(A' \otimes_A B)^G \simeq A' \otimes_A B^G$ ($A \to A'$ is finite and flat).
But we have an exact sequence $0 \to B^G \to B \to \prod_{g \in G} B$ where the rightmost map is

$$B^G = \ker \left( \left[ b \mapsto (b)_{g \in G} - (gb)_{g \in G} \right] : B \to \prod_{g \in G} B \right).$$

Tensoring this sequence with $A'$ and using flatness of $A \to A'$ we get the thesis. $\square$

Now by proposition 3.2.11 there exists a faithfully flat base change $A \to A'$ such that $B' := A' \otimes_A B \simeq A'^n$ as $A'$-algebra.
The $A'$-automorphisms of $B'$ are just permutations of the copies of $A'$ (since they must fix the identity of $B$). This means that $\operatorname{Aut}_{A'}(B') \simeq S_n$ and the action of $G$ is determined by it's image in $S_n$. Then it is easy to see that $B'^G \simeq A'^m$ for some $m \leq n$, so that $\operatorname{Spec} B'^G \to \operatorname{Spec} A'$ is an étale cover.

Finally we consider the case of an arbitrary base scheme $S$.
Write $S = \bigcup_i \operatorname{Spec} A_i$ and $X = \bigcup_i X_i$ where $X_i := \operatorname{Spec} B_i = f^{-1}(\operatorname{Spec} A_i)$.
By covering each $\operatorname{Spec} A_i$ by standard affine open, we may assume that $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j = D(f_{ij})$ for some $f_{ij} \in A_i$.
Define

$$X_{ij} := f^{-1}(\operatorname{Spec} A_i \cap \operatorname{Spec} A_j) = f^{-1}\big(D(f_{ij})\big) = D\big(\varphi(f_{ij})\big).$$

Using again the last lemma (base changing along $A \to A_{f_{ij}}$) we see that $X_{ij}^G$ is an open subscheme of $X_i^G$.
Now by uniqueness of the quotient by action of finite groups, we have that $X_{ij}^G \simeq X_{ji}^G$, and thus the $X_i^G$ glue along these isomorphisms, yielding an $S$-scheme Y.
We claim this is the quotient of $X$ by the action of $G$.
In fact if $f : X \to Z$ is an arrow fixed by $G$, then the restrictions $f|_{X_i} : X_i \to Z$ are fixed by $G$ too. Hence they factor as $X_i \to X_i^G \xrightarrow{g_i} Z$, and the morphisms $g_i$ clearly glue yielding a morphism $g : Y \to Z$ via which $f$ factors.
This prove that $Y$ is the quotient of $X$ by the action of $G$. $\square$

**(G3)**

Consider a morphism $u : Y \to X$ in $\mathbf{Fét}_S$.
We know by proposition 3.2.8 that this is finite étale. Thus it is open and closed by propositions 3.2.4 and 3.2.5.
As a consequence $X' := u(Y)$ and $X'' := X \setminus X'$ are disjoint open and closed subscheme of $X$. This means that $X = X' \amalg X''$ and $u$ clearly factors as $Y \to X' \to X = X' \amalg X''$.
Since open immersions are monomorphisms, the second arrow is a monomorphism inducing an isomorphism into a component of $X$.
We're just left to prove that $Y \to u(Y)$ is a strict epimorphism. The following lemma does the job.

**Lemma 3.2.15** *In* $\mathbf{Fét}_S$ *a morphism is surjective (at the level of topological spaces) if and only if it is a strict epimorphisms.*

**Proof**
First let $u : Y \to X$ be a strict epimorphism (actually epimorphism is enough).
We know that $X' := u(Y)$ is open and closed in $X$. Hence we can write $X = X' \amalg X''$ with $X'' := X \setminus u(Y)$. The claim now follows since composing $u$ with the two natural maps $X = X' \amalg X'' \rightrightarrows X' \amalg X'' \amalg X''$ yields the same morphism. Thus, since $u$ is an epimorphism, these two maps must coincide and this can only be the case if $X'' = \emptyset$.

Conversely take a surjective morphism $u : Y \to X$ in $\mathbf{Fét}_S$ and denote with $p$ and $q$ the projections $Y \times_X Y \rightrightarrows Y$.
The general case easily reduce to the affine, thus we may assume that $S = \operatorname{Spec} A$, so that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} C$.
We need to prove that

$$\operatorname{Spec}\big(C \otimes_B C\big) \rightrightarrows \operatorname{Spec} C \to \operatorname{Spec} B$$

is exact, or in other words that it is an equalizer diagram.
This is equivalent to the exactness of $B \to C \rightrightarrows C \otimes_B C$.
Firstly $\varphi : B \to C$ is faithfully flat (proposition 3.1.7), thus injective (remark 3.1.6).
It just remains to prove that $B = \ker(C \rightrightarrows C \otimes_B C)$.
Since it is sufficient to prove this after localizing at primes, and $B \to C$ is flat and thus locally free, we may assume that $C \simeq B^n$ as $B$-module. But then $C \otimes_B C \simeq B^n \otimes_B B^n \simeq \mathrm{Mat}_n(B)$ as $B$-module. Moreover the maps $C \rightrightarrows C \otimes_B C$ maps an element $(b_i) \in C$ to the matrices

$$\begin{pmatrix} b_1 & \cdots & b_1 \\ \vdots & \ddots & \vdots \\ b_n & \cdots & b_n \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} b_1 & \cdots & b_n \\ \vdots & \ddots & \vdots \\ b_1 & \cdots & b_n \end{pmatrix}$$

respectively.
Clearly these are equal if and only if $b_i = b_j$ for all $i, j$, i.e. if and only if $(b_i) \in B$. □

