ALGANT Master Thesis in Mathematics

Cohomology of Commutative Rings and the Cotangent Complex

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Introduction

Michel André and Daniel Quillen developed independently the ("correct" [1]) cohomology theory for commutative rings. Our main goal is to understand this cohomology. We will see different ways to define it; as a cotriple cohomology, using the cotangent complex or via more general simplicial resolutions. We will also compute explicit descriptions of this cohomology in degrees 0 and 1 as well as some more general properties.

In the first chapter we define model categories, and we discuss one main example of a model category, the category of chain complexes on an abelian category (with "enough projectives"). In particular, we provide a full proof showing that the category of R-modules for a commutative ring R is a model category. We also see how to provide a model category structure on the category of simplicial sets, and on the category of simplicial objects in an abelian category (with "enough projectives").

We develop a homotopy theory on simplicial sets in such a way that it is equivalent in a strong sense to the ordinary homotopy theory of topological spaces. The construction of this homotopy theory looks natural once we realize there is a pair of adjoint functors between these two categories. On the other hand, we also construct a (co-) homotopy theory on simplicial objects of an abelian category using the (co-) homology theory of chain complexes over that abelian category. In this case, we do not only have a pair of adjoints relating these two categories, but they are also inverse equivalences. This is the content of the Dold-Kan correspondence. For the model category and homotopy theory part, the main references are Goerss and Jardine [3], Hovey [6] and Quillen [12]. In the last section, where we prove the Dold-Kan correspondence, we follow Weibel [17].

In the second chapter, we mainly work in the category of commutative rings. As in any abelian category, the derived functors are defined using resolutions of rings and applying homology and cohomology to the image of resolutions under those functors. The Dold-Kan correspondence allows us to look at these resolutions as augmented simplicial rings. Thus, we can define homology and cohomology on rings using the homotopy and cohomotopy theory of simplicial rings that we built in the previous chapter. We define cotriples ("comonads" in Mac Lane [8]) on a category, and we use them to define certain augmented simplicial objects. In the category of rings, we construct an explicit cotriple that provides (cofibrant) augmented simplicial rings over any given ring. This important feature exhibits the relation between cotriple cohomology and the (André-Quillen) cohomology for commutative rings. In the last section of this chapter, we describe how the cohomology of rings looks like in low degrees. More specifically, for a k-algebra R, we see that the cohomology of R with values in an R-module M in degree 0 is just the module of k-derivations $\mathrm{Der}_k(R,M)$, and in degree one

[1] In words of Daniel Quillen.

is precisely $\text{Exalcomm}_k(R,M)$, the equivalence classes of extensions of R by M (as defined in Grothendieck [5]).

In the last chapter, we see how the cotangent complex can help us understand homology and cohomology of rings in higher degrees. In particular, we realize that the cotangent complex is a free simplicial R-module in the sense of Quillen [14] and therefore we can express homology and cohomology using the derived functors Tor and Ext respectively. Moreover, for any $A \to B \to C$ morphisms of rings, we show that the respective cotangent complexes form a distinguished triangle in the derived category of R-modules. The long exact sequence for cohomology follows from this fact. We also prove how the cotangent complex behaves under flat base changes and its consequences for homology and cohomology. The main references here are Iyengar [7] and Quillen [14].

Finally, we generalize the construction of homology and cohomology of commutative rings for some other categories. The motivation behind this is realizing that in the category of k-algebras over R, the abelianization functor is left adjoint to the natural faithful functor. In particular, this is also true for the category of universal algebras defined by a set of operations and relations. We use this link to extend the definitions of homology and cohomology for universal algebras. At the end, we include some remarks on how this cohomology can also be seen as a cotriple cohomology and as a special case of a more general sheaf cohomology using Grothendieck topologies. In this last part, we follow Quillen [12] and [13].

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Chapter 1

The Dold-Kan Correspondence

In order to provide a consistent homotopy theory on abelian categories we define model categories, which axiomatize homotopy properties of well-known homotopy theories for topological spaces or even homology over chain complexes.

1.1 Model categories

Definition. Let $\mathcal{C}$ be a category. A map f in $\mathcal{C}$ is a *retract* of a map $g \in \mathcal{C}$ if there is a commutative diagram of the form

$$\begin{array}{ccccc} & & \overset{\mathrm{id}}{\frown\!\frown\!\frown} & & \\ A & \longrightarrow & C & \longrightarrow & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & B \\ & & \underset{\mathrm{id}}{\smile\!\smile\!\smile} & & \end{array}$$

Definition. A *model category* is a category $\mathcal{C}$ which is equipped with three classes of maps called weak equivalences, fibrations and cofibrations, subject to the following axioms:

M1: The category $\mathcal{C}$ is closed under finite limits and colimits.

M2: For every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & C & \end{array}$$

in $\mathcal{C}$, where any two of f, g, h are weak equivalences, then so is the third one.

M3: The three distinguished classes of maps are closed under retracts.

M4: For any commutative diagram of solid arrows in $\mathcal{C}$

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow{\scriptstyle j} & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

where j is a cofibration, p is a fibration, and one of them is also a weak equivalence, then the dotted arrow exists making the diagram commutative.

M5: Any map $f : X \longrightarrow Y$ in $\mathcal{C}$ can be factored in two ways:

(i) $X \xrightarrow{i} Z \xrightarrow{q} Y$, where i is a cofibration, and q is a weak equivalence and a fibration,

(ii) $X \xrightarrow{j} Z \xrightarrow{p} Y$, where j is a weak equivalence and a cofibration, and p is a fibration.

Definition. A map in a model category $\mathcal{C}$ which is both a weak equivalence and a cofibration is called an *acyclic cofibration*. Analogously, a map in $\mathcal{C}$ which is both a weak equivalence and a fibration is called an *acyclic fibration*.

By M1, any model category $\mathcal{C}$ has an initial object ϕ (colimit of the empty diagram) and a terminal object $\star$ (limit of the empty diagram).

Definition. An object X in a model category $\mathcal{C}$ is called *cofibrant* if the canonical map $\phi \longrightarrow X$ is a cofibration. It is called *fibrant* if $X \longrightarrow \star$ is a fibration.

Definition. For $i : A \to B$, $p : X \to Y$, maps in a category $\mathcal{C}$, we say that i has the *left lifting property* (LLP) with respect to p, or that p has the *right lifting property* (RLP) with respect to i if for any commutative square of solid arrows

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

there exists the dotted arrow making the whole diagram commutative.

Remark. Using these definitions we can characterize fibrations and cofibrations of a model category $\mathcal{C}$. A cofibration in $\mathcal{C}$ has the LLP with respect to all acyclic fibrations (by M4). On the other hand, if f is a map in $\mathcal{C}$ having the LLP w.r.t. all acyclic fibrations, then we can factor $f = q\,i$ where i is a cofibration and q an acyclic fibration. Then, f has the LLP w.r.t. q, so there is some u solving the lifting problem

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ {\scriptstyle f}\downarrow & \overset{u}{\nearrow} & \downarrow{\scriptstyle q} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

and we get a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{id}} & X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{u} & Z & \xrightarrow{q} & Y \end{array} \qquad (q \circ u = \mathrm{id})$$

which means that f is a retract of the cofibration i. Thereofore it is also a cofibration. This shows that a map in $\mathcal{C}$ is a cofibration if and only if it has the LLP with respect to all acyclic fibrations in $\mathcal{C}$. Analogously we see that a map in $\mathcal{C}$ is a fibration if and only if it has the LLP with respect to all acyclic cofibrations.

Corollary 1.1.1. *If the following commutative diagram in a model category* $\mathcal{C}$

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B & \longrightarrow & D \end{array}$$

is a pushout square, and f is a cofibration, then g is also a cofibration. If, on the other hand, it is a pullback square and g is a fibration, then f is a fibration too.

Proof. Direct consequence of the previuos remark. □

Let R be a commutative ring (with 1) and let $\mathbf{Mod}_R$ be the category of R-modules. Then $\mathbf{Ch}_{\geqslant 0}(R)$ is the category of non-negative chain complexes in $\mathbf{Mod}_R$. This provides the first example of a model category.

Theorem 1.1.2. *The category $\mathbf{Ch}_{\geqslant 0}(R)$ has the structure of a model category where a morphism $f : M_\bullet \to N_\bullet$ is*

- *a weak equivalence if $H_\star f$ is an isomorphism;*
- *a fibration if $f_n : M_n \to N_n$ is surjective for $n \geqslant 1$, and;*
- *a cofibration if f_n is injective with projective cokernel for $n \geqslant 0$.*

In order to prove this theorem we use the following result that characterizes acyclic fibrations in $\mathbf{Ch}_{\geqslant 0}(R)$. For a chain complex $M_\bullet$ we denote by $Z_nM \subseteq M_n$ the cycles in M_n, setting $Z_{-1}M = 0$. The differential map is always denoted by ∂^M.

Lemma 1.1.3. *Let $f : M_\bullet \longrightarrow N_\bullet$ be a morphism in $\mathbf{Ch}_{\geqslant 0}(R)$. The following are equivalent:*

(a) $H_\star f$ is an isomorphism and $f_n : M_n \to N_n$ is surjective for $n \geqslant 1$.

(b) The induced map

$$f' : \; M_n \longrightarrow Z_{n-1}M \times_{Z_{n-1}N} N_n$$

is surjective for $n \geqslant 0$.

Proof. (a) ⇒ (b): For $n = 0$, f' is just $f_0 : M_0 \longrightarrow N_0$. Let $n \in N_0$. By surjectivity of $H_0 f$ there are some $m \in M_0$, $n' \in N_1$ such that $n = f_0(m) + \partial_1^N(n')$. By surjectivity of f_1 there is some $m' \in M_1$ with $f_1(m') = n'$, and therefore $f_0(m + \partial_1^M(m')) = f_0(m) + \partial_0^M(f_1(m')) = n$. Thus, f_0 is surjective. Now let $n \geqslant 1$ and form the diagram

$$\begin{array}{ccccc} M_n & & & & \\ & \searrow^{f'} & & \overset{f_n}{\curvearrowright} & \\ & & Z_{n-1}M \times_{Z_{n-1}N} N_n & \xrightarrow{p_2} & N_n \\ {\scriptstyle \partial_n^M}\searrow & & {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle \partial_n^N} \\ & & Z_{n-1}M & \xrightarrow{f_{n-1}} & Z_{n-1}(N) \end{array}$$

For any $(m,n) \in Z_{n-1}M \times_{Z_{n-1}N_n}$, there is some $m' \in M_n$ with $f_n(m') = n$. Then

$$\partial_n^M(m') - m \in Z_{n-1}M$$

and $f_{n-1}(\partial_n^M(m') - m) = 0$. Since f_n is surjective for all $n \geqslant 0$ we get a short exact sequence of chain complexes

$$0 \longrightarrow \ker(f)_\bullet \longrightarrow M_\bullet \xrightarrow{f} N_\bullet \longrightarrow 0$$

and since $H_\star f$ is an isomorphism, applying the long exact sequence of homology we get that the complex $\ker(f)_\bullet$ is acyclic. Hence, there is some $m'' \in \ker(f_n)$ such that

$$\partial_n^M(m'') = \partial_n^M(m') - m,$$

and therefore $f'(m' - m'') = (m,n)$.
(b) $\Rightarrow$ (a): Let $n \geqslant 1$. For any $n \in Z_n(N)$, let $(0,n) \in Z_{n-1}M \times_{Z_{n-1}N} N_n$ and by surjectivity of f' there is some $m \in M_n$ such that $f'(m) = (0,n)$. Then $\partial_n^M(m) = 0$, so $m \in Z_nM$ and $f_n(\partial_n^M(m)) = n$, so $f_n : Z_nM \longrightarrow Z_nN$ is surjective for all $n \geqslant 1$. But then, the map p_2 is also surjective, and thus f_n is surjective. This also gives surjectivity of $H_\star f$. To see injectivity, let $m \in Z_{n-1}M$, such that $f_{n-1}(m) = \partial_n^N(n)$ for some $n \in N_n$. Then $(m,n) \in Z_{n-1}M \times_{Z_{n-1}N} N_n$, and we can take some $m' \in M_n$ such that $f'(m') = (m,n)$. But then $\partial_n^M(m') = m$ so m is in fact a boundary of $Z_{n-1}M$. □

Proof of Theorem 1.1.2. First note that $\mathbf{Ch}_{\geqslant 0}(R)$ is closed under finite limits and colimits. Let us see that the classes of maps defined as in the statement satisfy the rest of the axioms. Axioms M2 and M3 are clear.

Let us denote by $D(n)$, $n \geqslant 1$ the chain complex given by

$$D(n)_k = \begin{cases} R & \text{for } k = n-1, n \\ 0 & \text{for } k \neq n-1, n \end{cases} \qquad \partial_k = \begin{cases} \mathrm{id}_R & \text{for } k = n \\ 0 & \text{for } k \neq n. \end{cases}$$

In particular, note that $H_m(D(n)) = 0$ for all $m \geqslant 0$. There is a natural isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Ch}_{\geqslant 0}(R)}(D(n), N_\bullet) & \longrightarrow & N_n \\ h & \longmapsto & h_n(1_R). \end{array}$$

If $f : M_\bullet \longrightarrow N_\bullet$ is a fibration, then for every $n \geqslant 1$ there is a solution to the lifting problem

$$\begin{array}{ccc} 0 & \longrightarrow & M_n \\ \downarrow & \nearrow & \downarrow f_n \\ R & \longrightarrow & N_n \end{array}$$

since R is a projective R-module. Hence, f has the RLP w.r.t. the maps $0 \longrightarrow D(n)$, $n \geqslant 1$. On the other hand, if $f : M_\bullet \longrightarrow N_\bullet$ is a map having the RLP w.r.t. all the maps $0 \longrightarrow D(n)$, for $n \geqslant 1$, then for any $n \geqslant 1$ and $x \in N_n$, x determines a map $h : D(n) \longrightarrow N_\bullet$ such that $h_n(1_R) = x$. The solution to the lifting problem

$$\begin{array}{ccc} 0 & \longrightarrow & M_\bullet \\ \downarrow & \overset{u}{\nearrow} & \downarrow f \\ D(n) & \xrightarrow{h} & N_\bullet \end{array}$$

provides a map u such that $f_n(u_n(1_R)) = h_n(1_R) = x$. Hence, the map f_n is surjective for all $n \geqslant 1$. Thus, a map is a fibration if and only if it has the RLP w.r.t. all maps $0 \longrightarrow D(n)$, $n \geqslant 1$.

For a chain complex $M_\bullet$, we define the chain complex

$$P(N_\bullet) = \bigoplus_{n>0} \bigoplus_{x\in N_n} D(n)[x]$$

where $D(n)[x]$ denotes a copy of $D(n)$. We define an evaluation morphism $\varepsilon : P(N_\bullet) \longrightarrow N_\bullet$, where any $r \in D(n)[x]_n$ is sent to $x \in N_n$. Hence, ε is surjective in every degree, so it is a fibration. Moreover, $H_m(P(N_\bullet)) = 0$ for all $m \geqslant 0$, so it is an acyclic complex. For any map $X_\bullet \longrightarrow Y_\bullet$ which is degree-wise surjective, the lifting problem on the left

$$\begin{array}{ccc} & & X_\bullet \\ & \overset{u}{\nearrow} & \downarrow \\ P(N_\bullet) & \longrightarrow & Y_\bullet \end{array} \qquad\qquad \begin{array}{ccc} & & X_n \\ & \overset{u_n}{\nearrow} & \downarrow\!\!\downarrow \\ P(N_\bullet)_n & \longrightarrow & Y_n \end{array}$$

has a solution u whose degree n is given by the solution of the lifting problem on the right, which exists since $P(N_\bullet)_n$ is a free R-module with basis $\{x \mid x \in N_n \cup N_{n+1}\}$.

We prove first axiom M5. Let $f : M_\bullet \longrightarrow N_\bullet$ be a morphism of chain complexes. By the universal property of the coproduct we can factorize f as

$$M_\bullet \xrightarrow{\ j\ } M_\bullet \oplus P(N_\bullet) \xrightarrow{\ p\ } N_\bullet$$

where q is surjective in every degree since it is the composite

$$P(N_\bullet) \longrightarrow M_\bullet \oplus P(N_\bullet) \xrightarrow{\ p\ } N_\bullet, \qquad \text{composite } \varepsilon$$

and ε is surjective in every degree. So p is a fibration. On the other hand, j is injective in every degree, and $\operatorname{coker}(j_n) \cong P(N_\bullet)_n$ which is a projective module. Since $P(N_\bullet)$ is acyclic,

$$H_n(M_\bullet \oplus P(N_\bullet)) \cong H_n(M_\bullet)$$

and $H_\star j$ is indeed an isomorphism. Thus j is an acyclic cofibration. This proves the factorization in M5 (ii). For the other one, we proceed by the following induction on $n \geqslant 0$: for all $0 \leqslant k \leqslant n-1$ we assume there are R-modules Q_k and maps $i_k : M_k \longrightarrow Q_k$, $q_k : Q_k \longrightarrow N_k$ and $\partial_k^Q : Q_k \longrightarrow Q_{k-1}$ such that $f_k = q_k i_k$, $(\partial_k^Q)^2 = 0$, i_k is a cofibration and the induced map

$$Q_k \longrightarrow Z_{k-1}Q \times_{Z_{n-1}N} N_k$$

is surjective for all $0 \leqslant k \leqslant n-1$. For the case $n = 0$ we just choose a surjection $P_0 \twoheadrightarrow N_0$ with P_0 projective module and factorize f_0 as

$$M_0 \xrightarrow{\ i_0\ } M_0 \oplus P_0 \xrightarrow{\ q_0\ } N_0$$

i_0 is injective with projective cokernel P_0 and q_0 is surjective since the map $P_0 \twoheadrightarrow N_0$ is surjective. We set $Q_0 = M_0 \oplus P_0$. Finally

$$Q_0 \longrightarrow Z_{-1}Q \times_{Z_{-1}N} N_0 \cong N_0$$

is isomorphic to the surjective map q_0.

For the inductive step, we consider the commutative diagram

$$\begin{array}{ccccc} M_n & \xrightarrow{\partial_n^M} & Z_{n-1}M & \xrightarrow{Z_{n-1}\, i_{n-1}} & Z_{n-1}Q \\ {\scriptstyle f_n}\downarrow & & & & \downarrow{\scriptstyle Z_{n-1}\, q_{n-1}} \\ N_n & & \xrightarrow{\qquad\partial_n^N\qquad} & & Z_{n-1}N \end{array}$$

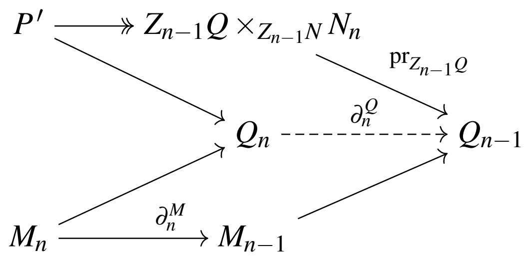
which gives a map

$$f'\colon\ M_n \longrightarrow Z_{n-1}Q \times_{Z_{n-1}N} N_n$$

factoring f_n. Let us call $N_n{}' = Z_{n-1}Q \times_{Z_{n-1}N} N_n$, and choose a surjection $P_n{}' \twoheadrightarrow N_n{}'$ with P'_n a projective module such that f' is factorize as

$$M_n \xrightarrow{i_n} M_n \oplus P'_n \xrightarrow{q'_n} N_n{}'$$

where i_n is injective with projective cokernel P'_n and q'_n is surjective since $P'_n \twoheadrightarrow N_n{}'$ is surjective. Setting $Q_n = M_n \oplus P'_n$, and $q_n = \mathrm{pr}_{N_n} q'_n : Q_n \longrightarrow N_n$. This completes the induction, where the differential map ∂_n^Q is given by the diagram



Induction gives a chain of R-modules $Q_\bullet$ and chain maps $i : M_\bullet \longrightarrow Q_\bullet$, $q : Q_\bullet \longrightarrow N_\bullet$ such that $f = qi$. The map i is degree-wise injective with projective cokernel, so it is a cofibration. The map q induces a surjection $M_n \longrightarrow Z_{n-1}Q \times_{Z_{n-1}N} N_n$ for $n \geqslant 0$, so by Lemma 1.1.3 is an acyclic fibration.

For axiom M4 assume we are given a commutative diagram

$$\begin{array}{ccc} A_\bullet & \longrightarrow & M_\bullet \\ {\scriptstyle f}\downarrow & \overset{u}{\nearrow} & \downarrow{\scriptstyle g} \\ B_\bullet & \longrightarrow & N_\bullet \end{array} \tag{1.1}$$

where g is an acyclic fibration, and f is a cofibration. We construct the map u by induction on its degree. Lemma 1.1.3 for $n = 0$ says that the map $g_0 : M_0 \longrightarrow N_0$ is surjective. Since f_0 is injective, $B_0 \cong A_0 \oplus \mathrm{coker}(f_0)$, where $\mathrm{coker}(f_0)$ is a projective module. Hence, the following lifting problem

$$\begin{array}{ccc} A_0 & \longrightarrow & M_0 \\ \big\downarrow & \nearrow & \downarrow{\scriptstyle g_0} \\ A_0 \oplus \mathrm{coker}(f_0) & \longrightarrow & N_0 \end{array}$$

has a solution which is precisely u_0. Assume now that $u_k : B_k \to M_k$ is given for $k < n$. To build u_n we need to solve the lifting problem

$$\begin{array}{ccc} A_n & \longrightarrow & M_n \\ {\scriptstyle f_n}\downarrow & \overset{u_n}{\nearrow} & \downarrow \\ B_n & \longrightarrow & Z_{n-1}M \times_{Z_{n-1}N} N_n \end{array}$$

where the map on the right is surjective by Lemma 1.1.3. Moreover, f_n is injective, so we have $B_n \cong A_n \oplus \operatorname{coker}(f_n)$, where $\operatorname{coker}(f_n)$ is a projective module. Hence the problem has a solution which is the map u_n.

Assume now that in (1.1) g is a fibration and f is an acyclic cofibration. We can apply the acyclic cofibrant-fibrant factorization to the map f and we have

$$\begin{array}{ccc} A_\bullet & \xrightarrow{j} & A_\bullet \oplus P(B_\bullet) \\ {\scriptstyle f}\downarrow & \overset{u}{\nearrow} & \downarrow{\scriptstyle p} \\ B_\bullet & \xrightarrow{\text{id}} & B_\bullet \end{array}$$

where j is an acyclic cofibration and p is a fibration. Since both j, f are weak equivalences, by M2 p is a weak equivalence too, so it an acyclic fibration, and therefore the map u exists. We get a commutative diagram

$$\begin{array}{ccccccc} A_\bullet & \xrightarrow{\text{id}} & A_\bullet & \xrightarrow{\text{id}} & A_\bullet & \longrightarrow & M_\bullet \\ {\scriptstyle f}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B_\bullet & \xrightarrow{u} & A_\bullet \oplus P(B_\bullet) & \xrightarrow{p} & B_\bullet & \longrightarrow & N_\bullet \end{array}$$
$$(p \circ u = \text{id})$$

and by the lifting property of $P(B_\bullet)$ we get a map $A_\bullet \oplus P(B_\bullet) \longrightarrow M_\bullet$ making the diagram commutative, whose composition with u gives the desired map $B_\bullet \longrightarrow M_\bullet$ solution of (1.1). $\square$

There is another important example of model category, which is the category of topological spaces, **Top**. More precisely, we focus our attention on the category of complactly generated Hausdorff spaces, **CGH**, for reasons that will be clear later on. Weak equivalences in **CGH** are the weak homotopy equivalences (whence the name), and fibrations are the Serre fibrations. Cofibrations are uniquely determined by the maps having the LLP with respect to all acyclic fibrations.

1.2 Fibrant simplicial sets

We define the category Δ whose objects are totally ordered sets $\mathbf{n} = \{0 < 1 < \ldots < n\}$ with $n+1$ elements, and whose morphisms $f : \mathbf{m} \to \mathbf{n}$ are order-preserving set functions.

Definition. For any category $\mathcal{A}$, a *simplicial object* $A_\star$ in $\mathcal{A}$ is a functor $A_\star : \Delta^{\text{op}} \longrightarrow \mathcal{A}$. Equivalently, a *cosimplicial object* $C^\star$ in $\mathcal{A}$ is a functor $C^\star : \Delta \longrightarrow \mathcal{A}$. For simplicity, we will denote $A_n = A_\star(\mathbf{n})$, whose elements are called *n-simplicies* (we also say *vertices* for the 0-simplicies), $C^n = C^\star(\mathbf{n})$ and $A_\star(f) = f^\star$ for f a map in Δ. A *simplicial map* is just a natural transformation. We will denote by $\mathcal{SA}$ the category of simplicial objects in $\mathcal{A}$ together with these simplicial maps as morphisms.

Example 1.2.1. For any object A in a category $\mathcal{A}$ we can construct a "constant" simplicial object $cA = (cA)_\star \in \mathcal{SA}$ given by $(cA)_n = A$ for all $n \geqslant 0$, and taking $f^\star = \mathrm{id}_A$ for every map f in Δ.

The following result shows a way to characterize simplicial objects that will be useful.

Proposition 1.2.2. *Let $\mathcal{A}$ be a category. A simplicial object $A_\star$ in $\mathcal{A}$ is just a sequence of objects A_n, $n \geqslant 0$ together with maps*

$$\begin{aligned} d_i : A_n \to A_{n-1}, &\quad 0 \leqslant i \leqslant n \quad \textit{(face maps)} \\ s_j : A_n \to A_{n+1}, &\quad 0 \leqslant j \leqslant n \quad \textit{(degeneracy maps)} \end{aligned}$$

which satisfy the following simplicial identities

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \textit{if } i < j \\ s_i s_j &= s_{j+1} s_i \quad \textit{if } i \leqslant j \end{aligned}$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \textit{if } i < j \\ \textit{identity} & \textit{if } i = j, j+1 \\ s_j d_{i-1} & \textit{if } i > j+1 \end{cases}$$

Proof. In the category Δ we define the coface maps $d^i : \mathbf{n-1} \to \mathbf{n}$ and codegeneracy maps $s^i : \mathbf{n+1} \to \mathbf{n}$ as follows:

$$d^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geqslant i \end{cases}, \quad s^i(j) = \begin{cases} j & \text{if } j \leqslant i \\ j-1 & \text{if } j > i \end{cases},$$

i.e., d^i is the unique (order-preserving) injective map whose image does not contain $i \in \mathbf{n}$, and s^i is the unique surjective map that sends two different elements in $\mathbf{n+1}$ to $i \in \mathbf{n}$. These maps satisfy the following *cosimplicial identities*:

$$\begin{aligned} d^j d^i &= d^i d^{j-1} \quad \text{if } i < j \\ s^j s^i &= s^i s^{j+1} \quad \text{if } i \leqslant j \end{aligned}$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{if } i < j \\ \text{identity} & \text{if } i = j, j+1 \\ d^{i-1} s^j & \text{if } i > j+1 \end{cases}$$

Moreover, for any map $f \in \mathrm{Hom}_\Delta(\mathbf{n}, \mathbf{m})$ which is not the identity map, we can write $i_s, \ldots, i_1$ for the elements in $\mathbf{m}$ which are not in the image of f (in that order respectively), and $j_1, \ldots, j_t$ the elements in $\mathbf{n}$ such that $f(j) = f(j+1)$. Then,

$$f = d^{i_1} \cdots d^{i_s} s^{j_1} \cdots s^{j_t}, \quad 0 \leqslant i_s < \cdots < i_1 \leqslant m, \quad 0 \leqslant j_1 < \cdots < j_t < n, \quad n - t + s = m.$$

This factorization is unique. If $A_\star$ is a simplicial object according to our initial definition, i.e., a functor $A_\star : \Delta^{op} \to \mathcal{A}$, then we just set $A_n = A_\star(\mathbf{n})$, and $d_i = A_\star(d^i)$, $s_i = A_\star(s^i)$. On the other hand, for a sequence of objects A_n in $\mathcal{A}$ and maps d_i, s_i satisfying the simplicial identities we define a functor $A_\star : \Delta^{op} \to \mathcal{A}$ by setting $A_\star(\mathbf{n}) = A_n$. For any map $f \in \mathrm{Hom}_\Delta(\mathbf{n}, \mathbf{m})$, if f is the identity map, then we send f to the identity map in A_n, and if it is not the identity map, then we use the previous factorization $f = d^{i_1} \cdots d^{i_s} s^{j_1} \cdots s^{j_t}$ and define $A_\star(f) = s_{j_t} \cdots s_{j_1} d_{i_s} \cdots d_{i_1}$. □

A *simplicial set* is a simplicial object in the category of sets. Analogously, we can talk of *simplicial groups, simplicial modules* and so on depending on the choice of the category $\mathcal{A}$.

Example 1.2.3. For any $k \geqslant 0$, we let $\Delta^k : \Delta^{\text{op}} \longrightarrow \mathbf{Set}$ be the contravariant functor which is represented by $\mathbf{k} \in \Delta$. In other words, for any $\mathbf{n} \in \Delta$,

$$\Delta^k(\mathbf{n}) = \mathrm{Hom}_\Delta(\mathbf{n}, \mathbf{k}),$$

and for any map $f : \mathbf{n} \to \mathbf{m}$ in Δ,

$$\begin{array}{rccc} \Delta^k(f): & \mathrm{Hom}_\Delta(\mathbf{m}, \mathbf{k}) & \longrightarrow & \mathrm{Hom}_\Delta(\mathbf{n}, \mathbf{k}). \\ & g & \longmapsto & gf \end{array}$$

Thus, we get a simplicial set Δ^k for all $k \geqslant 0$ which is called *standard k-simplex*. Moreover, any map $f : \mathbf{n} \to \mathbf{m}$ in Δ induces a map of standard simplicies $f : \Delta^n \longrightarrow \Delta^m$ by composition with f.

Definition. Let $K_\star$ be a simplicial set and let $x \in K_n$, then

(i) x is called *degenerate* if it the image of some degeneracy map, i.e., $x = s_i(y)$ for some s_i and $y \in K_{n-1}$,

(ii) x is called *non-degenerate* if it is not of the form $s_i(y)$ for any $y \in K_{n-1}$ and $s_i : K_{n-1} \to K_n$ for $i = 0, \dots, n-1$,

(iii) x is a *face (of K_n)* if it is in the image of some face map $d_i : K_{n+1} \to K_n$.

Remark. Yoneda Lemma [1] tells us that for a simplicial set $K_\star$, there is a natural bijection

$$\mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(\Delta^n, K_\star) \cong K_n.$$

In particular, any vertex $k \in K_0$ can be seen as a simplicial map $k : \Delta^0 \longrightarrow K_\star$. More generally, in order to define a map of simplicial sets $f : K_\star \longrightarrow L_\star$, it is enough to define the image for the simplicies of $K_\star$ which are not faces and non-degenerate. Indeed, all degenerate and face simplicies are uniquely determined by the naturality of the map f.

We define two subcomplexes of Δ, the *boundary of* Δ^n is a simplicial set $\partial\Delta^n$ which is the smallest subcomplex of Δ^n containing the faces $d_j(\mathrm{id}_\mathbf{n})$, $0 \leqslant j \leqslant n$, so

$$(\partial\Delta^n)_j = \begin{cases} (\Delta^n)_j & \text{if } 0 \leqslant j \leqslant n-1 \\ \text{degenerate elements of } (\Delta^n)_j & \text{if } j \geqslant n. \end{cases}$$

We set $\partial\Delta^0 = \emptyset$ to be the simplicial set with the empty set in every degree. On the other hand, the *k-th horn* of Δ^n is the simplicial set Λ^n_k for $0 \leqslant k \leqslant n$, which the the subcomplex of Δ^n generated by all faces $d_j(\mathrm{id}_\mathbf{n})$ except the k-th face $d_k(\mathrm{id}_\mathbf{n})$. Intuitively, one may think of Δ^0 as the one point space, and Δ^1 as the interval. The maps $d^0, d^1 : \mathbf{0} \to \mathbf{1}$ induce maps $d^0, d^1 : \Delta^0 \longrightarrow \Delta^1$ which can be though as the inclusions of "end points".

[1] Mac Lane [8] III.2.

Let us see how can we obtain a topological space $|K_\star|$ out of a simplicial set $K_\star$. For any $n \geqslant 0$, we denote by $|\Delta^n|$ the geometric n-simplex, i.e.,

$$|\Delta^n| = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geqslant 0, \sum_{i=0}^{n} t_i = 1 \right\}.$$

Notice that any map $f : \mathbf{n} \to \mathbf{m}$ in Δ induces a map $f_\star : |\Delta^n| \to |\Delta^m|$ given by

$$f_\star(t_0, \dots, t_n) = \left(\sum_{i \in f^{-1}(0)} t_i, \ \dots, \sum_{i \in f^{-1}(m)} t_i \right).$$

Definition. Let $K_\star$ be a simplicial set. We define the *simplex category* $\Delta \downarrow K_\star$ (the category of objects over $K_\star$) whose objects are maps $\sigma : \Delta^n \longrightarrow K_\star$ (or simplicies), and whose morphisms are commutative diagrams of simplicial maps

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\theta} & \Delta^m \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & K_\star & \end{array}$$

where θ is a map in Δ.

Lemma 1.2.4. *For a simplicial set $K_\star$ there is a natural bijection*

$$K_\star \cong \varinjlim_{\substack{\Delta^n \to K_\star \\ in\ \Delta \downarrow K_\star}} \Delta^n.$$

Proof. Note that Δ is a small category, and $\mathcal{S}\mathbf{Set}$ is cocomplete. Hence, there is a pair of adjoints [2]

$$\begin{array}{ccc} \mathcal{S}\mathbf{Set} & \longrightarrow & \mathcal{S}\mathbf{Set} \\ K_\star & \longmapsto & (\mathbf{n} \mapsto \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(\Delta^n, K_\star)), \end{array} \qquad \begin{array}{ccc} \mathcal{S}\mathbf{Set} & \longrightarrow & \mathcal{S}\mathbf{Set} \\ K_\star & \longmapsto & \varinjlim_{\substack{\Delta^n \to K_\star \\ \text{in } \Delta \downarrow K_\star}} \Delta^n. \end{array}$$

But by Yoneda Lemma, the first map is the identity, so by uniqueness of the adjoint, the other map should also be isomorphic to the identity map, which gives the result. □

This lemma motivates the following definition for the geometric realization of a simplicial set.