**(G4)**

<u>$\mathcal{F}_s$ preserve the final object.</u>

Obviously $S(\bar{s}) = \{\bar{s}\}$ is a singleton, which is the final object in **Set**. □

<u>$\mathcal{F}_s$ commute with fibered product.</u>

This is just universal property of fibered product. □

**(G5)**

<u>$\mathcal{F}_s$ commutes with finite coproduct.</u>

This is true since an element in $\mathcal{F}_{\bar{s}}(X \amalg Y)$ corresponds biunivocally to a point $z \in X \amalg Y$ (i.e. $z \in X$ or $z \in Y$) together with a field inclusion $k(z) \to \Omega_s$. □

<u>$\mathcal{F}_s$ commutes with quotients by actions of finite groups.</u>

Consider an étale cover $S' \to S$. For any $s \in S$ there exists an $s' \in S'$ lying over $s$, hence a finite separable field extension $k(s')/k(s)$. Then any geometric point $\bar{s} : \operatorname{Spec} \Omega_s \to S$ (i.e. any inclusion $k(s) \hookrightarrow \Omega_S$) lift to a geometric point $\bar{s}' : \operatorname{Spec} \Omega_s \to S'$ (i.e. an inclusion $k(s') \hookrightarrow \Omega_s$ extending $k(s) \hookrightarrow k(s')$).
Moreover we have

$$\mathcal{F}_{s'}(X \otimes_S S') = \operatorname{Spec} \Omega_s \times'_S (X \times_S S') \simeq \operatorname{Spec} \Omega_s \times_S X \simeq \mathcal{F}_s(X).$$

Then using proposition 3.2.11 we can reduce to the case where $X \simeq S^n$ and we find

$$\mathcal{F}_s\big(X^G\big) = \mathcal{F}_s\big(S^{n^G}\big) = \mathcal{F}_s(S^m) = \mathcal{F}_s(S)^m = \big(\mathcal{F}_s(S)^n\big)^G = \mathcal{F}_s(S^n)^G = \mathcal{F}_s(X)^G.$$

We're done. $\square$

**(G6)**

Let $u : X \to Y$ be a strict epimorphism. We want to prove that $\mathcal{F}_{\bar{s}}(u) = u \circ (-) : X(\bar{s}) \to Y(\bar{s})$ is surjective.
In fact take a point in $Y(\bar{s})$, i.e. a point $y \in Y$ together with a field inclusion $k(y) \hookrightarrow \Omega_s$. Since strict epimorphisms in $\mathbf{Fét}_S$ are surjective (lemma 3.2.15), there exists $x \in X$ such that $u(x) = y$. For $u$ being unramified, we have a finite separable field extension $k(y) \hookrightarrow k(x)$. Moreover since $\Omega_s$ is separably closed, this extend to an embedding $k(x) \hookrightarrow \Omega_s$, i.e. a point in $X(\bar{s})$.
By construction we have the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}\Omega_s \arrow[r] \arrow[dr] & \operatorname{Spec} k(x) \arrow[r] \arrow[d] & X \arrow[d] \\
& \operatorname{Spec} k(y) \arrow[r] & Y
\end{tikzcd}$$

which means that $\operatorname{Spec}\Omega_s \to \operatorname{Spec} k(x) \to X$ maps via $\mathcal{F}_s(u)$ to the point in $Y(\bar{s})$ we started with. $\square$

**(G7)**

Consider a morphism $u : Y \to X$ in $\mathbf{Fét}_s$ such that $\mathcal{F}_{\bar{s}}(u) : \mathcal{F}_{\bar{s}}(Y) \to \mathcal{F}_{\bar{s}}(X)$ is an isomorphism.
First we will prove it is surjective.
Using the factorization in (G3) and the fact that $\mathcal{F}_{\bar{s}}$ commute with direct sum, we see that $\mathcal{F}_{\bar{s}}(u)$ factorizes as

$$\mathcal{F}_{\bar{s}}(Y) \to \mathcal{F}_{\bar{s}}(X') \to \mathcal{F}_{\bar{s}}(X') \amalg \mathcal{F}_{\bar{s}}(X'').$$

Since $\mathcal{F}_{\bar{s}}(u)$ is surjective, it must be $\mathcal{F}_{\bar{s}}(X'') = \emptyset$.
We claim that this implies $X'' = \emptyset$, and thus $u$ surjective.
In fact if $X'' \neq \emptyset$, the structure morphism $X'' \to S$ is surjective, thus the map $\mathcal{F}_{\bar{s}}(X'') \to \mathcal{F}_{\bar{s}}(S)$ is surjective by lemma 3.2.15 and (G6). This imply $\mathcal{F}_{\bar{s}}(X'') \neq \emptyset$.
This proves that $u$ is surjective, thus an étale cover.

It just remains to prove that $[Y : X] = 1$ (lemma 3.2.10).
It will suffice to prove that for any étale cover $X \to S$ we have $\deg(X \to S) = \left|\mathcal{F}_{\bar{s}}(X)\right|$, since then we would have

$$\deg(X \to S) = \left|\mathcal{F}_{\bar{s}}(X)\right| = \left|\mathcal{F}_{\bar{s}}(Y)\right| = \deg(Y \to S).$$

But an element in $\mathcal{F}_{\bar{s}}(X)$ correspond biunivocally to a point $x$ lying over $s$ and a field inclusion $k(x) \hookrightarrow \Omega_s$ extending $k(s) \hookrightarrow \Omega_s$. In other word

$$\mathcal{F}_s(X) = \coprod_{x \mapsto s} \mathrm{Hom}_{k(s)}\big(k(x), \Omega_s\big).$$

Now since $k(x)/k(s)$ is finite separable by unramifiedness, we have that $\left|\mathrm{Hom}_{k(s)}\big(k(x), \Omega_s\big)\right| = \dim_{k(s)} k(x)$.
Thus

$$\left|\mathcal{F}_s(X)\right| = \sum_{x \mapsto s} \dim_{k(s)} k(x) = \mathrm{rk}_{k(s)}\big(B \otimes_A k(s)\big) = \mathrm{rk}(s).$$

We're done. □

## 3.3 Two arithmetic applications

Since when they were invented étale cohomology and Galois theory for schemes gave large contribution to many fields of mathematics. One of these is for sure arithmetic geometry.
To conclude this thesis we want to give two examples to show how the theory we studied so far applies in arithmetical contexts.