Definition. Let $K_\star$ be a simplicial set. The *geometric realization* of $K_\star$ is the colimit

$$|K_\star| = \varinjlim_{\substack{\Delta^n \to K_\star \\ \text{in } \Delta \downarrow K_\star}} |\Delta^n|.$$

in the category of topological spaces.

[2] Mac Lane and Moerdijk [9] I. Theorem 2.

Remark. Note that any simplicial map $f : K_\star \to J_\star$ induces a map $f_\star : \Delta \downarrow K_\star \to \Delta \downarrow J_\star$, where

$$(f_\star)(\Delta^n \longrightarrow K_\star) = \Delta^n \longrightarrow K_\star \xrightarrow{f} J_\star \in \Delta \downarrow J_\star.$$

This way, the geometric realization becomes in fact a functor

$$|\cdot| : \mathcal{S}\mathbf{Set} \longrightarrow \mathbf{Top}.$$

Example 1.2.5. As notation suggests, for any $n \geqslant 0$, the geometric realization of the standard n-simplex Δ^n is precisely the geometric n-simplex $|\Delta^n|$, since the simplex category $\Delta \downarrow \Delta^n$ has $\mathrm{id}_{\Delta^n} : \Delta^n \longrightarrow \Delta^n$ as terminal object.

Let us go now the other way around, so building a simplicial set starting from a topological space. Let X be a topological space, and $n \geqslant 0$. A *singular n-simplex* is a continuous map $\sigma : |\Delta^n| \to X$. If we denote the set of n-simplices by

$$\mathcal{S}(X)_n = \{\sigma : |\Delta^n| \to X \ , \ \sigma \text{ continuous}\},$$

we can obtain a simplicial set $\mathcal{S}(X)_\star$ viewed as a functor $\mathcal{S}(X)_\star : \Delta \longrightarrow \mathbf{Set}$. For any $\mathbf{n} \in \Delta$, $\mathcal{S}(X)_\star(\mathbf{n}) = \mathcal{S}(X)_n$, and for any map $f : \mathbf{n} \to \mathbf{m}$ in Δ, recall the induced function $f : \Delta^n \to \Delta^m$ and define

$$\begin{array}{rccc} \mathcal{S}(X)_\star(f) : & \mathcal{S}(X)_m & \longrightarrow & \mathcal{S}(X)_n \\ & \sigma & \longmapsto & \sigma f \end{array}$$

This is a well-defined functor and $\mathcal{S}(X)_\star$ is in fact a simplicial set. Moreover, any continuous map $g : X \to Y$ between topological spaces induces maps

$$\begin{array}{rccc} \mathcal{S}(f)_n : & \mathcal{S}(X)_n & \longrightarrow & \mathcal{S}(Y)_n \\ & \sigma & \longmapsto & g\sigma \end{array}$$

Thus, we get a functor

$$\mathcal{S} : \mathbf{Top} \longrightarrow \mathcal{S}\mathbf{Set}.$$

Proposition 1.2.6. *There is a pair of adjoint functors*

$$|\cdot| : \mathcal{S}\mathbf{Set} \rightleftarrows \mathbf{Top} : \mathcal{S},$$

where, with this notation, we always mean that $|\cdot|$ *is the left adjoint and* $\mathcal{S}$ *the right adjoint.*

Proof. Let $K_\star$ be a simplicial set, X a topological space. First of all, notice that for any $n \geqslant 0$ we have a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X) \cong \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(\Delta^n, \mathcal{S}(X)_\star)$$

since any continuous map $\sigma : |\Delta^n| \to X$, defines a simplicial map that sends any $\theta \in (\Delta^n)_m$ to the composite

$$|\Delta^m| \xrightarrow{|\theta_\star|} |\Delta^n| \xrightarrow{\sigma} X \qquad \in \mathcal{S}(X)_m.$$

The inverse is given by $\sigma_n(\mathrm{id}_{\mathbf{n}})$, for σ a simplicial map $\Delta^n \to \mathcal{S}(X)_\star$. Hence, there are a natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Top}}(|K_\star|, X) &\cong \varprojlim_{\Delta^n \to K_\star} \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X) \cong \varprojlim_{\Delta^n \to K_\star} \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(\Delta^n, \mathcal{S}(X)_\star) \\ &\cong \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(K_\star, \mathcal{S}(X)_\star). \end{aligned}$$

□

Proposition 1.2.7. *For any simplicial set $K_\star$, $|K_\star| \in$* **CGH**.

Proof. See Goerss and Jardine [3] I, Proposition 2.3. □

The category of simplicial sets $\mathcal{S}\mathbf{Set}$ is closed under finite limits and colimits. The realization functor $|\cdot|$ preserves colimits since it is left adjoint by Proposition 1.2.6, and finite limits (see Hovey [6] Lemma 3.2.4.). We can define a model structure on $\mathcal{S}\mathbf{Set}$ using the model category **CGH**, and the realization functor. Formally, this is done using a Quillen equivalence.

Definition. Let $\mathcal{C}$, $\mathcal{D}$ be two model categories. A *Quillen functor* from $\mathcal{C}$ to $\mathcal{D}$ is a pair of adjoint functors

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

such that

- the functor F preserves cofibrations and weak equivalences between cofibrant objects,
- the functor G preserves fibrations and weak equivalences between fibrant objects.

A Quillen functor is a *Quillen equivalence* if for all cofibrant objects $X \in \mathcal{C}$ and all fibrant objects $Y \in \mathcal{D}$, a morphism

$$X \longrightarrow G(Y)$$

is a weak equivalence if and only if the adjoint map

$$F(X) \longrightarrow Y$$

is a weak equivalence in $\mathcal{D}$.

Theorem 1.2.8. *The geometric realization functor and the singular set functor give a Quillen equivalence*

$$|\cdot| : \mathcal{S}\mathbf{Set} \rightleftarrows \mathbf{CGH} : \mathcal{S}$$

*for the model category structure on $\mathcal{S}$***Set** *where a morphism $f : K \to J$ is*

- *a weak equivalence if $|f| : |K| \to |J|$ is a weak equivalence of topological spaces;*
- *a cofibration if $f_n : K_n \to J_n$ is injective for $n \geqslant 0$, and;*
- *a fibration if f has the RLP with respect to all the inclusions $\Lambda^n_k \subseteq \Delta^n$, for $n \geqslant 1$ and $0 \leqslant k \leqslant n$.*

Proof. See Quillen [12] I.4 and II.3. □

Remark. Let $X \in$ **CGH**, and consider the lifting problem in $\mathcal{S}\mathbf{Set}$ given by

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathcal{S}(X)_\star \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \star \end{array}$$

for some $0 \leqslant k \leqslant n$, which by adjointness is the same as the lifting problem in **CGH**

$$\begin{array}{ccc} |\Lambda^n_k| & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ |\Delta^n| & \longrightarrow & \star \end{array}$$

for which the dotted arrow always exists since $|\Lambda^n_k|$ is a strong deformation retract of $|\Delta^n|$. Hence, the canonical map $\mathcal{S}(X)_\star \longrightarrow \star$ has the RLP with respect all inclusions $\Lambda^n_k \subseteq \Delta^n$, so it is a fibration. Thus, $\mathcal{S}(X)_\star$ is a fibrant simplicial set for all $X \in \mathbf{CGH}$.

Lemma 1.2.9. *For every $0 \leqslant k \leqslant n$, $Hom_{\mathcal{S}\mathbf{Set}}(\Lambda^n_k, K_\star)$ is in bijective correspondence with the set of n-tuples $(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$, $x_i \in K_{n-1}$ for all $i \neq k$, such that $d_i x_j = d_{j-1} x_i$ for all $i < j$ (i, j not equal to k).*

Proof. For any $0 \leqslant i < j \leqslant n$, with i, $j \neq k$, consider the fibre product

$$\begin{array}{ccc} \Delta^{n-1} \times_{\Lambda^n_k} \Delta^{n-1} & \xrightarrow{p_1} & \Delta^{n-1} \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle d^j} \\ \Delta^{n-1} & \xrightarrow{d^i} & \Lambda^n_k \end{array}$$

where d^i, d^j are the induced maps $\Delta^{n-1} \to \Delta^n$, but since i, $j \neq k$ they always lie in Λ^n_k. We get a coequalizer

$$\coprod_{i<j} \Delta^{n-1} \times_{\Lambda^n_k} \Delta^{n-1} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \coprod_{i \neq k} \Delta^{n-1} \longrightarrow \Lambda^n_k.$$

Let us consider now the following commutative diagram

$$\begin{array}{ccc} \mathbf{n-2} & \xrightarrow{d^{j-1}} & \mathbf{n-1} \\ {\scriptstyle d^i}\downarrow & & \downarrow{\scriptstyle d^i} \\ \mathbf{n-1} & \xrightarrow{d^j} & \mathbf{n} \end{array} \tag{1.2}$$

For any maps $f_1, f_2 : \mathbf{m} \to \mathbf{n-1}$ in Δ such that $f_1\, d^i = f_2\, d^j = h$, we have that i, $j \notin \operatorname{im} h$, and hence we can factor h as

$$\begin{array}{ccc} \mathbf{m} & \xrightarrow{h} & \mathbf{n} \\ {\scriptstyle g}\downarrow & \nearrow_{d^j d^i = d^i d^{j-1}} & \\ \mathbf{n-2} & & \end{array}$$

and we get a commutative diagram

$$\begin{array}{ccccc} \mathbf{m} & & \overset{f_1}{\curvearrowright} & & \\ & \searrow^{g} & & & \\ & & \mathbf{n-2} & \xrightarrow{d^{j-1}} & \mathbf{n-1} \\ {\scriptstyle f_2}\curvearrowright & & {\scriptstyle d^i}\downarrow & & \downarrow{\scriptstyle d^i} \\ & & \mathbf{n-1} & \xrightarrow{d^j} & \mathbf{n} \end{array}$$

which means that (1.2) is a pullback in Δ. Thus, we get

$$\Delta^{n-1} \times_{\Lambda^n_k} \Delta^{n-1} \cong \Delta^{n-1} \times_{\Delta^n} \Delta^{n-1} \cong \Delta^{n-2}.$$

and the coequalizer can be rewritten as

$$\coprod_{i<j} \Delta^{n-2} \underset{d^i}{\overset{d^{j-1}}{\rightrightarrows}} \coprod_{i \neq k} \Delta^{n-1} \longrightarrow \Lambda^n_k.$$

□

Corollary 1.2.10 (Kan condition)**.** *A simplicial set $K_\star$ is fibrant if and only if for all $0 \leqslant k \leqslant n$, and any set of n-tuples*

$$(x_0,\dots,x_{k-1},x_{k+1},\dots,x_n),\, x_i \in K_{n-1}, \quad \textit{with} \;\; d_i x_j = d_{j-1}x_i \;\; \forall i<j(i,\, j\neq k),$$

there is some $y \in K_n$ such that $d_i y = x_i$ for all $i \neq k$.

Proof. Assume first that $K_\star$ is fibrant, and we are given a set of n-tuples as in the heading. By the previous lemma we get a map $u : \Lambda^n_k \longrightarrow K_\star$, and since $K_\star$ is fibrant, it can be extended to a map $v : \Delta^n \longrightarrow K_\star$. Taking $y = v_n(\mathrm{id}_\mathbf{n})$ we get that for all $i \neq k$,

$$d_i(y) = d_i(v_n(\mathrm{id}_\mathbf{n})) = v_{n-1}(d_i(\mathrm{id}_\mathbf{n})) = u_{n-1}(d_i(\mathrm{id}_n)) = x_i.$$

On the other hand, assume we are given a lifting problem

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\;p\;} & K_\star \\ \big\downarrow & \nearrow_{q} & \\ \Delta^n & & \end{array}$$

and consider the n-tuple

$$(p_{n-1}(d_1(\mathrm{id}_\mathbf{n})),\dots,p_{n-1}(d_{k-1}(\mathrm{id}_\mathbf{n})),p_{n-1}(d_{k+1}(\mathrm{id}_\mathbf{n})),\dots,p_{n-1}(d_n(\mathrm{id}_\mathbf{n}))) \in (K_{n-1})^n$$

which satisfies

$$d_i p_{n-1} d_j(\mathrm{id}_\mathbf{n}) = d_i d_j p_n(\mathrm{id}_\mathbf{n}) = d_{j-1} d_i p_n(\mathrm{id}_\mathbf{n}) = d_{j-1} p_{n-1} d_i(\mathrm{id}_\mathbf{n}), \quad \forall i<j.$$

Hence, we get some $y \in K_n$ such that $d_i y = x_i$ for all $i \neq j$. The map q is then well defined by sending id_n to $y \in K_n$. □

Example 1.2.11 (Moore)**.** Let $G_\star$ be a simplicial group. Then, its underlying simplicial set is fibrant. To see this let $0 \leqslant k \leqslant n+1$, and let $x_i \in G_n$ for $i \neq k$ such that $d_i x_j = d_{j-1} x_i$ for all $i<j$. We proceed by induction on r, such that there is some $g_r \in G_{n+1}$ with $d_i(g_r) = x_i$ for $i \leqslant r$, $i \neq k$. We set $g_{-1} = 1$, and assume g_{r-1} is given. If $r = k$, then we just set $g_r = g_{r-1}$. In other case, we let $u = x_r^{-1} d_r(g_{r-1})$. Then, for $i<r$, $i \neq k$ we have

$$d_i(u) = d_i(x_r^{-1} d_r(g_{r-1})) = d_i(x_r^{-1}) d_{r-1} d_i(g_{r-1}) = d_i(x_r^{-1}) d_{r-1}(x_i) = d_i(x_r^{-1}) d_i(x_r) = 1,$$

and so $d_i(s_r u) = 1$ too. Therefore, taking $g_r = g_{r-1}\, s_r(u)^{-1}$ we have

$$\begin{aligned} d_i(g_r) &= d_i(g_{r-1}) = x_i, && \text{for } i<r, i\neq k,\\ d_r(g_r) &= d_r(g_{r-1})u^{-1} = x_r, && \text{for } r \neq k, \end{aligned}$$

which completes the induction step.

1.3 Simplicial homotopy groups

Given two simplicial sets $K_\star$, $L_\star$, the product $K_\star \times L_\star$ is the simplicial set given by

$$(K_\star \times L_\star)_n = K_n \times L_n,$$

and for any map $f : \mathbf{n} \to \mathbf{m}$ in Δ,

$$f^\star = f^\star \times f^\star : K_m \times L_m \to K_n \times L_n.$$

Definition. Let $K_\star, L_\star$ be simplicial sets. The *function complex* $\mathbf{Hom}(K,L)_\star$ is the simplicial set given by

$$\mathbf{Hom}(K,L)_n = \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(K_\star \times \Delta^n, L_\star), \quad n \geqslant 0,$$

and for any map $\theta : \mathbf{m} \to \mathbf{n}$ in Δ,

$$\begin{array}{rccc} \theta^\star : & \mathbf{Hom}(K,L)_n & \longrightarrow & \mathbf{Hom}(K,L)_m \\ & (K_\star \times \Delta^n \xrightarrow{f} L_\star) & \longmapsto & (K_\star \times \Delta^m \xrightarrow{\mathrm{id}\times\theta} K_\star \times \Delta^n \xrightarrow{f} L_\star). \end{array}$$

Remark. Let $i : J_\star \hookrightarrow K_\star$ be an inclusion of simplicial sets. Precomposition with i gives rise to a map

$$i^\star : \mathbf{Hom}(K,L)_\star \longrightarrow \mathbf{Hom}(J,L)_\star\,.$$

Let $f, g : K_\star \longrightarrow L_\star$ be maps of simplicial sets. They induce vertices $\widehat{f}, \widehat{g} : \Delta^0 \longrightarrow \mathbf{Hom}(K,L)_\star$ of $\mathbf{Hom}(K,L)_\star$ sending $\mathrm{id}_\mathbf{0}$ to f and g respectively. If their restriction to $J_\star$ is the same, i.e.,

$$u = f|_{J_\star} = g|_{J_\star} : J_\star \longrightarrow L_\star$$

then $i^\star(\widehat{f}) = i^\star(\widehat{g}) = \widehat{u}$, where $\widehat{u} : \Delta^0 \longrightarrow \mathbf{Hom}(J,L)_\star$ is the vertex of $\mathbf{Hom}(J,L)_\star$ that sends any $\mathbf{n} \to \mathbf{0}$ to

$$J_\star \times \Delta^n \xrightarrow{\mathrm{pr}_{J_\star}} J_\star \xrightarrow{u} L_\star\,.$$

where $\mathrm{pr}_{J_\star}$ denotes the projection onto $J_\star$. Thus, $\widehat{f}$ and $\widehat{g}$ are just vertices on the fibre of $\widehat{u}$.