### The case $S = \operatorname{Spec} k$

The case in which the base scheme is the spectrum of a field is already quite interesting.
In fact the theory of étale cohomology over $\operatorname{Spec} k$ is just a geometric reformulation of classical Galois cohomology.
The main reason is that étale $k$-schemes of finite type are necessairily finite (by quasi compactness these are disjoint union of finitely many spectrum of finite field extensions of the base field). This makes things way easier then in the general case, since all the theory of Galois category may be applied.
A nice consequence of this fact is that the étale topology coincide with the canonical topology.

First we introduce some notion of modern Galois theory.
The main difference from the classical one is that extensions are allowed to be infinite. Still the definition of Galois extension adapt easily to this case:

**Definition 3.3.1** *An algebraic (non necessarily finite) field extension $K/k$ is called* **Galois** *if $K^{\mathrm{Aut}(K/k)} = k$.*
*In this case we write $G(K/k) := \mathrm{Aut}(K/k)$ for the Galois group of $K/k$.*

In particular let $k$ be a field and fix an algebraic and a separable closure of $k$, say

$$\bar{k}/k_s/k$$

We make the following definition.

**Definition 3.3.2** *We call absolute Galois group of $k$ the group $G_k := \mathrm{Gal}(k_s/k)$.*

Now write $S = \operatorname{Spec} k$, $\bar{S} := \operatorname{Spec} \bar{k}$ and $\bar{s} : \bar{S} \to S$.

The fundamental group of $S$ in $\bar{s}$ is then defined and the following result holds:

**Proposition 3.3.3** $\pi_1(S, \bar{s}) \simeq G_k$

**Proof**
Rephrasing definition 2.4.1 we see that Galois objects in $\mathbf{Fét}_S$ correspond to separable field extensions $K/k$ such that $\mathrm{Aut}_k(K)$ act transitively on $\mathrm{Hom}_S(\operatorname{Spec} \bar{k}, X)$.
This happen if and only if $\left|\mathrm{Aut}_k(K)\right| = \left|\mathrm{Hom}_S(\operatorname{Spec} \bar{k}, X)\right| = [K : k]$. Hence Galois object in $\mathbf{Fét}_S$ are just Galois fields extension in the usual sense.
Now by proposition 2.5.4 we have $\pi_1(S, s) \simeq \varprojlim G(K/k)$ (here $G(K/k)$ stands for the Galois group of $K$ over $k$), where the limit runs over all finite Galois field extensions of $k$.
We just need to prove that $G_k \simeq \varprojlim \mathrm{Gal}_k(K)$.
Consider the natural map

$$\phi : G_k \to \prod G(K/k)$$

whose component on the $K$ factor is the restriction $G_k \to G(K/k)$ (we wrote $G(K/k)$ for $\mathrm{Gal}(K/k)$).
This is well defined: if $\sigma \in G_k$ then $\sigma$ maps an element in $K$ to a root of its minimal polynomial over $k$. But since $K$ is Galois this splits into linear factor in $L$, hence $\sigma$ is indeed a $k$-automorphism of $L$.
Moreover the image of $\phi$ is clearly contained in the inverse limit.
We just need to prove that $\phi$ maps $G(K/k)$ injectively onto $\varprojlim G(L/k)$.
First we prove that $\phi$ is injective.
Let $\sigma \in G(K/k)$, $\sigma \neq id_K$, so that $\sigma(\alpha) \neq \alpha$ for some $\alpha \in K$. But $k(\alpha)$ is contained in the splitting field of the minimal polynomial for $\alpha$, say $L$. This is a Galois extension and $\sigma_{|_L} \neq id_L$. Thus $\phi$ is injective.

Finally we prove surjectivity.
Take an element $(\sigma_L) \in \varprojlim G(L/k)$. Let $\sigma : K \to K$ be the automorphism sending $\alpha$ to $\sigma_L(\alpha)$, where $L$ is a finite Galois extension containing $\alpha$.
Then $\sigma$ is clearly well defined and it clearly maps to $(\sigma_L)$ via $\phi$. $\square$

This fact already suggests that étale cohomology over the spectrum of a field is just a new insight into classical Galois cohomology.

The next proposition close the loop, showing that the étale topology on $(\operatorname{Spec} k)_{\text{ét}}$ is equivalent to the canonical topology on $G$-**mod** .

**Proposition 3.3.4** *$\mathcal{F}_{\bar{s}}$ becomes an equivalence of topologies if we equip* $\mathbf{Fét}_S$ *with the étale topology and $\pi_1(S,s)$-*$\underline{\mathbf{FSet}}$ *with the canonical topology.*
*In particular* $(\operatorname{Spec} k)_{\acute{e}t}$ *is the canonical topology over* $\operatorname{Spec} k$.

**Proof**
We just need to prove that $\{X_i \to X\}$ is a covering in $\mathbf{Fét}_S$ if and only if $\{\mathcal{F}_{\bar{s}}(X_i) \to \mathcal{F}_{\bar{s}}(X)\}$ is a covering in $\pi_1(S,s)$-$\underline{\mathbf{FSet}}$.
Since $\mathcal{F}_{\bar{s}}$ preserve coproduct it suffices to prove that $Y \to X$ is surjective in $\mathbf{Fét}_S$ if and only if $\mathcal{F}_{\bar{s}}(Y) \to \mathcal{F}_{\bar{s}}(X)$ is surjective.
But this was already proven in (G6) and (G7). $\square$

**Remark 3.3.5** This is not true for a general base scheme $S$. Nevertheless it can be shown ([22] II 3.1.2) that coverings in $S_{\text{ét}}$ are universal effectively surjective (definition 1.5.7). This implies in particular that the étale topology is coarser than the canonical topology, which means that all representable presheaves are sheaves.

From the previous proposition, together with remark 1.6.4, we obtain equivalences of categories

$$\mathcal{S}\big(\text{ét}_S\big) \to \mathcal{S}\big(\mathcal{T}_G\big) \to G\text{–}\mathbf{mod}.$$

We can obtain an explicit form for this functor.
Recall from the proof of proposition 1.6.3 that $\mathcal{S}(\mathcal{T}_G) \to G$-**mod** is the functor

$$\mathscr{F} \to \varinjlim \mathscr{F}(G/H),$$

the limit taken over the open normal subgroup of $G$.
But the quotients $G/H$ corresponds to group of automorphisms of some finite Galois extension $K/k$, which correspond via the eqivalence $\text{ét}_S \to G$-**Set** to $\operatorname{Spec} K$.
Hence we obtain for the functor $\mathcal{S}\big(\text{ét}_S\big) \to G$-**mod** the formula

$$\mathscr{F} \to \varinjlim \mathscr{F}(\operatorname{Spec} K).$$

the limit taken over the finite Galois extension of $k$.
This means that any abelian sheaf on $(\operatorname{Spec} k)_{\text{ét}}$ is represented by $\varinjlim \mathscr{F}(\operatorname{Spec} K)$.

As a last remark we point out what this imply at the level of cohomology. Since $\operatorname{Spec} k$ is sent to a singleton $e$ by the equivalence $S_{ét} \to G_k$-**mod** , the section functor $\Gamma_{\operatorname{Spec} k}$ identifies with the functor $\Gamma_e$.
Then we have $\partial$-functorial isomorphisms

$$H^q_{\text{ét}}(\operatorname{Spec} k, \mathscr{F}) \simeq H^q\big(G, \varinjlim \mathscr{F}(\operatorname{Spec} K)\big).$$

where the right hand side represent Galois cohomology of $k$ (see the discussion at the end of section 1.6).
Moreover it can be shown ([29]) that every profinite group arises as a Galois group of some field extension. This means that the theory of étale sheaves over the spectrum of a field is equivalent to the theory of modules over profinite groups.

This is a great result, cause it gives a dictionary among three different ambit: the first one is étale cohomology over the spectrum of a field; the second one is Galois cohomology; the third one is cohomology theory for modules over profinite groups.
This is an example of how modern algebraic geometry bridges the gap among three different sphere of mathematics: geometry, arithmetic and algebra.

## Hilbert theorem 90

In this last section we want to give another example of how étale cohomology gives a new insight into arithmetic problems. We will give a geometric interpretation to a classical theorem, Hilbert theorem 90.

In its classical form the theorem states what follows:

**Theorem 3.3.6** (Hilbert theorem 90, arithmetic version)
*Let $K/k$ be a cyclic extension of fields (i.e. a Galois extensions with cyclic Galois group) and let $x \in K$ be an element of norm 1.*
*Then $x = y/gy$ for some $y \in L$ and $g \in \operatorname{Gal}(K/k)$.*

The theorem was already generalized and stated in cohomological terms by saying that

$$H^1(G, K^\times) = 0.$$

To see that this implies theorem 90 in its classical form, recover the description of $H^1(G, K^\times)$ by means of co-cycles.
With this set-up it is the quotient of the group of 1-cocycles (i.e. maps $f : G \to K^\times$ such that $f(gh) = gf(h) + f(g)$ for all $g, h \in G$) over the group of 1-coboundaries (i.e. maps of the form $[g \to x^{-1}g(x)] : G \to K^\times$).
Now if $G = \langle g \rangle$ is a cyclic group it's not difficult to see that a 1-cocycle is uniquely determined by the image of $g$, which has to be an element $x \in K^\times$ having norm 1.
Hence the map $f : g \mapsto x$ determines a 1-cocycle and since $H^1(G, K^\times) = 0$ it has to be a 1-coboundary too. This means that $f(g) = x = y^{-1}g(y)$ for some $y \in L^\times$, which is indeed Hilbert theorem 90.

In the language of the previous section, this means that if $S = \operatorname{Spec} k$ then

$$H^1_{ét}\big(S, \mathbb{G}_{m,S}\big) = 0$$

where $\mathbb{G}_{m,S}$ is the étale sheaf $[X \to S] \mapsto \Gamma(X, \mathcal{O}_X)^\times$.

We will generalize this result even further:

**Theorem 3.3.7** (Hilbert theorem 90, geometric version)
*There is an isomorphism*

$$H^1_{ét}\big(S, \mathbb{G}_{m,S}\big) \simeq \operatorname{Pic}(S)$$

*where* $\operatorname{Pic}(S)$ *is the group of invertible (i.e. locally free of rank 1)* $\mathcal{O}_S$*-modules.*

**Remark 3.3.8** This indeed generalizes the arithmetic version of the theorem. In fact $\operatorname{Pic}(\operatorname{Spec} A) = 0$ for any local ring $A$ (since by proposition 3.1.9 locally free $\implies$ free for modules over local rings).

First we shall briefly recall some basic facts about spectral sequences, which are a central tool in computing cohomology groups.
For the proofs of the theorems we refer to section 5 of [30].