If $K_\star, L_\star$ are simplicial sets, then there is a canonical evaluation map

$$\mathrm{ev} : \quad K_\star \times \mathbf{Hom}(K,L)_\star \longrightarrow L_\star$$

which is given in degree $n \geqslant 0$ by

$$\begin{array}{ccc} K_n \times \mathbf{Hom}(K,L)_n & \longrightarrow & L_n \\ (x,g) & \longmapsto & g_n(x, \mathrm{id}_\mathbf{n}). \end{array}$$

Theorem 1.3.1 (Exponential law)**.** *Let $J_\star, K_\star, L_\star$ be simplicial sets. The function*

$$\begin{array}{rccc} ev_\star : & Hom_{\mathcal{S}\mathbf{Set}}(J_\star, \mathbf{Hom}(K,L)_\star) & \longrightarrow & Hom_{\mathcal{S}\mathbf{Set}}(K_\star \times J_\star, L_\star) \\ & (J_\star \xrightarrow{f} \mathbf{Hom}(K,L)_\star) & \longmapsto & (K_\star \times J_\star \xrightarrow{id\times f} K_\star \times \mathbf{Hom}(K,L)_\star \xrightarrow{ev} L_\star). \end{array}$$

is a bijection which is natural in $J_\star, K_\star, L_\star$.

Proof. Let $g : K_\star \times J_\star \longrightarrow L_\star$ be a map of simplicial sets. For any n-simplex $x \in J_n$, we can define a map $u : K_\star \times \Delta^n \longrightarrow L_\star$, given in degree m by

$$(k, \mathbf{m} \xrightarrow{\theta} \mathbf{n}) \longmapsto g_m(k, \theta^\star(x)) \in L_m$$

This way we can construct the map

$$\begin{array}{rccc} \mathrm{ev}_\star^{-1} : & \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(K_\star \times J_\star, L_\star) & \longrightarrow & \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(J_\star, \mathbf{Hom}(K,L)_\star) \\ & g & \longmapsto & (x \mapsto u) \end{array}$$

which is the inverse of $\mathrm{ev}_\star$. For instance, let $g = \mathrm{ev}(\mathrm{id} \times f)$ for some $f : J_\star \to \mathbf{Hom}(K,L)_\star$. By naturality of f there is a commutative diagram

$$\begin{array}{ccc} J_n & \xrightarrow{\theta^\star} & J_m \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_m} \\ \mathbf{Hom}(K,L)_n & \xrightarrow{\theta^\star} & \mathbf{Hom}(K,L)_m \end{array}$$

and thus,

$$f_n(x)\,(\mathrm{id}_{K_\star} \times \theta) = f_m(\theta^\star(x))$$

for any $x \in J_n$ and $\theta \in (\Delta^n)_m$. Hence, for any $x \in J_n$, $(k, \theta) \in K_m \times (\Delta^n)_m$ we have

$$\begin{aligned} ((\mathrm{ev}_\star^{-1}(g))_n(x))_m(k,\theta) &= g_m(k,\theta^\star(x)) = \mathrm{ev}_m(\mathrm{id}_{K_m} \times f_m)(k,\theta^\star(x)) = \mathrm{ev}_m(k, f_m(\theta^\star(x))) \\ &= (f_m(\theta^\star(x)))_m(k,\mathrm{id}_{\mathbf{m}}) = (f_n(x)\,(\mathrm{id}_{K_\star} \times \theta))_m(k,\mathrm{id}_{\mathbf{m}}) \\ &= (f_n(x))_m(k,\theta), \end{aligned}$$

so in fact $\mathrm{ev}_\star^{-1}(g) = f$. On the other hand, let $g \in \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(K_\star \times J_\star, L_\star)$, let us call $h = \mathrm{ev}_\star^{-1}(g)$, and for any $(x,k) \in K_n \times J_n$

$$\begin{array}{ccccc} K_n \times J_n & \xrightarrow{\mathrm{id}\times h_n} & K_n \times \mathbf{Hom}(K,L)_n & \xrightarrow{\mathrm{ev}_n} & L_n \\ (k,x) & \longmapsto & (k, h_n(x)) & \longmapsto & (h_n(x))_n(k,\mathrm{id}_{\mathbf{n}}) = g_n(k, \mathrm{id}_{\mathbf{n}}^\star(x)) = g_n(k,x). \end{array}$$

This composition is precisely $\mathrm{ev}_\star(h)_n(k,x)$, and therefore $\mathrm{ev}_\star(h) = g$. □

Proposition 1.3.2. *Let $i : K_\star \hookrightarrow L_\star$ be an inclusion of simplicial sets and let $p : X_\star \to Y_\star$ be a fibration. Then the map*

$$\mathbf{Hom}(L,X)_\star \xrightarrow{(i^\star, p_\star)} \mathbf{Hom}(K,X)_\star \times_{\mathbf{Hom}(K,Y)_\star} \mathbf{Hom}(L,Y)_\star\,,$$

induced by the diagram

$$\begin{array}{ccc} \mathbf{Hom}(L,X)_\star & \xrightarrow{p_\star} & \mathbf{Hom}(L,Y)_\star \\ {\scriptstyle i^\star}\downarrow & & \downarrow{\scriptstyle i^\star} \\ \mathbf{Hom}(K,X)_\star & \xrightarrow{p_\star} & \mathbf{Hom}(K,Y)_\star\,, \end{array}$$

is a fibration. Moreover, it is a weak equivalence if either i or p is a weak equivalence.

Proof. See Goerss and Jardine [3] I, Proposition 5.2. □

Remark. For the "model category" reader, this property is just saying that $\mathcal{S}$**Set** is a closed simplicial model category.

Corollary 1.3.3. *If $X_\star$ is a fibrant simplicial set, and $i : K_\star \hookrightarrow L_\star$ is an inclusion of simplicial sets, then the induced map*

$$i^\star : \mathbf{Hom}(L,X)_\star \longrightarrow \mathbf{Hom}(K,X)_\star$$

is a fibration.

Proof. We just apply the last proposition to the fibration $X_\star \longrightarrow \star$. □

Recall that in **Top** (or **CGH**) two continuous maps $f,g : X \to Y$ are homotopic if there is a continuous map $H : X \times |\Delta^1| \to Y$ such that the following diagram commutes

$$\begin{array}{ccccc} X \cong X \times |\Delta^0| & \xrightarrow{\mathrm{id}\times(d^0)_\star} & X \times |\Delta^1| & \xleftarrow{\mathrm{id}\times(d^1)_\star} & X \times |\Delta^0| \cong X. \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle H} & \swarrow{\scriptstyle g} & \\ & & Y & & \end{array}$$

Definition. Let $f,\, g : K_\star \longrightarrow L_\star$ be maps of simplicial sets. A *simplicial homotopy* from f to g is a map of simplicial sets $h : K_\star \times \Delta^1 \to L_\star$ such that the following commutes

$$\begin{array}{ccccc} K_\star \cong K_\star \times \Delta^0 & \xrightarrow{\mathrm{id}\times d^1} & K_\star \times \Delta^1 & \xleftarrow{\mathrm{id}\times d^0} & K_\star \times \Delta^0 \cong K_\star \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle g} & \\ & & L_\star & & \end{array}$$

The maps $f,\, g$ are said *(simplicially) homotopic* and we write $f \simeq g$. Moreover, if $i : J_\star \hookrightarrow K_\star$ denotes the inclusion of a subcomplex $J_\star$ of $K_\star$, and $f|_{J_\star} = g|_{J_\star}$, a *simplicial homotopy* from f to g (*rel J*) is a simplicial homotopy from f to g, $h : K_\star \times \Delta^1 \longrightarrow L_\star$, such that the following diagram

$$\begin{array}{ccc} J_\star \times \Delta^1 & \xrightarrow{\mathrm{pr}_{J_\star}} & J_\star \\ {\scriptstyle i\times\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle g|_{J_\star}=f|_{J_\star}} \\ K_\star \times \Delta^1 & \xrightarrow{h} & L_\star \end{array}$$

commutes. In this case we will also write $f \simeq g$ (rel $J_\star$). A homotopy that can be factored as

$$K_\star \times \Delta^1 \xrightarrow{\mathrm{pr}_{K_\star}} K_\star \xrightarrow{k} L_\star$$

for some map k is called a *constant homotopy* (at k).

Remark. A homotopy $h : K_\star \times \Delta^1 \longrightarrow L_\star$ is just a 1-simplex of $\mathbf{Hom}(K,L)_\star$. Moreover, if $f \simeq q : K_\star \longrightarrow L_\star$ (rel $J_\star$) via h, then calling $k = f|_{J_\star} = g|_{J_\star} : J_\star \to L_\star$ there is a commutative diagram

$$\begin{array}{ccc} J_\star \times \Delta^1 & \xrightarrow{\mathrm{pr}_{J_\star}} & J_\star \\ {\scriptstyle i\times\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle k} \\ K_\star \times \Delta^1 & \xrightarrow{h} & L_\star \end{array}$$

and $(i^{\star})_1(h)$ is the 1-simplex of $\mathbf{Hom}(J,L)_\star$ which is a constant homotopy at k. Moreover, by the exponential law (Theoreom 1.3.1), there is a bijection

$$\mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(\Delta^1, \mathbf{Hom}(K,L)_\star) \cong \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(K_\star \times \Delta^1, L_\star).$$

Recall a previous remark where we obtained vertices $\widehat{f}$, $\widehat{g}$ of the function complex $\mathbf{Hom}(K,L)_\star$ out of maps $f,g : K_\star \longrightarrow L_\star$. This bijection is telling us that homotopies between maps f,g are just homotopies between vertices $\widehat{f}$, $\widehat{g}$.

Example 1.3.4. For any $n \geqslant 0$, the simplicial set Δ^n is homotopy equivalent to Δ^0, that is, there are maps $f : \Delta^n \to \Delta^0$ and $g : \Delta^0 \to \Delta^n$ such that $fg \simeq \mathrm{id}_{\Delta^0}$, and $gf \simeq \mathrm{id}_{\Delta^n}$. To see this, let $g_m(\mathbf{m} \to \mathbf{0}) = \mathbf{m} \to \mathbf{0} \xrightarrow{i} \mathbf{n}$, where i sends $\mathbf{0}$ to $n \in \mathbf{n}$, and $\mathbf{m} \to \mathbf{0}$ is the only m-simplex in Δ^0. f is the only map $\Delta^n \to \Delta^0$. Then, $fg = \mathrm{id}_{\Delta^0}$. Let $h : \Delta^n \times \Delta^1 \to \Delta^n$ be given for any $m \geqslant 0$ by

$$\begin{array}{rccc} h_m : & (\Delta^n)_m \times (\Delta^1)_m & \longrightarrow & (\Delta^n)_m \\ & (\alpha, \beta) & \longmapsto & \left(i \mapsto \begin{cases} \alpha(i), & \text{if } \beta(i) = 0, \\ m, & \text{if } \beta(i) = 1 \end{cases} \right). \end{array}$$

Then, $h(\mathrm{id} \times d^1) = \mathrm{id}_{\Delta^n}$ and $h(\mathrm{id} \times d^0) = gf$.

Lemma 1.3.5. *If $K_\star$ is a fibrant simplicial set, then simplicial homotopy is an equivalence relation on the vertices $\Delta^0 \to K$ of $K_\star$.*

Proof. For an n-simplex σ, we denote its boundary by $\partial\sigma = (d_0\sigma, \dots, d_n\sigma)$. Let $x, y : \Delta^0 \to K_\star$ be two vertices. A homotopy from x to y is just a 1-simplex $v \in K_1$ such that $\partial v = (y,x)$. If we take the 1-simplex s_0x, then $\partial(s_0x) = (x,x)$, so $x \simeq x$ and the relation is reflexive. If $x \simeq y$, then there is a 1-simplex v_2 such that $\partial v_2 = (y,x)$. Take $v_1 = s_0x$ such that $d_1v_1 = x = d_1v_2$. By Lemma 1.2.9 this defines a map $u : \Lambda^2_0 \to K_\star$, and we have

$$\begin{array}{ccc} \Lambda^2_0 & \xrightarrow{u} & K_\star \\ \big\downarrow & \overset{v}{\nearrow} & \big\downarrow \\ \Delta^2 & \longrightarrow & \star \end{array}$$

where the dotted arrow v exists since $K_\star$ is fibrant. Moreover, we have

$$d_0 d_0 v = d_0 d_1 v = x, \quad d_1 d_0 v = d_0 d_2 v = y,$$

so the 1-simplex d_0v satisfies $\partial(d_0v) = (x,y)$, so $y \simeq x$ and the relation is symmetric. Finally, let $z : \Delta^0 \to K_\star$ be another vertex such that $x \simeq y$ via $v_2 \in K_1$, and $y \simeq z$ via $v_0 \in K_1$. Then, $d_0v_2 = y = d_1v_0$, so it defines a map $u' : \Lambda^2_1 \to K_\star$, and there is a map v' making the following diagram commutative

$$\begin{array}{ccc} \Lambda^2_1 & \xrightarrow{u'} & K_\star \\ \big\downarrow & \overset{v'}{\nearrow} & \big\downarrow \\ \Delta^2 & \longrightarrow & \star \end{array}$$

since K is fibrant. Moreover, v' satisfies

$$d_0 d_1 v' = d_0 d_0 v' = z, \quad d_1 d_1 v' = d_1 d_2 v' = x$$

so taking the 1-simplex d_1v', we have $\partial(d_1v') = (z,x)$, so $x \simeq z$ and the relation is also transitive. $\square$

Definition. For a fibrant simpliciat set $K_\star$ we define $\pi_0\, K_\star$ to be the set of homotopy classes of vertices of $K_\star$. For any vertex $k \in K_0$, we denote by $\pi_0(K_\star, k)$ the pointed set $\pi_0\, K_\star$ with basepoint the homotopy class $[k]$ of k.

Example 1.3.6 (Δ^n is not fibrant)**.** Let $n \geqslant 1$, and take the vertices ι_0, $\iota_1 : \Delta^0 \to \Delta^n$, where for any $m \geqslant 0$,

$$\iota_0(\mathbf{m} \to \mathbf{0}) = \underset{i \mapsto 0}{\mathbf{m} \to \mathbf{n}} \qquad \iota_1(\mathbf{m} \to \mathbf{0}) = \underset{i \mapsto 1}{\mathbf{m} \to \mathbf{n}}.$$

Now, we can consider the map $\widehat{u} : \mathbf{1} \to \mathbf{n}$ sending $0 \mapsto 0$, and $1 \mapsto 1$, and the induced 1-simplex $u : \Delta^1 \to \Delta^n$ given by composition. Then $d_0\, u = \iota_1$, and $d_1\, u = \iota_0$, so $\iota_0 \simeq \iota_1$. But in order to have $\iota_1 \simeq \iota_0$ we need a 1-simplex $v : \Delta^1 \to \Delta^n$, which we can see as a map $\widehat{v} : \mathbf{1} \to \mathbf{n}$ such that $d^0\beta = \iota_0$ and $d^1\beta = \iota_1$. But this means that $\widehat{v} d_0 : \mathbf{0} \to \mathbf{n}$ sends $0 \mapsto 0$ and $\widehat{v} d_1 : \mathbf{0} \to \mathbf{n}$ sends $0 \mapsto 1$, i.e., $\widehat{v}(0) = 1$, and $\widehat{v}(1) = 0$, which cannot happen since $\widehat{v}$ is a map in Δ. Hence, $\iota_0 \not\simeq \iota_1$. By Lemma 1.3.5, it follows that Δ^n is not a fibrant simplicial set.

Proposition 1.3.7. *The functor $\mathcal{S} : \mathbf{Top} \longrightarrow \mathcal{S}\mathbf{Set}$ preserves homotopy.*

Proof. Let $f, g : X \to Y$ be homotopic continuous maps between topological spaces such that $H : X \times\ [0,1] \to Y$ is a homotopy from f to g. There is a canonical map of simplicial sets $u : \Delta^1 \to \mathcal{S}(|\Delta^1|)_\star$ given by $(f : \mathbf{n} \to \mathbf{1}) \mapsto (f_\star : |\Delta^n| \to |\Delta^1|)$. Let us call by h the composite

$$\begin{array}{ccccc}
\mathcal{S}(X)_\star \times \mathcal{S}(|\Delta^1|)_\star & \xrightarrow{\mathrm{id}\times\iota_\star} & \mathcal{S}(X)_\star \times \mathcal{S}([0,1])_\star & \xrightarrow{\cong} & \mathcal{S}(X \times [0,1])_\star \\
{\scriptstyle \mathrm{id}\times u}\uparrow & & & & \downarrow{\scriptstyle H_\star} \\
\mathcal{S}(X)_\star \times \Delta^1 & & \overset{h}{\dashrightarrow} & & \mathcal{S}(Y)_\star.
\end{array}$$

The third map is an isomorphism since any map $\sigma : |\Delta^n| \to X \times [0,1]$ corresponds to a pair of continuous maps $|\Delta^n| \to X$, and $|\Delta^n| \to [0,1]$. For $i = 0,1$, the image of $(\sigma, 0_n) \in \mathcal{S}(X)_n \times (\Delta^0)_n$ under the composition

$$\mathcal{S}(X)_\star \times \Delta^0 \xrightarrow{\mathrm{id}\times d^i} \mathcal{S}(X)_\star \times \Delta^1 \xrightarrow{\mathrm{id}\times u} \mathcal{S}(X)_\star \times \mathcal{S}(|\Delta^1|)_\star \xrightarrow{\mathrm{id}\times\iota_\star} \mathcal{S}(X)_\star \times \mathcal{S}([0,1])_\star$$

is $(\sigma, \sigma_{1-i}) \in \mathcal{S}(X)_n \times \mathcal{S}([0,1])_n$ where $\sigma_{1-i} : \Delta^n \to [0,1]$ has constant value $1-i$. Hence, $h\,(\mathrm{id} \times d^1) = (H|_{X\times\{0\}})_\star = f_\star$ and $h\,(\mathrm{id} \times d^0) = (H|_{X\times\{1\}})_\star = g_\star$. So h is a simplicial homotopy from $f_\star$ to $g_\star$.

□

This shows that our definition of homotopy for simplicial sets "makes sense", since it "agrees" with the topological notion of homotopy. At this point, one may be tempted to define homotopy groups in $\mathcal{S}$**Set** following its construction in **Top** (or more precisely in **CGH**), which we could attempt grosso modo by setting $\pi_n(K)$ to be the set of maps $\sigma : \Delta^n \to K$ modulo the relation $\simeq$ (rel $\partial\Delta^n$). But in order to do so, we need the simplicial homotopy relation $\simeq$ to be an equivalence relation, and this is not necessarily true, since it fails in general to be symmetric and transitive. Nevertheless, for fibrant simplicial sets, $\simeq$ is an equivalence relation. Recall that in fact $\mathcal{S}(X)$ is fibrant for all $X \in$ **CGH**.

Proposition 1.3.8. *Let $L_\star$ be a fibrant simplicial set and let $J_\star \subseteq K_\star$ be an inclusion of simplicial sets, then*

(a) the homotopy relation $\simeq$ is an equivalence relation in $Hom_{\mathcal{S}\mathbf{Set}}(K_\star, L_\star)$, and

(b) the homotopy relative relation $\simeq$ (rel $J_\star$) is an equivalence relation in $Hom_{\mathcal{S}\mathbf{Set}}(K_\star, L_\star)$.

Proof. Part (a) is a particular case of part (b) taking $J_\star = \emptyset$. Therefore, we only need to prove (b). We have seen that homotopy of maps $f, g : K_\star \to L_\star$ are just homotopy of vertices $\widehat{f}, \widehat{g}$ of $\mathbf{Hom}(K,L)_\star$, and in this case they are also on the fibre of the vertex $\widehat{u}$, with $u = f|_{J_\star} = g|_{J_\star}$ in the map

$$i^\star : \mathbf{Hom}(K,L)_\star \longrightarrow \mathbf{Hom}(J,L)_\star.$$

By Corollary 1.3.3 the map $i^\star$ is a fibration. Calling $X_\star$ the fibre of $\widehat{u}$ in $i^\star$, we get a pullback diagram

$$\begin{array}{ccc} X_\star & \longrightarrow & \mathbf{Hom}(K,L)_\star \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i^\star} \\ \widehat{u} & \hookrightarrow & \mathbf{Hom}(J,L)_\star \end{array}$$

Hence by Corollary 1.1.1, i' is also a fibration, so $X_\star$ is a fibrant simplicial set. By Lemma 1.3.5 simplicial homotopy is an equivalence relation on the vertices of $X_\star$. □

This allows us to define homotopy groups for simplicial sets.

Definition. Let $K_\star$ be a fibrant simplicial set, and let $k \in K_0$ be a vertex in $K_\star$. For $n \geqslant 1$ we define

$$\pi_n K_\star = \pi_n(K_\star, k) = [(\Delta^n, \partial\Delta^n), (K_\star, k)]_\simeq$$

the set of homotopy classes (rel $\partial\Delta^n$) of simplicial maps $f : \Delta^n \to K_\star$ such that

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^0 \\ \downarrow & & \downarrow{\scriptstyle k} \\ \Delta^n & \xrightarrow{f} & K_\star \end{array}$$

commute. Moreover, we also denote the composition

$$\Delta^n \longrightarrow \Delta^0 \xrightarrow{k} K_\star$$

by k, and we call it the *constant map* at k. We write $[f]$ for the equivalence class in $\pi_n K_\star$ of f.

Theorem 1.3.9. *$\pi_n(K_\star, k)$ has a group structure for $n \geqslant 0$, which is abelian for $n \geqslant 2$. It is called the n-th homotopy group of $K_\star$. Moreover, the neutral element of the group is the homotopy class $[k]$.*

Proof. See Goerss [3] I, Theorem 7.2. □

Remark. As one might guess from the construction, these homotopy groups are isomorphic to the homotopy groups over topological spaces, in the sense that there are natural isomorphisms

$$\begin{aligned} \pi_0(K_\star, k) &\cong \pi_0(|K_\star|, |k|), \\ \pi_n(K_\star, k) &\cong \pi_n(|K_\star|, |k|), \quad \text{for } n \geqslant 1. \end{aligned}$$

For a complete proof one can see Lemma 3.4.2 and Proposition 3.6.3 in Hovey [6].

1.4 The Dold-Kan correspondence

The main goal of this section is to show that for every abelian category $\mathcal{A}$, the category of non-negative chain complexes in $\mathcal{A}$, $\mathbf{Ch}_{\geqslant 0}(\mathcal{A})$, is equivalent to $\mathcal{SA}$. Moreover, this equivalence will preserve homotopy. In the next chapter we will see a nice application regarding resolution of objects. First we need to define homotopy on simplicial objects in an abelian category.

Definition. Let $A_\star$ be a simplicial object in an abelian category $\mathcal{A}$. The *unnormalized chain complex* associated to $A_\star$ is a chain complex $C_\bullet = C(A_\star)$ with $C_n = A_n$ as n-chains and with boundary map $\partial : C_n \to C_{n-1}$ given by

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}.$$

This definition makes sense since $\partial^2 = 0$ (as a direct consequence of the simplicial identities satisfied by the d_i's) so C is in fact a chain complex. Moreover, this defines a functor from $\mathcal{SA}$ to $\mathbf{Ch}_{\geqslant 0}(\mathcal{A})$.

Definition. Let $A_\star$ be a simplicial object in an abelian category $\mathcal{A}$. We define

$$\pi_n(A_\star) = H_n(C(A_\star)) \quad \text{for } n \geqslant 0.$$

This being done, we need now to refine the unnormalised chain complex $C(A_\star)$ into a new chain complex whose homology will be naturally isomorphic to the homology of $C(A_\star)$ but that will behave nicer with chain homotopic maps. Notice that we have only used the face maps in $A_\star$ to define the chain complex $C(A_\star)$. Hence, if we are given a simplicial object $A_\star$ in $\mathcal{A}$ and we "forget" about the degeneracy maps we are still able to compute the chain complex $C(A_\star)$. This motivates the following definition.

Definition. Let Δ_s denote the category whose objects are the objects in Δ and whose morphisms are order-preserving injective set functions. A *semisimplicial object* in a category $\mathcal{A}$ is a functor $A_\star : \Delta_s^{op} \longrightarrow \mathcal{A}$. The category of semisimplicial objects in a category $\mathcal{A}$ will be denoted by $s\mathcal{SA}$.

Remark. The characterization of a simplicial object given in Proposition 1.2.2 tells us that a semisimplicial object is just a simplicial object with no degeneracy maps and therefore the only simplicial identities that face maps d_i satisfy now are $d_i d_j = d_{j-1} d_i$ for $i < j$.

We define the forgetful functor

$$F : \mathcal{SA} \longrightarrow s\mathcal{SA}$$

that makes any simplicial object into a semisimplicial object by forgetting degeneracies. This functor has a left adjoint

$$G : s\mathcal{SA} \longrightarrow \mathcal{SA}$$

when the category $\mathcal{A}$ has finite coproducts, that is defined as follows. For any $B_\star \in s\mathcal{A}$, we set

$$GB_n = \coprod_{f:\mathbf{n} \twoheadrightarrow \mathbf{k}} B_k[f],$$

where the coproduct runs through all possible surjections $f : \mathbf{n} \twoheadrightarrow \mathbf{k}$ in Δ, and $B_k[f]$ denotes a copy of B_k. Also, for any morphism $g : \mathbf{n} \to \mathbf{m}$ in Δ, we define the map $G(g) : GB_m \to GB_n$ by defining its restrictions to each of the components of GB_m. We do it as follows: let $B_k[f]$ be one of them for a surjection $f : \mathbf{m} \twoheadrightarrow \mathbf{k}$. The map fg factors as

$$\begin{array}{ccc} \mathbf{n} & \xrightarrow{g} & \mathbf{m} \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{q} & \xhookrightarrow{d} & \mathbf{k} \end{array}$$

with s surjective and d injective (as seen in the proof of Proposition 1.2.2). Then, the restriction of $G(g)$ to $B_k[f]$ is the map $B(d) : B_k \to B_q = B_q[s] \subseteq GB_n$. This makes GB into a simplicial object of $\mathcal{A}$: for $\mathrm{id}_{\mathbf{n}} : \mathbf{n} \to \mathbf{n}$ identity map, then $G(\mathrm{id}_{\mathbf{n}}) : GB_n \to GB_n$ is clearly the identity map in GB_n. On the other hand, for $g_1 : \mathbf{n} \to \mathbf{n}'$, $g_2 : \mathbf{n}' \to \mathbf{m}$ maps in Δ, let

$$\mathbf{n}' \xrightarrow{p} \mathbf{q} \xhookrightarrow{s} \mathbf{k}$$

be the epi-monic factorization of fg_2 and let

$$\mathbf{n} \xrightarrow{p'} \mathbf{q}' \xhookrightarrow{s'} \mathbf{q}$$

be the epi-monic factorization of pg_1. We get a commutative diagram

$$\begin{array}{ccccc} \mathbf{n} & \xrightarrow{g_1} & \mathbf{n}' & \xrightarrow{g_2} & \mathbf{m} \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ \mathbf{q}' & \xhookrightarrow{s'} & \mathbf{q} & \xhookrightarrow{s} & \mathbf{k} \end{array}$$

and

$$\mathbf{n} \xrightarrow{p'} \mathbf{q}' \xhookrightarrow{ss'} \mathbf{k}$$

is the epi-monic factorization of $f\, g_2\, g_1$. Commutativity of the diagram means that

$$G(g_2\, g_1) = G(g_2)\, G(g_1).$$

Lemma 1.4.1. *G is left adjoint to F,*

$$G : s\mathcal{SA} \rightleftarrows \mathcal{SA} : F.$$

Proof. Let $A_\star \in \mathcal{SA}$, and $B_\star \in s\mathcal{SA}$. We have to show

$$\mathrm{Hom}_{\mathcal{SA}}(G(B_\star), A_\star) \cong \mathrm{Hom}_{s\mathcal{SA}}(B_\star, F(A_\star)).$$

If $k < n$ and $f : \mathbf{n} \twoheadrightarrow \mathbf{k}$ is a surjection in Δ, it can be factorized as

$$\mathbf{n} \xrightarrow{s^i} \mathbf{n-1} \xrightarrow{g} \mathbf{k}, \qquad f = g\, s^i$$

for some s^i and g. We get a commutative diagram

$$\begin{array}{ccc} \mathbf{n} & \xrightarrow{s^i} & \mathbf{n-1} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \mathbf{k} & \xhookrightarrow{\mathrm{id}} & \mathbf{k} \end{array}$$

and hence the map $s_i : G(B)_{n-1} \to G(B)_n$ identifies the factor $B_k[g]$ of $G(B)_{n-1}$ with $B_k[f]$ of $G(B)_n$, and we write $B_k[f] = s_i B_k[g]$. Now, if $n-1 = k$, then $g = \mathrm{id}_{\mathbf{k}}$, and if $n-1 < k$ we can repeat this procedure. It follows that any factor $B_k[f]$ of $G(B)_n$, for $f : \mathbf{n} \twoheadrightarrow \mathbf{k}$ with $f \neq \mathrm{id}$, is of the form $(s_{i_\ell} \cdots s_{i_1} B_k)[\mathrm{id}_{\mathbf{k}}]$ where $f = s^{i_1} \cdots s^{i_\ell}$. Therefore, in order to define a morphism $h_n : G(B)_n \to A_n$, we only need to specify the image of $B_n[\mathrm{id}_{\mathbf{n}}]$ for all $n \geqslant 0$. Hence, any morphism $h : B_\star \to F(A_\star)$ defines a map $\widehat{h} : G(B_\star) \to A_\star$ where the restriction of $\widehat{h}_n$ to $B_n[\mathrm{id}_{\mathbf{n}}]$ is just $h_n : B_n \to A_n$. Moreover, any morphism $\widehat{h} : G(B_\star) \to A_\star$ defines a map $h : B_\star \to F(A_\star)$ in a natural way, given by $h_n : B_n \to A_n$ to be just $\widehat{h}_n|_{B_n[\mathrm{id}_{\mathbf{n}}]}$. These maps are well-defined and they are clearly inverse of each other. □

For a simplicial object $A_\star$ in an abelian category $\mathcal{A}$ we denote by $D = D(A_\star)$ be the subcomplex of $C(A_\star)$ such that $D(A_\star)_n$ is generated by the images of the degeneracy maps $s_i : A_{n-1} \to A_n$ for $i = 0, \ldots, n-1$.

Proposition 1.4.2. *$D(A_\star)$ is a chain subcomplex of the unnormalized chain complex $C(A_\star)$.*

Proof. The only thing we need to check is that for any $a \in D_n$, $\partial(a)$ is in fact an element in D_{n-1}, i.e., it can be written as a sum of elements in the image of the degeneracy maps $s_j : A_{n-2} \to A_{n-1}$. But this is clear if we use the simplicial identities, since for $s_j : A_{n-1} \to A_n$, we have

$$\begin{aligned}
\partial s_j &= \sum_{i=0}^{n} (-1)^i d_i s_j \\
&= \sum_{i=0}^{j-1} (-1)^i s_{j-1} d_i \;+\; (-1)^j d_j s_j \;+\; (-1)^{j+1} d_{j+1} s_j \;+\; \sum_{i=j+2}^{n} (-1)^i s_j d_{i-1} \\
&= \sum_{i=0}^{j-1} (-1)^i s_{j-1} d_i + \sum_{i=j+2}^{n} (-1)^i s_j d_{i-1}
\end{aligned}$$

□

The singularity of this chain complex is stated in the following lemma.

Lemma 1.4.3. *$D(A_\star)$ is an acyclic chain complex, that is,*

$$H_n(D(A_\star)) = 0 \quad \textit{for all } n \geqslant 0.$$

Proof. Let us consider the following filtration of $D = D(A_\star)$ [3]

$$F_0 D_n = 0, \quad F_p D_n = s_0(A_{n-1}) + \ldots + s_p(A_{n-1}),\ n \leqslant p, \quad F_n D_n = D_n.$$

Using the computation of the last proposition we see that ∂s_j can be written in terms of s_{j-1} and s_j and therefore $F_p D_n$ is indeed a subcomplex of D_n. This filtration is bounded, so by the Classical Convergence Theorem [4], we get a bounded spectral sequence that converges to $H_\star(D)$:

$$E^1_{pq} = H_{p+q}(F_p D / F_{p-1} D) \Longrightarrow H_{p+q}(D).$$

Hence, we just need to show that the complexes $F_p D_n / F_{p-1} D_n$ are acyclic. Note that $F_{n-1} D_n = F_n D_n = D_n$, so if $n \leqslant p$ the quotient is zero. For $n > p$, let us see that $(-1)^p s_p : A_{n-1} \to A_n$

[3] For a proof avoiding the use of spectral sequences see Jardine and Goerss [3] III. Theorem 2.4.
[4] Weibel [17] Theorem 5.5.1.

induces a chain homotopy from the identity map to 0 in $F_pD_n/F_{p-1}D_n$, i.e., that in F_pD_n we have

$$(\partial s_p + s_p\partial)s_p \equiv (-1)^p s_p \mod F_{p-1}D_n.$$

Using again the computation from the last result, we see that

$$\partial s_p \equiv \sum_{i=p+2}^{n} (-1)^i s_p d_{i-1} \mod F_{p-1}D_n,$$

and therefore modulo $F_{p-1}D_n$ we have

$$\begin{aligned}
\partial s_p^2 + s_p\partial s_p &\equiv \sum_{i=p+2}^{n+1}(-1)^i s_p d_{i-1} s_p + \sum_{i=p+2}^{n}(-1)^i s_p^2 d_{i-1} \\
&= (-1)^{p+2} s_p + \sum_{i=p+3}^{n+1}(-1)^i s_p^2 d_{i-2} + \sum_{i=p+2}^{n}(-1)^i s_p^2 d_{i-1} = (-1)^p s_p.
\end{aligned}$$

Hence the identity map in $F_pD_n/F_{p-1}D_n$ is null homotopic, so $H_m(F_pD_n/F_{p-1}D_n) = 0$ for all $m \geqslant 0$. □

Due to this result, it makes sense to consider the complex $C(A_\star)/D(A_\star)$ since it will have the same homology as the unnormalized chain complex $C(A_\star)$. We will call this resulting complex $N(A_\star)$, and it can be expressed as follows.