**Definition 3.3.9** *A cohomology* **spectral sequence** *in an abelian category* $\mathcal{C}$ *consists of*

*a) objects $E_r^{pq} \in \mathcal{C}$ for all $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ and $r \geq 2$;*

*b) morphisms (called differentials) $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ such that $d_r^{pq} \circ d_r^{p+r,q-r+1} = 0$;*

*c) isomorphisms $\alpha_r^{pq} : \ker(d_r^{pq})/\mathrm{im}(d^{p-r,q+r-1}) \simeq E_{r+1}^{pq}$.*

*We will denote this with $(E_r^{pq}, d_r^{pq})$.*

The spectral sequence we are interested in satisfies the following additional condition:

**Definition 3.3.10** *We call $(E_r^{pq}, d_r^{pq})$ a* **first quadrant** *cohomology spectral sequence if $E_2^{pq} = 0$ for all $p < 0$ and $q < 0$.*

**Remark 3.3.11** If this happen then we readily see using condition **c)** that $E_r^{pq} = 0$ for all $p < 0$, $q < 0$ and $r \geq 2$.
Then for fixed $p, q \in \mathbb{Z}$ we see that for sufficiently large $r$ the differentials $d_r^{pq}$ and $d^{p-r,q+r-1}$ are the zero morphisms (they land and start in the fourth and second quadrant respectively). Hence using again condition **c)** we obtain that $E_r^{pq} \simeq E_{r+1}^{pq}$ for sufficiently large $r$.
We call these **limit terms** of the spectral sequence and denote them with $E_\infty^{pq}$.

The most interesting spectral sequences are the converging one.
First we need to introduce the notion of filtration of an object:

**Definition 3.3.12** *Given an object $A$ in an abelian category $\mathcal{C}$ we call (decreasing)* **filtration of** *$A$ a family $\left(F^p(A)\right)_{p \in Z}$ of subobjects of $A$ such that $F^{p+1}(A) \subseteq F^p(A)\ \forall p \in Z$.*
*We say that the filtration is finite if $F^N(A) = 0$ and $F^n(A) = A$ for $N$ big enough and $n$ small enough.*

**Definition 3.3.13** *We say that a cohomology spectral sequence* **converges** *if we are given:*

*d) objects $(E^n)_{n \in \mathbb{Z}}$ together with finite decreasing filtrations $F^p(E^n)$*

*e) isomorphisms $\beta^{pq} : E_\infty^{pq} \simeq F^p(E^{p+q})/F^{p+1}(E^{p+q})$*

*In this case we will write $E_2^{pq} \implies E^{p+q}$.*

We states now some important results:

**Proposition 3.3.14 (Edge morphisms)**
*For any convergent first-quadrant cohomology spectral sequence* $E_2^{pq} \implies E^{p+q}$ *there exist the so-called* **edge morphisms**

$$\begin{cases} \varphi_n : E_2^{n,0} \to E^n \\ \psi_n : E^n \to E_2^{0,n} \end{cases}$$

**Proposition 3.3.15 (Exact sequence of low-degree terms)**
*With the same hypothesis as before, the edge morphisms fit in an exact sequence*

$$0 \longrightarrow E_2^{1,0} \xrightarrow{\varphi_1} E^1 \xrightarrow{\psi_1} E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \xrightarrow{\varphi_2} E^2$$

*which is called* **exact sequence of low-degree terms** *or* **five terms exact sequence**.

With some further assumption we can ensure some edge morphism to be isomorphisms:

**Proposition 3.3.16** *With the same hypothesis as before, assume furthermore that* $E^{pq}$ *for all* $0 < q < n$ *(respectively* $0 < p < n$*).*
*Then the edge morphisms* $\varphi_m : E_2^{m,0} \to E^m$ *(respectively* $E^m \to E_2^{0,m}$*) are isomorphisms for all* $m < n$*.*
*In particular if* $n = 1$ *the spectral sequence is called trivial and* $\varphi_m$ *(respectively* $\psi_m$*) is an isomorphism for all* $m$*.*

Finally the next theorem gives a procedure to construct an important type of spectral sequence:

**Theorem 3.3.17** *Consider abelian categories* $\mathcal{C}, \mathcal{C}'$ *and* $\mathcal{C}''$ *and right exact additive functors* $F : \mathcal{C} \to \mathcal{C}'$ *and* $G : \mathcal{C}' \to \mathcal{C}''$*.*
*Moreover assume that* $\mathcal{C}$ *and* $\mathcal{C}'$ *have enough injective objects and that* $F$ *maps injective objects to* $G$*-acyclic object (i.e. objects annihilated by* $R^qG$*).*
*Then for each object* $A \in \mathcal{C}$ *there is a convergent first-quadrant cohomology spectral sequence*

$$R^pG\big(R^qF(A)\big) \implies R^{p+q}(G \circ F)(A).$$

This theorem is a central tool for computing cohomology groups.
We give some examples to illustrate how.

**Example 3.3.18** Let $(\mathcal{C}, \mathcal{T})$ be a site and $i$ be the inclusion $\mathcal{S}(\mathcal{T}) \to \mathcal{P}(\mathcal{T})$, which is left exact by proposition 1.3.1.