Definition. Let $A_\star$ be a simplicial object in an abelian category $\mathcal{A}$. The *normalized chain complex* is a chain complex $N_\bullet = N(A_\star)$ with

$$N_n = \bigcap_{i=0}^{n-1} \ker(d_i : A_n \longrightarrow A_{n-1})$$

as n-chains and with boundary map ∂ given by

$$\partial = (-1)^n d_n : N_n \longrightarrow N_{n-1}.$$

Note that $N(A_\star)$ is in fact a subcomplex of $C(A_\star)$ since the boundary map ∂^N in N is just the restriction of the boundary map in C to N, since $\partial(a) = (-1)^n d_n(a) = \partial^N(a)$ for every $a \in N_n$. This also justifies the abuse of notation we make here by calling both boundary maps in C and N by the same name. Moreover, this defines a functor

$$N : \mathcal{SA} \longrightarrow \mathbf{Ch}_{\geqslant 0}(\mathcal{A}),$$

where face maps in the simplicial objects of $\mathcal{A}$ are not needed, so it can also be seen as a functor

$$N : s\mathcal{SA} \longrightarrow \mathbf{Ch}_{\geqslant 0}(\mathcal{A}).$$

Proposition 1.4.4. *Let $A_\star$ be a simplicial object in an abelian category $\mathcal{A}$. Then*

$$C(A_\star) = N(A_\star) \oplus D(A_\star).$$

Proof. For $n \geqslant 0$, we will show that the natural map induced by the inclusions

$$\varphi : N_n \oplus D_n \longrightarrow A_n$$

is an isomorphism of chain complexes. For $n = 0$, $D_0 \cong 0$, and $N_0 \cong A_0$. Let $n > 0$, and $y \in N_n \cap D_n$, so that $y = \sum_{j=0}^{n-1} s_j(y_j)$ for some $y_i \in A_{n-1}$. Let i be the smallest integer such that $s_i(y_i) \neq 0$, we claim that $d_i(y) = y_i$. If $n = 1$, then $y = s_0(y_0) + s_1(y_0)$ and

$$\begin{cases} d_0(y) = d_0 s_0(y_0) + d_0 s_1(y_1) = y_0 & \text{if } s_0(y_0) \neq 0, \\ d_1(y) = d_1 s_1(y_1) = y_1 & \text{if } s_0(y_0) = 0. \end{cases}$$

For $n > 1$, we can rewrite the sum so that $d_i s_j(y_j) = 0$ for $i < j$. To see this, notice that $d_i s_i = \mathrm{id}_{A_{n-2}}$ means that we can write

$$A_{n-1} \cong \ker(d_i) \oplus \mathrm{im}(s_i).$$

Thus, we write $y_j = a_j + s_i(b_j)$ for $a_j \in \ker(d_i)$, $b_j \in A_{n-2}$ and for any $j > i$ we have

$$s_j(y_j) = s_j(a_j) + s_j(s_i(b_j)) = s_j(a_j) + s_i(s_{j-1}(b_j)),$$

and so

$$y = \sum_{j=i}^{n-1} s_j(y_j) = s_i\left(y_i + \sum_{j=i+1}^{n-1} s_{j-1}(b_j)\right) + \sum_{j=i+1}^{n-1} s_j(a_j)$$

where in fact $d_i s_j(a_j) = s_{j-1} d_i(a_j) = 0$ for $i < j$. With this remark the claim follows,

$$d_i(y) = \sum_{j=i}^{n-1} d_i s_j(y_j) = d_i s_i(y_i) + \sum_{j=i+1}^{n-1} d_i s_j(y_j) = y_i.$$

Hence, for $i < n$, $d_i(y) = y_i \neq 0$, so y is not in the kernel of d_i, contradicting $y \in N_n$. This shows that $N_n \cap D_n = 0$, and hence φ is injective. Let

$$N_j A_n = \bigcap_{i=0}^{j} \ker(d_i) \subseteq A_n.$$

Clearly $N_{n-1}A_n = N_n \subseteq \mathrm{im}\,\varphi$. We now proceed by downward induction on j to show that

$$N_j A_n \subseteq \mathrm{im}\,\varphi \quad \text{ for all } j = n-1, \ldots, 0.$$

Assume $N_j A_n \subseteq \mathrm{im}\,\varphi$ and let $y \in N_{j-1}A_n$. Then $y' = y - s_j d_j(y)$ satisfies

$$d_j(y') = d_j(y) - d_j(y) = 0, \quad d_i(y') = d_i(y) - s_{j-1} d_{j-1} d_i(y) = 0 \text{ for } i < j.$$

So $y' \in N_j \subseteq \mathrm{im}\,\varphi$. Recall that $s_j d_j(y) \in \mathrm{im}\, s_j \subseteq D_n \subseteq \mathrm{im}\,\varphi$ too, so $y = y' + s_j d_j(y) \in \mathrm{im}\,\varphi$. So the induction works, and we get $N_0 A_n = \ker(d_0) \subseteq \mathrm{im}\,\varphi$. Finally, using again the decomposition $A_n \cong \ker(d_0) \oplus \mathrm{im}(d_0)$, we see that $A_n \subseteq \mathrm{im}\,\varphi$.

□

Corollary 1.4.5. *Let A be a simplicial object in an abelian category $\mathcal{A}$. Then, there is a natural isomorphism*

$$\pi_\star(A_\star) = H_\star(C(A_\star)) \cong H_\star(N(A_\star)).$$

Proof. Direct consequence of the last proposition and Lemma 1.4.3. □

In Section 1.3 we defined simplicial homotopies for simplicial sets so that it behaved nicely with homotopy in **CGH**. We can now extend this definition for simplicial objects in an abelian category and check that it also behaves nicely with homology of chain complexes. For $K_\star$ a simplicial set and $A_\star$ a simplicial object in a category $\mathcal{A}$ having products, we define the product $A_\star \otimes K_\star$ to be the simplicial object in $\mathcal{A}$ where

$$(A_\star \otimes K_\star)_n = A_n \times K_n$$

is just the product of K_n copies of A_n. Notice that if each K_n is finite, then $\mathcal{A}$ need only to have finite products (as it is the case for abelian categories).

Definition. Let $f, g : A_\star \to B_\star$ be maps of simplicial objects in an abelian category $\mathcal{A}$. A *simplicial homotopy* from f to g is a simplicial map $h : A_\star \otimes \Delta^1 \to B_\star$ such that the following commutes

$$\begin{array}{ccccc} A_\star \cong A_\star \otimes \Delta^0 & \xrightarrow{\mathrm{id}\times d^1} & A_\star \otimes \Delta^1 & \xleftarrow{\mathrm{id}\times d^0} & A_\star \otimes \Delta^0 \cong A_\star \\ & \searrow^{f} & \downarrow h & {}^{g}\swarrow & \\ & & B_\star & & \end{array}$$

The maps f, g are said *(simplicially) homotopic* and we will write $f \simeq g$.

Lemma 1.4.6. *Let $f, g : A_\star \longrightarrow B_\star$ be homotopic maps of simplicial objects in an abelian category $\mathcal{A}$. Then,*

$$N(f) \simeq N(g) : N(A_\star) \longrightarrow N(B_\star)$$

are chain homotopic maps.

Proof. For any $n \geqslant 0$, we can characterize the maps in $(\Delta^1)_n$ by the preimage of $0 \in \mathbf{1}$, i.e.,

$$(\Delta^1)_n = \{f_i : \mathbf{n} \to \mathbf{1} \in \mathrm{Hom}(\mathbf{n}, \mathbf{1}) \mid f_i^{-1}(0) = \{0, \ldots, i\}\}_{i=-1,\ldots,n}$$

Let $h : A_\star \otimes \Delta^1 \to B_\star$ be the simplicial homotopy from f to g. Then, $h_n : A_n \times (\Delta^1)_n \to B_n$ is just a collection of maps $h_n^i : A_n \times \{f_i\} \cong A_n \to B_n$ for $i = -1, \ldots, n$, such that

$$d_i h_n^j = \begin{cases} h_{n-1}^{j-1} d_i, & i \leqslant j \\ h_{n-1}^{j} d_i, & i > j \end{cases}, \quad s_i h_n^j = \begin{cases} h_{n+1}^{j+1} s_i, & i \leqslant j \\ h_{n+1}^{j} s_i, & i > j \end{cases}, \quad \begin{cases} h_n^{-1} = g \\ h_n^{n} = f \end{cases}.$$

We define $k_n = \sum_{j=0}^{n} (-1)^j (h_{n+1}^j s_j - s_j f_n) : A_n \to B_{n+1}$. For $i < n+1$ we have

$$\begin{aligned} d_i k_n &= \sum_{j=0}^{i-2} (-1)^j (h_n^j s_j d_{i-1} - s_j d_{i-1} f_n) + (-1)^{i-1} (h_n^{i-1} - f_n) + (-1)^i (h_n^{i-1} - f_n) \\ &\quad + \sum_{j=i+1}^{n} (-1)^j (h_n^{j-1} s_{j-1} d_i - s_{j-1} d_i f_n) \\ &= \sum_{j=0}^{i-2} (-1)^j (h_n^j s_j d_{i-1} - s_j f_{n-1} d_{i-1}) + \sum_{j=i+1}^{n} (-1)^j (h_n^{j-1} s_{j-1} d_i - s_{j-1} f_{n-1} d_i) \end{aligned}$$

and therefore the restriction of k_n to $N_n(A)$ lies in $N_{n+1}(B)$. Moreover,

$$\begin{aligned} d_{n+1} k_n &= \sum_{j=0}^{n-1} (-1)^j (h_n^j s_j d_n - s_j d_n f_n) + (-1)^n (h_n^n - f_n) \\ &= \sum_{j=0}^{n-1} (-1)^j (h_n^j s_j d_n - s_j f_{n-1} d_n) + (-1)^n (g_n - f_n) \\ &= k_{n-1} d_n + (-1)^n (g_n - f_n). \end{aligned}$$

Hence,

$$\begin{aligned}\partial_{n+1}k_n + k_{n-1}\partial_n &= (-1)^{n+1}d_{n+1}k_n + (-1)^n k_{n-1}d_n = (-1)^{n+1}(d_{n+1}k_n - k_{n-1}d_n)\\ &= (-1)^{n+1}(-1)^n(g_n - f_n) = f_n - g_n\end{aligned}$$

and $\{k_n\}$ is a chain homotopy from $N(f)$ to $N(g)$. □

For an abelian category $\mathcal{A}$, let $C_\bullet$ be a chain complex in $\mathbf{Ch}_{\geqslant 0}(\mathcal{A})$ with boundary map ∂. For any injective map $f : \mathbf{n} \to \mathbf{m}$ in Δ, we define the map $L(f) : C_m \longrightarrow C_n$ to be

$$L(f) = \begin{cases} \mathrm{id}_{C_n}, & \text{if } n = m,\\ (-1)^{n+1}\partial_{n+1}, & \text{if } m = n+1, \text{ and } f = d^n,\\ 0, & \text{else.}\end{cases}$$

Taking C_n as n-simplicies, we get a semisimplicial object $C_\star$ in $\mathcal{A}$. This defines a functor

$$L : \mathbf{Ch}_{\geqslant 0}(\mathcal{A}) \longrightarrow s\mathcal{SA}.$$

Construction of maps may seem odd, but one may note similarities with the definition of the normalized chain complex $N(A)$. They are not casual, in fact, N and $G\,L$ are inverse equivalences.

Theorem 1.4.7 (Dold-Kan)**.** *Let $\mathcal{A}$ be an abelian category. There is an equivalence of categories*

$$\mathcal{SA} \underset{G\,L}{\overset{N}{\rightleftarrows}} \mathbf{Ch}_{\geqslant 0}(\mathcal{A})$$

under which simplicial homotopy in $\mathcal{SA}$ corresponds to homology in $\mathbf{Ch}_{\geqslant 0}(\mathcal{A})$ and simplicially homotopic morphisms correspond to chain homotopic maps.

Proof. Let $K = GL$ and $C \in \mathbf{Ch}_{\geqslant 0}(\mathcal{A})$ with boundary map ∂. There is a natural inclusion of the normalized chain complex into the unnormalised chain complex

$$\Psi_n(C) : N_n(K(C)) \hookrightarrow C_n(K(C)) = K(C)_n = \bigoplus_{f:\mathbf{n}\to\mathbf{k}} C_k[f], \quad n \geqslant 0.$$

Here, we adopt the usual convention of denoting by $\oplus$ the coproduct in an abelian category. Consider the face maps $d_i : KC_n \to KC_{n-1}$ in the simplicial object $K(C)$ and in particular, their restriction to the factor $C_n[\mathrm{id}_{\mathbf{n}}] = C_n$ of $K(C)_n$. For any $d^i : \mathbf{n-1} \to \mathbf{n}$ the epi-monic factorization

$$\begin{array}{ccc}\mathbf{n-1} & \overset{d^i}{\hookrightarrow} & \mathbf{n}\\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}}\\ \mathbf{n-1} & \overset{d^i}{\hookrightarrow} & \mathbf{n}\end{array}$$

shows that the image of d_i restricted to C_n is the zero map for all $i = 0, \dots, n-1$, and for $i = n$ $(-1)^n\partial_n : C_n \to C_{n-1}[\mathrm{id}_{\mathbf{n-1}}]$. Hence,

$$C_n \subseteq \bigcap_{i=0}^{n-1} \ker d_i = N_n(K(C)) = \mathrm{im}\,\Psi_n(C).$$

Any other factor in $K(C)_n$ which is not C_n is of the form $C_k[f]$ for $f : \mathbf{n} \to \mathbf{k}$ surjective and $k < n$. Thus, it can be factorized as

$$\mathbf{n} \xrightarrow{s^i} \mathbf{n-1} \xrightarrow{g} \mathbf{k}, \qquad f = g \circ s^i$$

for some s^i and g. Hence, the degeneracy map $s_i : KC_{n-1} \to KC_n$ restricted to $C_k[g]$ has image $C_k[f]$, so $C_k[f] \in D_n(K(C))$ and by Proposition 1.4.4 it cannot be in the image of $\Psi_n(C)$. Therefore, im $\Psi_n(C) = C_n$, i.e, $N_n(K(C)) \cong C_n$ is a natural isomorphism. Moreover, the differential in $NK(C)$ is given by $(-1)^n d_n : N_n(K(C)) \to N_{n-1}(K(C))$ which is precisely the map $(-1)^n(-1)^n\partial_n = \partial_n : C_n \to C_{n-1}$. Thus,

$$NK(C) \cong C.$$

On the other hand, let $A_\star \in \mathcal{SA}$. There is a natural map

$$\Phi_n(A_\star) : K_n(N(A_\star)) = \bigoplus_{f:\mathbf{n}\to\mathbf{k}} N_k(A_\star)[f] \longrightarrow A_n, \quad n \geqslant 0,$$

whose restriction to $N_k(A_\star)[f]$ for a surjection $f : \mathbf{n} \to \mathbf{k}$ is defined as the composite

$$N_k(A_\star)[f] \hookrightarrow C_k(A_\star) = A_k \xrightarrow{f^\star} A_n.$$

For $g : \mathbf{m} \to \mathbf{n}$ a map in Δ, with decomposition

$$\begin{array}{ccc} \mathbf{m} & \xrightarrow{g} & \mathbf{n} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{l} & \xhookrightarrow{j} & \mathbf{k} \end{array}$$

there is a commutative diagram

$$\begin{array}{ccccccc} K_n(N(A_\star)) & \hookleftarrow & N_k(A_\star)[f] & \hookrightarrow & A_k & \xrightarrow{f^\star} & A_n \\ \downarrow{\scriptstyle g^\star} & & {\scriptstyle g^\star|_{N_k(A_\star)}}\downarrow{\scriptstyle j^\star|_{N_k(A_\star)}} & & \downarrow{\scriptstyle j^\star} & & \downarrow{\scriptstyle g^\star} \\ K_m(N(A_\star)) & \hookleftarrow & N_l(A_\star)[h] & \hookrightarrow & A_l & \xrightarrow{h^\star} & A_m \end{array}$$

So $\Phi_n(A_\star)$ is a simplicial map. We prove by induction on n that it is an isomorphism. For $n = 0$, note that $N_0(A_\star) = A_0$, and hence

$$\Phi_0(A_\star) : K_0(N(A_\star)) = N_0(A_\star)[\mathrm{id}_\mathbf{0}] \xrightarrow{\cong} A_0$$

is an isomorphism. Now assume $\Phi_j(A_\star)$ is an isomorphism for $j < n-1$. Let us show that $\Phi_n(A_\star)$ is also an isomorphism. $\Phi_n(A_\star)$ restricted to the factor $N_n(A_\star)[\mathrm{id}_\mathbf{n}]$ is the natural identification of $N_n(A_\star)[\mathrm{id}_\mathbf{n}]$ with $N_n(A_\star)$. Hence $N_n(A_\star) \subseteq$ im $\Phi_n(A_\star)$. Since $\Phi_{n-1}(A_\star)$ is surjective, $A_{n-1} \subseteq$ im $\Phi_{n-1}(A_\star)$, and for $0 \leqslant i \leqslant n$,

$$s_i \Phi_{n-1}(A_\star) = \Phi_n(A_\star)\, s_i,$$

so $s_i(A_{n-1}) \subseteq$ im $(\Phi_n(A_\star)\, s_i)$. Thus $D(A_\star) \subseteq$ im $\Phi_n(A_\star)$. Finally, by Proposition 1.4.4,

$$A_n = N_n(A_\star) \oplus D_n(A_\star) \subseteq \text{im } \Phi_n(A_\star),$$

so it is a surjection. To see injectivity we use elements. For $0 \leqslant k < n$, and any surjection $f : \mathbf{n} \twoheadrightarrow \mathbf{k}$ we can choose a section $f' : \mathbf{k} \hookrightarrow \mathbf{n}$ such that

$$f'(i) = \max\{j \mid f(j) = i\}.$$

For any two $f, g : \mathbf{n} \twoheadrightarrow \mathbf{k}$, we define the equivalence relation

$$f \leqslant g \Longleftrightarrow f'(i) \leqslant g'(i) \quad \text{for all } 0 \leqslant i \leqslant k.$$

In particular, notice that if $g f' = \mathrm{id}_{\mathbf{k}}$ then we must have $f \leqslant g$. Now, let $(x_h) \in \ker \Phi_n(A_\star)$. If a component $x_f \neq 0$ for some $f : \mathbf{n} \twoheadrightarrow \mathbf{k}$, we take the maximal such f with respect to $\leqslant$. We have a commutative factorization

$$\begin{array}{ccc} \mathbf{k} & \xhookrightarrow{f'} & \mathbf{n} \\ \downarrow{\scriptstyle \mathrm{id}} & & {\scriptstyle g}\downarrow\ \downarrow{\scriptstyle f} \\ \mathbf{k} & \xhookrightarrow{\mathrm{id}} & \mathbf{k} \end{array}$$

and if there is some other $g : \mathbf{n} \twoheadrightarrow \mathbf{k}$ making the diagram commute, then $f \leqslant g$, so $0 = x_g \in (x_h)$. Therefore, the component of $(f')^\star((x_h)) \in K_k N(A_\star)$ in the factor $N_k(A_\star)[\mathrm{id}_{\mathbf{k}}]$ is just x_f. But $(f')^\star((x_h)) \in \ker \Phi_k(A_\star) = 0$ by induction, so we must have $x_f = 0$. Hence, we get $x_f = 0$ for all $f \neq \mathrm{id}_{\mathbf{n}}$. Finally, the restriction of $\Phi_n(A_\star)$ to $N_n(A_\star)[\mathrm{id}_{\mathbf{n}}]$ is just the natural inclusion $N_n(A_\star) \hookrightarrow A_n$, so $x_{\mathrm{id}_{\mathbf{n}}} = 0$. Hence, $(x_h) = 0$ and $\Phi_n(A_\star)$ is injective.
We are just left to show that simplicially homotopic maps correspond to chain homotopic maps. We have already proven one direction in Lemma 1.4.6. Let $f, g : C \to D$ be chain maps, and let $\{k_n\}$ be a chain homotopy from f to g. We define maps $h_i : K(C)_n \to K(D)_{n+1}$. As we saw in the proof of Lemma 1.4.1, we only need to specify the restriction of h_i to the factor $C_n[\mathrm{id}_{\mathbf{n}}]$ of $K(C)_n$, which is given by

$$h_i|_{C_n[\mathrm{id}_{\mathbf{n}}]} : C_n[\mathrm{id}_{\mathbf{n}}] \longrightarrow K(D)_{n+1} = \begin{cases} s_i f & \text{if } i < n-1 \\ s_{n-1} f - s_n k_{n-1} d & \text{if } i = n-1 \\ s_n (f - s_{n-1} d) - k_n & \text{if } i = n \end{cases}.$$

We set now $h_n^i : K(C)_n \to K(D)_n$ for $i = -1, \ldots, n$ by

$$\begin{cases} h_n^{-1} = K(g), \quad h_n^n = K(f) \\ h_n^i = d_{i+1} h_i, \quad \text{for } 0 \leqslant i < n \end{cases}.$$

By the description of simplicial homotopies we did in the poof of Lemma 1.4.6, this gives a simplicial homotopy $h : K(C) \otimes \Delta^1 \to K(D)$ from $K(g)$ to $K(f)$ and ends the proof. □

Proposition 1.4.8. *Let $\mathcal{A}$ be an abelian category, then there are pairs of adjoints*

(a)

$$L : \mathbf{Ch}_{\geqslant 0}(\mathcal{A}) \rightleftarrows s\mathcal{SA} : N,$$

(b)

$$K = GL : \mathbf{Ch}_{\geqslant 0}(\mathcal{A}) \rightleftarrows \mathcal{SA} : N.$$

Proof. (a) Let $C \in \mathbf{Ch}_{\geqslant 0}(\mathcal{A})$, and $A_\star \in s\mathcal{SA}$. We have to see

$$\mathrm{Hom}_{s\mathcal{SA}}(L(C), A_\star) \cong \mathrm{Hom}_{\mathbf{Ch}}(C, N(A_\star)).$$

Let $h \in \mathrm{Hom}_{s\mathcal{SA}}(L(C), A_\star)$, since it is a natural transformation, the following diagram is commutative

$$\begin{array}{ccc} C_n & \xrightarrow{h_n} & A_n \\ \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} \\ C_{n-1} & \xrightarrow{h_{n-1}} & A_{n-1} \end{array}$$

For $i = 0, \ldots, n-1$, the left arrow in the diagram is the zero map (by definition of $L(C)$), and hence $d_i h_n = 0$, so

$$\mathrm{im}(h_n) \subseteq \bigcap_{i=0}^{n-1} \ker d_i = N_n.$$

Therefore, h induces a map $\widehat{h} : C \to N(A_\star)$ given by $\widehat{h}_n = h_n : C_n \to N_n$. Conversely, a map $\widehat{h} \in \mathrm{Hom}_{\mathbf{Ch}}(C, N(A_\star))$ also induces a map $h : L(C) \to A_\star$ given by the composition

$$h_n : C_n \xrightarrow{\widehat{h}_n} N_n \hookrightarrow A_n.$$

(b) Direct consequence of (a) and Lemma 1.4.1. $\square$

Chapter 2

(Co-) Homology of Commutative Rings

In this chapter we will use simplicial objects to define resolutions of an object in an abelian category. Then, we will apply the homotopy theory we have just constructed to the category of commutative rings in order to define homology and cohomology here.

Definition. An *augmented simplicial object* in a category $\mathcal{A}$ is a simplicial object $A_\star$ together with a morphism $\varepsilon : A_0 \to A_{-1}$ to a fixed object $A_{-1} \in \mathcal{A}$ such that $\varepsilon d_0 = \varepsilon d_1$. It is called *aspherical* if $\pi_0(A_\star) \cong A_{-1}$ and $\pi_n(A_\star) = 0$ for all $n \geqslant 1$. We will denote it by $A_\star \to A_{-1}$.

If $\mathcal{A}$ is an abelian category, then $A_\star$ is aspherical if the underlying unnormalized (or normalized) chain complex associated to $A_\star$ is exact. Therefore, $C(A_\star)$ (or $N(A_\star)$) is a resolution of A_{-1} in $\mathcal{A}$.

2.1 Cotriple homology and cohomology

Let $\mathcal{C}$ be a category, $T : \mathcal{C} \to \mathcal{C}$ a functor and $d : T \to T^2$ a natural transformation with components $d_C : T(C) \to T^2(C)$ for all $C \in \mathcal{C}$. We denote by Td, dT the natural transformations $T^2 \to T^3$ with components $(Td)_C = T(d_C)$, and $(dT)_C = d_{T(C)}$ respectively.

Definition. A *cotriple* $(\perp, \varepsilon, \delta)$ on a category $\mathcal{C}$ is a functor $\perp : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\varepsilon : \perp \to \mathrm{id}_\mathcal{C}$, $\delta : \perp \to \perp^2$ such that the following diagrams commute

$$\begin{array}{ccc} \perp & \xrightarrow{\delta} & \perp^2 \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\perp\delta} \\ \perp^2 & \xrightarrow{\delta\perp} & \perp^3 \end{array} \qquad \begin{array}{ccccc} & & \perp & & \\ & \swarrow{\scriptstyle =} & \downarrow{\scriptstyle\delta} & {\scriptstyle =}\searrow & \\ \perp & \xleftarrow{\varepsilon\perp} & \perp^2 & \xrightarrow{\perp\varepsilon} & \perp. \end{array}$$

Example 2.1.1. In any category $\mathcal{C}$ there is always a trivial cotriple taking $\perp = \mathrm{id}_\mathcal{C}$, and $\varepsilon = \delta$ to be the constant natural transformation $\mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C}$.

Remark. If we apply naturality of ε to the map $\perp^j \varepsilon_A : \perp^{j+1} A \to \perp^j A$ for $j \geq 0$ we a get commutative diagram

$$\begin{array}{ccc} \perp(\perp^{j+1}A) & \xrightarrow{\perp^{j+1}(\varepsilon_A)} & \perp(\perp^j A) \\ {\scriptstyle\varepsilon_{\perp^{j+1}A}}\downarrow & & \downarrow{\scriptstyle\varepsilon_{\perp^j A}} \\ \perp^{j+1}A & \xrightarrow{\perp^j \varepsilon_A} & \perp^j A \end{array}$$

which means that ε satisfies the identity $(\varepsilon\bot^j)(\bot^{j+1}\varepsilon) = (\bot^j\varepsilon)(\varepsilon\bot^{j+1})$ for all $j \geqslant 0$. Similarly, applying naturality of δ to $\bot^{j-1}\delta_A : \bot^j A \to \bot^{j+1}A$ we get

$$(\delta\bot^{j+1})(\bot^j\delta) = (\bot^{j+1}\delta)(\delta\bot^j), \quad \text{for } j > 0.$$

Finally, naturality of ε applied to $\bot^j\delta_A$, and naturality of δ applied to $\bot^{j-1}\varepsilon_A$ gives

$$\begin{aligned}(\varepsilon\bot^{j+2})(\bot^{j+1}\delta) &= (\bot^j\delta)(\varepsilon\bot^{j+1}), \quad \text{for } j \geqslant 0,\\ (\delta\bot^{j-1})(\bot^j\varepsilon) &= (\bot^{j+1}\varepsilon)(\delta\bot^j), \quad \text{for } j > 0,\end{aligned}$$

respectively.

Given an object $C \in \mathcal{C}$ we can use a cotriple $(\bot, \varepsilon, \delta)$ to construct an augmented simplicial object $\bot_\star C \to C$ in $\mathcal{C}$ by taking

$$\bot_n C = \bot^{n+1}C, \quad d_i = \bot^i\varepsilon\bot^{n-i}C : \bot^{n+1}C \to \bot^n C, \quad s_i = \bot^i\delta\bot^{n-i}C : \bot^{n+1}C \to \bot^{n+2}C,$$

and setting $\bot_0 C = \bot C \to C$ to be ε_C. The simplicial identities are satisfied:

$$\begin{aligned}
d_i d_j &= \bot^i(\varepsilon\bot^{j-i-1})(\bot^{j-i}\varepsilon)\bot^{n-j} = \bot^i(\bot^{j-i-1}\varepsilon)(\varepsilon\bot^{j-i})\bot^{n-j} = d_{j-1}d_i, \quad \text{for } i < j,\\
s_i s_j &= \bot^i(\delta\bot^{j-i+1})(\bot^{j-i}\delta)\bot^{n-j} = \bot^i(\bot^{j-i+1}\delta)(\delta\bot^{j-i})\bot^{n-j} = s_{j+1}s_i, \quad \text{for } i \leqslant j,\\
d_i s_j &= \bot^i(\varepsilon\bot^{j-i+1})(\bot^{j-i}\delta)\bot^{n-j} = \bot^i(\bot^{j-i-1}\delta)(\varepsilon\bot^{j-i})\bot^{n-j} = s_{j-1}d_i, \quad \text{for } i < j,\\
d_i s_i &= \bot^i(\varepsilon\bot)(\delta)\bot^{n-i} = \mathrm{id}_{\bot^{n+1}}, \quad d_{i+1}s_i = \bot^i(\bot\varepsilon)(\delta)\bot^{n-i} = \mathrm{id}_{\bot^{n+1}}\\
d_i s_j &= \bot^j(\bot^{i-j}\varepsilon)(\delta\bot^{i-j-1})\bot^{n+1-i} = \bot^j(\delta\bot^{i-j-2})(\bot^{i-j-1}\varepsilon)\bot^{n+1-i}\\
&= s_j d_{i-1}, \quad \text{for } i > j+1.
\end{aligned}$$

Finally, by the previous remark, $\varepsilon_C(\bot\varepsilon_C) = \varepsilon_C(\varepsilon_{\bot C})$, so $\varepsilon_C d_1 = \varepsilon_C d_0$. Thus $\bot_\star C \to C$ is indeed an augmented simplicial object.

If $E : \mathcal{C} \to \mathcal{A}$ is a functor to an abelian category, for every $C \in \mathcal{C}$, we get an augmented simplicial object in $\mathcal{A}$, $E\bot_\star C \to E(C)$.

Definition. The *cotriple homology of C with coefficients in E* (*relative to the cotriple* $\bot$) is the homotopy of the simplicial object $E\bot_\star C \to E(C)$, which by the previous chapter is the same as the homology of the associated chain complex $C(E\bot_\star C)$. It is denoted $H_\star(C;E)$, such that

$$H_n(C;E) = \pi_n(E\bot_\star C), \quad n \geqslant 0.$$

Remark. Although we have been working on simplicial objects, we can dualize everything for cosimplicial objects. In particular, for every abelian category $\mathcal{A}$, there is an equivalence $N^\star$ between the cosimplicial objects of $\mathcal{A}$ and $\mathbf{Ch}^{\geqslant 0}(\mathcal{A})$ (the category of cochain complexes $C^\bullet$ in $\mathcal{A}$ with $C^n = 0$ for $n < 0$), where $N^\star(A^\star)$ is a summand of the unnormalized cochain complex $C(A^\star)$ of $A^\star$. We define the *cohomotopy* of a cosimplicial object $A^\star$ to be the cohomology of $N^\star(A^\star)$, i.e.,

$$\pi^i(A^\star) = H^i(N^\star(A^\star)).$$

In particular, we also have $\pi^i(A^\star) \cong H^i(C(A^\star))$. Finally, if $\mathcal{A}$ has enough injectives, then the cohomotopy groups $\pi^\star(A^\star)$ are the right derived functors of the functor π^0.

If $E : \mathcal{C}^{op} \to \mathcal{A}$ is a functor to an abelian category, for every $C \in \mathcal{C}$ we get an augmented cosimplicial object in $\mathcal{A}$, $E(C) \to E(\perp_\star C)$.

Definition. The *cotriple cohomology of C with coefficients in E* (*relative to the cotriple* $\perp$) is the cohomotopy of the cosimplicial object $E(C) \to E(\perp_\star C)$. It is denoted $H^\star(C;E)$ such that

$$H^n(C;E) = \pi^n(E\perp_\star C), \quad n \geqslant 0.$$

There is an easy way to obtain cotriples using pairs of adjoint functors (besides, every cotriple arises from a pair of adjoint functors in this manner [1]). Let

$$F : \mathcal{C} \rightleftarrows \mathcal{B} : U$$

be a pair of adjoint functors having unit and counit $\eta : \mathrm{id}_\mathcal{C} \to UF$ and $\varepsilon : FU \to \mathrm{id}_\mathcal{B}$ respectively. Recall that they satisfies the identities

$$(\varepsilon F)(F\eta) = \mathrm{id}_F, \quad (U\varepsilon)(\eta U) = \mathrm{id}_U, \quad (UF\eta)\eta = (\eta UF)\eta.$$

Let $\perp = FU$, and $\delta = F\eta U$, such that $\varepsilon : \perp \to \mathrm{id}_\mathcal{B}$, and $\delta : \perp \to \perp^2$. It follows

$$\begin{aligned} (UF\eta)\eta = (\eta UF)\eta \Rightarrow (FUF\eta U)F\eta U = (F\eta UFU)F\eta U &\implies (\perp\delta)\delta = (\delta\perp)\delta, \\ (U\varepsilon)(\eta U) = \mathrm{id}_U \Rightarrow F((U\varepsilon)(\eta U)) = \mathrm{id}_{FU} &\implies (\perp\varepsilon)\delta = \mathrm{id}_\perp, \\ (\varepsilon F)(F\eta) = \mathrm{id}_F \Rightarrow ((\varepsilon F)(F\eta))U = \mathrm{id}_{FU} &\implies (\varepsilon\perp)\delta = \mathrm{id}_\perp, \end{aligned}$$

so this choice gives in fact a cotriple.