Denote by $\mathcal{H}^q$ the right derived functors $R^q i$.
Recall that we have defined in section 1.2 a functor $(-)^\dagger : \mathcal{P}(\mathcal{T}) \to \mathcal{P}(\mathcal{T})$, which applied twice is left adjoint to $i$.
We will use the formalism of spectral sequences to show that $\mathcal{H}^q(\mathscr{F})^\dagger = 0$ for all sheaves $\mathscr{F}$ and for all $q > 0$.
First note that by proposition 1.2.5 it is actually enough to show that $\mathcal{H}^q(\mathscr{F})^\# := \big(\mathcal{H}^q(\mathscr{F})^\dagger\big)^\dagger = 0$.
Now consider the factorization of the identical functor $\mathrm{id}_{\mathcal{S}(\mathcal{T})}$ given by

$$\mathcal{S}(\mathcal{T}) \xrightarrow{i} \mathcal{P}(\mathcal{T}) \xrightarrow{\#} \mathcal{S}(\mathcal{T}).$$

Since sheafification is an exact functor (see corollary 1.2.8), every presheaf is #-acyclic.
Hence the hypothesis of theorem 3.3.17 are fulfilled and for all sheaf $\mathscr{F}$ we obtain a spectral sequence

$$R^p\#\big(\mathcal{H}^q(\mathscr{F})\big) \implies R^{p+q}(\mathrm{id}_{\mathcal{S}(\mathcal{T})}).$$

For # being exact we have $R^p\# = 0\,\forall p > 0$ and we obtain by proposition 3.3.16 that the edge morphisms

$$R^q(\mathrm{id}_{\mathcal{S}(\mathcal{T})})(\mathscr{F}) \to \mathcal{H}^q(\mathscr{F})^\#$$

are isomorphisms for all $q > 0$.
But the identical functor is clearly exact, thus $R^q(\mathrm{id}_{\mathcal{S}(\mathcal{T})}) = 0$.
We're done.

**Example 3.3.19 (the spectral sequence for Čech cohomology)**
In this example we investigate the relation between cohomology of sheaves and Čech cohomology. In particular we will show that:

- $\check{H}^1(U, \mathscr{F}) \simeq H^1(U, \mathscr{F})$;
- $\check{H}^2(U, \mathscr{F}) \hookrightarrow H^2(U, \mathscr{F})$.

Recall that in section 1.2 we defined the Čech cohomology groups $\mathscr{F}^\dagger = \check{H}^q(U, \mathscr{F})$ for a presheaf $\mathscr{F}$ on a site $\mathcal{T}$ (remark 1.2.3).
Moreover we observed that if $\mathscr{F}$ is a sheaf then $\check{H}^0(U, \mathscr{F}) \simeq \mathscr{F}(U)$.
Hence the section functor $\Gamma_U : \mathcal{S}(\mathcal{T}) \to \underline{\mathbf{Ab}}$ factorizes as

$$\mathcal{S}(\mathcal{T}) \xrightarrow{i} \mathcal{P}(\mathcal{T}) \xrightarrow{\check{H}^0(U,-)} \underline{\mathbf{Ab}}.$$

Here $i : \mathcal{S}(\mathcal{T}) \to \mathcal{P}(\mathcal{T})$ is the inclusion, which is right adjoint to an exact functor (see again section 1.2, in particular corollary 1.2.8). Hence it

preserves injective objects, and in particular it maps injective objects to $\check{H}^0(U,-)$-acyclic objects.
Thus the hypothesis of theorem 3.3.17 are fulfilled and we get for each $U \in \mathcal{T}$ a spectral sequences

$$\check{H}^p\big(U, \mathcal{H}^q(\mathscr{F})\big) \implies H^{p+q}(U, \mathscr{F}).$$

The corresponding exact sequence of low-degree terms is

$$0 \to \check{H}^1(U, \mathscr{F}) \to H(U, \mathscr{F}) \to \check{H}^0\big(U, \mathcal{H}^1(\mathscr{F})\big) \to \check{H}^2(U, \mathscr{F}) \to H^2(U, \mathscr{F}).$$

But the previous example shows us that $\check{H}^0\big(-, \mathcal{H}^1(\mathscr{F})\big) = \mathcal{H}^1(\mathscr{F})^\dagger = 0$.
The thesis follows at once.

**Example 3.3.20 (Leray spectral sequence)**
In this example we show how spectral sequences may be used to compare low degree cohomology groups on different sites.
Let $f : (\mathcal{C}', \mathcal{T}') \to (\mathcal{C}, \mathcal{T})$ be a morphism of sites with underlying functor $F : \mathcal{C} \to \mathcal{C}'$.
Then there is a functor $f_* : \mathcal{S}(\mathcal{T}') \to \mathcal{S}(\mathcal{T})$ (see remark 1.1.9) which is easily seen to be left exact.
Next consider a category $\mathcal{C}''$ consisting of a single object $o$ and a single arrow $1_o$; denote with $\mathcal{T}''$ the unique possible structure of site on it.
Then for any fixed $U \in \mathcal{C}$ there is a continuous functor $G : \mathcal{C}'' \to \mathcal{C}$ mapping $o$ to $U$. Denote with $g : (\mathcal{C}, \mathcal{T}) \to (\mathcal{C}'', \mathcal{T}'')$ the corresponding morphism of sites.
It is easy to see that the category of sheaves over $\mathcal{T}''$ is equivalent to the category of abelian groups, and the functor $g_* : \mathcal{S}(\mathcal{T}) \to \mathcal{S}(\mathcal{T}'')$ identifies with $\Gamma_U$ under this equivalece.
Moreover one can prove (see [22], chapter I, (3.7.1)) that $f_*$ maps injective sheaves to $g_*$-acyclic objects.
Hence we have functors

$$\mathcal{S}(\mathcal{T}') \xrightarrow{f_*} \mathcal{S}(\mathcal{T}) \xrightarrow{g_*} \underline{\mathbf{Ab}}$$

satisfying the hypothesis of theorem theorem 3.3.17 and for each sheaf $\mathscr{F}'$ on $\mathcal{T}'$ we obtain a spectral sequence

$$H^p\big(U, R^q f_*(\mathscr{F}')\big) \implies H^{p+q}(F(U), \mathscr{F}').$$

Its exact sequence of low-degree terms is

$$0 \to H^1\big(U, f_*(\mathscr{F}')\big) \to H^1(F(U), \mathscr{F}') \to \cdots$$

$$\cdots \to R^1 f_*(\mathscr{F}')(U) \to H^2\big(U, f_*(\mathscr{F}')\big) \to H^2(F(U), \mathscr{F}')$$

and it is a useful tool to compare the first cohomology groups of $\mathscr{F}'$ and $f_*(\mathscr{F}')$.

Now fix a scheme $S$. If in the last example we define $f$ to be the natural morphism of sites $S_{ét} \to S_{\mathrm{Zar}}$ we get the following:

**Proposition 3.3.21** *For any sheaf $\mathscr{F} \in \mathcal{S}(S_{ét})$ there is a convergent first-quadrant cohomology spectral sequence*

$$H^p_{\mathrm{Zar}}\big(S, R^q f_*(\mathscr{F})\big) \implies H^{p+q}_{ét}(S, \mathscr{F}).$$

This spectral sequence is a central tool for comparing the Zariski and Étale cohomology groups.

Next we will define a particular abelian sheaf on $S_{ét}$, namely the **sheaf of invertible elements**.
Let $\mathbb{G}_{m,S}$ be the sheaf on $S_{ét}$ represented by the $S$-scheme

$$\operatorname{Spec} \mathbb{Z}[T, T^{-1}] \times_{\operatorname{Spec} \mathbb{Z}} S$$

(this is in fact a sheaf by remark 3.3.5). Notice that

$$\begin{aligned}
\mathbb{G}_{m,S}(X) &= \operatorname{Hom}_S(X, \operatorname{Spec} \mathbb{Z}[T, T^{-1}] \times_{\operatorname{Spec} \mathbb{Z}} S) \\
&= \operatorname{Hom}_{\mathfrak{Sch}}(X, \operatorname{Spec} \mathbb{Z}[T, T^{-1}]) \\
&= \operatorname{Hom}_{\underline{\mathbf{Rng}}}\big(\mathbb{Z}[T, T^{-1}], \Gamma(X, \mathcal{O}_X)\big) \\
&= \Gamma(X, \mathcal{O}_X)^\times
\end{aligned}$$

Hence $\mathbb{G}_{m,S}$ associates to each étale $S$-scheme $X$ the multiplicative group of invertible elements in $\Gamma(X, \mathcal{O}_X)$.

The following classical result about the first Zariski cohomology group of $\mathbb{G}_{m,S}$ holds:

**Proposition 3.3.22** *Denote with $\mathcal{O}_S^\times$ the Zariski sheaf*

$$[U \mapsto \Gamma(U, \mathcal{O}_S)^\times] : S_{\mathrm{Zar}} \to \underline{\mathbf{Ab}}.$$

*Then*

$$H^1_{\mathrm{Zar}}(S, \mathcal{O}_S^\times) \simeq \operatorname{Pic}(S).$$

**Proof** (sketch)
By example 3.3.19 we know that

$$H^1_{\mathrm{Zar}}(S, \mathcal{O}_S^\times) \simeq \check{H}^1_{\mathrm{Zar}}(S, \mathcal{O}_S^\times) = \varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{O}_S^\times).$$

Now recall (see remark 1.2.3) that $H^1(\{U_i \to U\}_{i\in I}, \mathcal{O}_S^\times)$ may be described as the first cohomology group of the complex

$$0 \to \prod_i \mathcal{O}_S^\times(U_i) \to \prod_{i,j} \mathcal{O}_S^\times(U_i \cap U_j) \to \prod_{i,j,h} \mathcal{O}_S^\times(U_i \cap U_j \cap U_h) \to \cdots .$$

Now consider an invertible $\mathcal{O}_S$-module $\mathscr{L}$ and a trivializing cover $\{U_i \subseteq U\}$. Then we have isomorphisms $\varphi_i : \mathcal{O}_S(U_i) \to \mathscr{L}|_{U_i}$.
By restricting them on double intersections we get isomorphisms of $\mathcal{O}_S(U_i \cap U_j)$-modules $\varphi_j^{-1} \circ \varphi_i : \mathcal{O}_S(U_i \cap U_j) \to \mathcal{O}_S(U_i \cap U_j)$. Being isomorphisms between free modules of rank 1, these are given by multiplication by an element $s_{ij} \in \mathcal{O}_S^\times(U_i \cap U_j)$.
One can check (using the cocycle condition on triple intersection) that $(s_{ij})_{i,j}$ is a Čech cocycle and hence it represent an element in

$$\check{H}^1_{\mathrm{Zar}}(S, \mathcal{O}_S^\times) = \varinjlim H^1(\mathcal{U}, \mathcal{O}_S^\times).$$

We have thus defined a map $\mathrm{Pic}(S) \to \check{H}^1_{\mathrm{Zar}}(S, \mathcal{O}_S^\times)$ and this can be proved to be an isomorphism (see [10] 5.4.7 for details ). □

Finally the proof heavily relies on a descent argument, which we shall briefly explain. We refer to [13] for a treatment about descent theory (in the affine case).
Let $A \to B$ be faithfully flat, $M$ be an $A$-module and $M' := B \otimes_A M$.
Then we have an isomorphism

$$\begin{aligned} \phi : M' \otimes_A B &\longrightarrow B \otimes_A M' \\ (b \otimes x) \otimes b' &\longmapsto b \otimes (b' \otimes x) \end{aligned}$$

Conversely, under some assumption, a pair $(M', \phi)$ as above arises in this way.
Precisely consider a $B$-module $M'$ and a $B \otimes_A B$-module isomorphism $\phi : M' \otimes_A B \to B \otimes_A M'$.
Define maps

$$\begin{aligned} \phi_1 &: B \otimes_A M' \otimes_A B \longrightarrow B \otimes_A B \otimes_A M' \\ \phi_2 &: M' \otimes_A B \otimes_A B \longrightarrow B \otimes_A B \otimes_A M' \\ \phi_3 &: M' \otimes_A B \otimes_A B \longrightarrow B \otimes_A M' \otimes_A B \end{aligned}$$

by tensoring $\phi$ with $1_B : B \to B$ respectively on the first, second and third factor.
Then we have the following result:

**Lemma 3.3.23** *If $\phi_2 = \phi_1\phi_3$ then the pair $(M', \phi)$ arises as we described above.*
*Precisely $M' \simeq B \otimes_A M$ for some $A$-module $M$ and $\phi$ identifies with the map $(b \otimes x) \otimes b' \mapsto b \otimes (b' \otimes x)$.*

**Proof**
See [13] II §8 or [18] I 2.21. $\square$.

We are ready to prove theorem 3.3.7:

**Proof of theorem 3.3.7 (sketch)**
The exact sequence of low-degree terms (proposition 3.3.15) of the Leray spectral sequence for $\mathbb{G}_{m,S}$ (proposition 3.3.21) starts with

$$0 \to H^1_{\mathrm{Zar}}\big(X, f_*\mathbb{G}_{m,S}\big) \to H^1_{\acute{e}t}\big(S, \mathbb{G}_{m,S}\big) \to H^0_{\mathrm{Zar}}\big(X, R^1 f_*\mathbb{G}_{m,S}\big) \to \cdots$$

Clearly $f_*\mathbb{G}_{m,S} = \mathcal{O}_S^\times$, hence by proposition 3.3.22 we just need to prove the following lemma:

**Lemma 3.3.24**

$$R^1 f_*\mathbb{G}_{m,S} = 0.$$

**Proof** (sketch)
First of all we claim that $R^1 f_*\mathbb{G}_{m,S}$ is the Zariski sheaf associated to the Zariski presheaf

$$U \mapsto H^1_{\acute{e}t}\big(U, \mathbb{G}_{m,S}\big).$$

This follows by a more general fact which proof relies on homological algebra formalism:
Consider for all $q > 0$ the functor $h^q : \mathcal{S}(S_{\acute{e}t}) \to \mathcal{S}(S_{\mathrm{Zar}})$ which associates to an étale sheaf $\mathscr{F}$ the Zariski sheaf associated to the Zariski presheaf $U \mapsto H^q_{\acute{e}t}\big(U, \mathscr{F}\big)$. One can prove that the $h^q$ define a collection of universal cohomological $\partial$-functors (see [18] III 1.13 for details). But $h^0$ is just $f_* : \mathcal{S}(S_{\acute{e}t}) \to \mathcal{S}(S_{\mathrm{Zar}})$ hence we find that $h^q \simeq R^q f_*$.
Moreover it is not difficult to see that $H^1_{\acute{e}t}\big(U, \mathbb{G}_{m,S}\big) = H^1_{\acute{e}t}\big(U, \mathbb{G}_{m,U}\big)$ (see [22] II 1.4.9 for details).
Since open affine subschemes form a base for the Zariski topology, we may assume that $U$ is affine and by passing to the stalks we may assume that it

is the spectrum of a local ring.
Using the result of example 3.3.19 we see that

$$H^1_{ét}\big(U, \mathbb{G}_{m,U}\big) \simeq \check{H}^1(U, \mathbb{G}_{m,U}\big) = \varinjlim H^1\big(\{U_i \to U\}, \mathbb{G}_{m,U}\big)$$

where the limits runs over all étale coverings.
But $U$ is affine, thus quasi compact, and the morphisms $U_i \to U$ are open (remark 3.2.6). Hence every such covering may be refined to a finite covering consisting of affine schemes and we can compute the limit over this family of covering.
Moreover $H^1\big(\{U_i \to U\}, \mathbb{G}_{m,U}\big) = H^1\big(\{\amalg U_i \to U\}, \mathbb{G}_{m,U}\big)$ hence we finally find

$$H^1_{ét}\big(U, \mathbb{G}_{m,U}\big) \simeq \varinjlim_{V} H^1\big(\{V \to U\}, \mathbb{G}_{m,U}\big),$$

where the limit now runs through all surjective affine étale morphisms.
The following proposition concludes the proof:

**Proposition 3.3.25** *Let $A$ be a local ring and let $V = \operatorname{Spec} B \to U = \operatorname{Spec} A$ be a faithfully flat morphism.*
*Then $H^1(\{V \to U\}, \mathbb{G}_{m,U}) = 0$.*

**Proof**
Recall the description of $H^1(\{V \to U\}, \mathbb{G}_{m,U}) = 0$ given in remark 1.2.3 and take a 1-cocycle $H^1(\{V \to U\}, \mathbb{G}_{m,U}) = 0$.
This is an element $b \in (B \otimes_A B)^\times$, which give rise by multiplication to an isomorphism $\phi : B \otimes_A B \to B \otimes_A B$. Moreover the cocycle condition immediately translate into the descent condition of lemma 3.3.23.
Hence $B$ seen as a $B$-module arise by extension of scalars, which means there exists an $A$-module $M$ such that $B \otimes_A M \simeq B$.
Now $B$ is faithfully flat, hence $M$ must be a flat $A$-module (remark 3.1.8).
But then, for $A$ being local, $M$ is a free $A$-module (proposition 3.1.9) of rank 1, i.e $M \simeq A$ as $A$-module.
Now this translates into the fact that $b$ is a coboundary.
For details and the proof of an even more general result see [18] III 4.10.

□

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