Example 2.1.2. We denote by $\mathbf{Alg}_k$ the category of commutative k-algebras for k a commutative ring (with 1). The forgetful functor $U : \mathbf{Alg}_k \longrightarrow \mathbf{Set}$ has a left adjoint $F : \mathbf{Set} \longrightarrow \mathbf{Alg}_k$ where $F(X) = k[X]$ is the polynomial k-algebra on the set X. We see it as a free k-algebra with basis $\{e_x \mid x \in X\}$. The unit η and counit ε are given by

$$\begin{array}{rccc} \eta_X : & X & \longrightarrow & UF(X) \\ & x & \longmapsto & e_x \end{array}, \qquad \begin{array}{rccc} \varepsilon_R : & k[R] & \longrightarrow & R \\ & e_r & \longmapsto & r \end{array}$$

for any $R \in \mathbf{Alg}_k$ and $X \in \mathbf{Set}$, where $k[R]$ denotes the polynomial algebra on the underlying set of R. We get a cotriple

$$\perp : \mathbf{Alg}_k \longrightarrow \mathbf{Alg}_k$$

sending a k-algebra R to $k[R]$. The underlying augmented simplicial set $U(\perp_\star R) \to UR$ is aspherical, i.e.,

$$\pi_0 U(\perp_\star R) \cong UR, \qquad \pi_n U(\perp_\star R) = 0, \quad \text{for all } n \geqslant 1.$$

To see this set

$$f_{-1} = \eta U : U \longrightarrow U\perp, \quad \text{and} \quad f_n = \eta U\perp^{n+1} : U\perp_n \longrightarrow U\perp_{n+1}, \ \text{for } n \geqslant 0.$$

Then

$$\begin{aligned} (U\varepsilon)f_{-1} &= (U\varepsilon)(\eta U) = \mathrm{id}_U \\ d_0 f_n &= (U\varepsilon\perp^{n+1})(\eta U\perp^{n+1}) = \mathrm{id}_{\perp^{n+1}}. \end{aligned}$$

[1] See Mac Lane [8] IV. 2.

Applying naturality of ηU to the maps $\varepsilon_R : \perp R \to R$ and $\perp^{i-1}\varepsilon\perp^{n-i+1}R : \perp^{n+1}R \to \perp^n R$, we get

$$\begin{aligned}(\eta U)(U\varepsilon) = (U\perp\varepsilon)(\eta U\perp) &\implies f_{-1}(U\varepsilon) = d_1 f_0\\ (\eta U\perp^n)(U\perp^{i-1}\varepsilon\perp^{n-i+1}) = (U\perp^i\varepsilon\perp^{n+1-i})(\eta U\perp^{n+1}) &\implies f_{n-1}d_{i-1} = d_i f_n\end{aligned}$$

respectively. Moreover,

$$\begin{aligned}(\eta U\perp)(\eta U) = (\eta UF)(\eta)U = (UF\eta)(\eta)U = (U\delta)(\eta U) &\implies f_0(\eta U) = s_0(\eta U),\\ (\eta U\perp^2)(U\delta)(\eta U) = (\eta UFUFU)(\eta UFU)(\eta U) = (U\delta\perp)(U\delta)(\eta U) &\implies f_1(s_0(\eta U)) = s_0^2(\eta U),\end{aligned}$$

and so on, so that

$$f_n(s_{n-1}s_{n-2}\dots s_0(\eta U)) = f_n(s_0^n(\eta U)) = s_0^{n+1}(\eta U) = s_n s_{n-1}\dots s_0(\eta U), \quad \text{for all } n \geqslant 0.$$

In order to define the homotopy groups we choose as basepoint the vertex $k = \eta_{UR}(0) \in U\perp_0 R$. The homotopy groups are

$$\pi_n\,(U(\perp_\star R), k) = [(\Delta^n, \partial\Delta^n), (U(\perp_\star R), k)]_\simeq\,, \quad \text{for } n \geqslant 1.$$

Let us consider an n-simplex $x \in U\perp_n R$ such that

$$\begin{array}{ccc}\partial\Delta^n & \longrightarrow & \Delta^0\\ \big\downarrow & & \big\downarrow k\\ \Delta^n & \xrightarrow{\;x\;} & U(\perp_\star R)\end{array}$$

is commutative. Hence $[x]$ is a homotopy class element in $\pi_n(U(\perp_\star R), k)$. We want to show that $[x] = [k]$. Take $y = f_n(x) \in U_{\perp_{n+1}}R$. Then,

$$\begin{aligned}&d_i(y) = d_i(f_n(x)) = f_{n-1}(d_{i-1}(x)) = f_{n-1}(k) = k, \quad \text{for } 0 < i \leqslant n+1\\ &d_0(y) = d_0(f_n(x)) = x.\end{aligned}$$

Hence, we get

$$y|_{\Lambda_0^{n+1}} = k|_{\Lambda_0^{n+1}} : \Lambda_0^{n+1} \longrightarrow U(\perp_\star R)$$

and we can define the map

$$g:\quad \Lambda_0^{n+1}\times\Delta^1 \bigsqcup_{\Lambda_0^{n+1}\times\partial\Delta^1} \Delta^{n+1}\times\partial\Delta^1 \longrightarrow U(\perp_\star R)$$

by setting g to be k on $\Lambda_0^{n+1}\times\Delta^1$ and on $\Delta^{n+1}\times\{1\}$, and to be x on $\Delta^{n+1}\times\{0\}$. Let us see that the map

$$\Lambda_0^{n+1}\times\Delta^1 \bigsqcup_{\Lambda_0^{n+1}\times\partial\Delta^1} \Delta^{n+1}\times\partial\Delta^1 \longrightarrow \Delta^{n+1}\times\Delta^1 \tag{2.1}$$

induced by the inclusions $i : \partial\Delta^1 \hookrightarrow \Delta^1$ and $\Lambda_0^{n+1} \hookrightarrow \Delta^{n+1}$ is a cofibration. Let $p : A_\star \longrightarrow B_\star$ be an acyclic fibration of simplicial sets, and consider the lifting problem

$$\begin{array}{ccc} \Lambda_0^{n+1}\times\Delta^1 \coprod_{\Lambda_0^{n+1}\times\partial\Delta^1} \Delta^{n+1}\times\partial\Delta^1 & \longrightarrow & A_\star \\ \downarrow & \nearrow & \downarrow p \\ \Delta^{n+1}\times\Delta^1 & \longrightarrow & B_\star \end{array}$$

Using the exponential law from Theorem 1.3.1 we see that this problem is equivalent to lifting problem

$$\begin{array}{ccc} \Lambda_0^{n+1} & \longrightarrow & \mathbf{Hom}(\partial\Delta^1, A) \\ \downarrow & \nearrow & \downarrow (i^\star, p_\star) \\ \Delta^{n+1} & \longrightarrow & \mathbf{Hom}(\partial\Delta^1, A) \times_{\mathbf{Hom}(\partial\Delta^1, B)} \mathbf{Hom}(\Delta^1, B). \end{array}$$

But in this last square the lift exists since the map on the right is an acyclic fibration by Proposition 1.3.2. Hence, (2.1) is indeed a cofibration of simplicial sets. From Example 1.2.11 we get that the simplicial set $U(\perp_\star R)$ is fibrant, thus

$$\begin{array}{ccc} \Lambda_0^{n+1}\times\Delta^1 \coprod_{\Lambda_0^{n+1}\times\partial\Delta^1} \Delta^{n+1}\times\partial\Delta^1 & \xrightarrow{\ g\ } & U(\perp_\star R) \\ \downarrow & \nearrow f & \\ \Delta^{n+1}\times\Delta^1 & & \end{array}$$

there is a map $f : \Delta^{n+1}\times\Delta^1 \longrightarrow U(\perp_\star R)$ making the diagram commutative. We define h to be the composition

$$h : \quad \Delta^n \times \Delta^1 \xrightarrow{d^0\times \mathrm{id}} \Delta^{n+1}\times\Delta^1 \xrightarrow{\ f\ } U(\perp_\star R).$$

Then, notice that the map $d^0 : \Delta^n \longrightarrow \Delta^{n+1}$ sends $\mathrm{id}_{\mathbf{n}}$ to $d_0(\mathrm{id}_{\mathbf{n+1}})$, which is the only face of the $(n+1)$-simplex $\mathrm{id}_{\mathbf{n+1}}$ missing in Λ_0^{n+1}. Therefore, the following diagram commutes

$$\begin{array}{ccccc} \Delta^n\times\Delta^0 & \xrightarrow{\mathrm{id}\times d^0} & \Delta^n\times\Delta^1 & \xleftarrow{\mathrm{id}\times d^1} & \Delta^n\times\Delta^0 \\ & & \downarrow d^0\times\mathrm{id} & & \\ & k & \Delta^{n+1}\Delta^1 & x & \\ & & \downarrow f & & \\ & \searrow & U(\perp_\star R) & \swarrow & \end{array}$$

meaning that h is a homotopy from x to k. Hence, $[x] = [k]$, and therefore

$$\pi_n\, U(\perp_\star R) = [k] = 0, \quad \text{for } n > 0.$$

In addition, recall that two vertices x, z in $\pi_0\, U(\perp_\star R)$ are in the same homotopy class if there is a 1-simplex $y \in \perp^2 R$ such that $\partial y = (d_0\, y, d_1\, z) = (x, z)$. Let us show the following equality,

$$\ker[(U\varepsilon)f_{-1}] = \ker(d_1 f_0) = \{x \in \perp R \mid \exists y \in \perp^2 R \text{ with } d_0(y) = x,\, d_1(y) = 0\}.$$

The containment $\subseteq$ is clear taking $y = f_0(x)$. The other one is also clear since $d_1 f_0(x) = d_1 f_0(d_0(y)) = d_1(y) = 0$ for every x in the second set. Thus, the induced map

$$f_{-1}(U\varepsilon) : \pi_0 U(\perp_\star R) \longrightarrow U(\perp R)$$

is well defined and injective, and it follows that $U\varepsilon : \pi_0 U(\perp_\star R) \to U(R)$ is also injective. Hence, it is a bijection and

$$\pi_0 U(\perp_\star R) \cong U(R).$$

2.2 André-Quillen homology and cohomology

We introduce a brief discussion to show how to define the model category structure on $\mathcal{S}\mathbf{Alg}_k$. We proved that for a commutative ring (with 1) R, the category $\mathbf{Ch}_{\geqslant 0}(R)$ has a model structure. The proof heavily relies on the fact that the category $\mathbf{Mod}_R$ has enough projectives. Actually, using a similiar proof as in Theorem 1.1.2 we can extend the result for abelian categories with "enough projectives".

Recall that an object P in an abelian category $\mathcal{A}$ is projective if it satisfies the following universal property: for any surjection $g : B \longrightarrow C$ in $\mathcal{A}$ and any diagram of solid arrows

$$\begin{array}{ccc} & & B \\ & \nearrow & \downarrow g \\ P & \longrightarrow & C \end{array}$$

there is at least one map $P \longrightarrow B$ solving the lifting problem (so that the diagram is commutative).

Definition. An abelian category $\mathcal{A}$ has *natural projective resolutions* if for every object $A \in \mathcal{A}$ there is a projective object $P_A \in \mathcal{A}$ and a surjection $P_A \to A$ which is natural in $\mathcal{A}$.

Theorem 2.2.1. *Let $\mathcal{A}$ be an abelian category with natural projective resolutions. Then $\mathbf{Ch}_{\geqslant 0}(\mathcal{A})$ has the structure of a model category where a morphism $f : X_\bullet \longrightarrow Y_\bullet$ is*

- *a weak equivalence if $H_\star f$ is an isomorphism;*
- *a fibration if $f_n : X_n \longrightarrow Y_n$ is surjective for all $n \geqslant 0$, and;*
- *a cofibration if f_n is injective with projective cokernel for $n \geqslant 0$.*

Proof. Similar as the proof for Theorem 1.1.2. For details see for instance Gillem [2] Theorem 5.5.2. □

Now, for any abelian category $\mathcal{A}$ with natural projective resolutions, the Dold-Kan correspondence

$$\mathcal{S}\mathcal{A} \underset{GL}{\overset{N}{\rightleftarrows}} \mathbf{Ch}_{\geqslant 0}(\mathcal{A})$$

gives a model structure on the category $\mathcal{S}\mathcal{A}$.

Proposition 2.2.2. *Let $\mathcal{A}$ be an abelian category with enough natural projective resolutions. Then $\mathcal{S}\mathcal{A}$ has a model category structure where a simplicial map $f : X_\star \longrightarrow Y_\star$ is a weak equivalence, fibration or cofibration if the map $N(f) : N(X) \longrightarrow N(Y)$ is a weak equivalence, fibration or cofibration in the model structure of $\mathbf{Ch}_{\geqslant 0}(\mathcal{A})$ respectively.*

Proof. Direct consequence of Theorem 2.2.1 and the equivalence of the categories given by Dold-Kan. □

There is another way to induce a model category structure on some categories of the form $\mathcal{S}\mathcal{A}$ using the model category structure of $\mathcal{S}\mathbf{Set}$. Let $\mathcal{C}$ be a category closed under finite limits and colimits and assume there is a pair of adjoint functors

$$F : \mathcal{S}\mathbf{Set} \rightleftarrows \mathcal{S}\mathcal{C} : G.$$

Moreover, assume that for any $\{X_i\}_{i\in I}$ diagram in $\mathcal{C}$ with I a filtered category the natural map

$$\varinjlim_I G(X_i) \xrightarrow{\cong} G\left(\varinjlim_I X_i\right)$$

is an isomorphism.

Theorem 2.2.3. *$\mathcal{S}\mathcal{C}$ has the structure of a model category where a morphism f in $\mathcal{S}\mathcal{C}$ is*

- *a weak equivalence if $G(f)$ is a weak equivalence in $\mathcal{S}\mathbf{Set}$;*
- *a fibration if $G(f)$ is a fibration in $\mathcal{S}\mathbf{Set}$, and;*
- *a cofibration if it has the LLP with respect to all acyclic fibrations in $\mathcal{S}\mathcal{A}$*

if every cofibration with the LLP with respect to all fibrations is an acyclic cofibration.

Proof. See Goerss and Jardine [3] II. Theorem 5.2. □

Remark. We have seen two different ways to provide a model structure on categories of the form $\mathcal{S}\mathcal{A}$ for an abelian category $\mathcal{A}$. In general, these two different structures have nothing in common. However, for the case $\mathcal{A} = \mathbf{Mod}_R$, a map f is a fibration (resp. weak equivalence) in the sense of Theorem 2.2.2 (which is precisely the model structure we showed in Theorem 1.1.2) if and only if the underlying map of simplicial sets $U(f)$ is a fibration (resp. weak equivalence) in the sense of Theorem 2.2.3. [2]

Example 2.2.4. Let us consider the abelian category $\mathbf{Alg}_k$ which is closed under finite limits and colimits, and the pair of adjoints from Example 2.1.2

$$F : \mathbf{Set} \rightleftarrows \mathbf{Alg}_k : U,$$

where U was the forgetful functor. We can extend this to a pair of adjoint functors

$$F : \mathcal{S}\mathbf{Set} \rightleftarrows \mathcal{S}\mathbf{Alg}_k : U,$$

taking $U(X_n) = U(X)_n$ and so on. We get in fact a model category structure on $\mathcal{S}\mathbf{Alg}_k$ with Theorem 2.2.3 (see Goerss and Jardine [3] II, or Quillen [12] II.4).

[2] See Quillen [12] II. 3.

Corollary 2.2.5. *The category $\mathcal{S}\mathbf{Alg}_k$ has the structure of a model category where a morphism $f : X_\star \longrightarrow Y_\star$ is*

- *a weak equivalence if $f_\star : \pi_\star X_\star \to \pi_\star Y_\star$ is an isomorphism;*
- *a fibration if as a map of simplicial sets, $U(f)$, it is a fibration, and;*
- *a cofibration if it has the LLP with respect to all acyclic fibrations in $\mathcal{S}\mathbf{Alg}_k$.*

Remark. Cofibrations are uniquely characterized by the lifting property they satisfy with respect to fibrations. Moreover, by the previous remark, a map of simplicial rings is an acyclic fibration if as a map of groups it is an acyclic fibration, i.e., if it induces isomorphisms on homology and it is surjective in each dimension. Also, for any simplicial k-algebra $A_\star$, there is a canonical map $ck \longrightarrow A_\star$ given in degree $n \geqslant 0$ by the structure map $k \longrightarrow A_n$. Thus ck is the initial object in $\mathcal{S}\mathbf{Alg}_k$.

Proposition 2.2.6. *Let $i : R_\star \longrightarrow S_\star$ be a cofibration and $p : X_\star \longrightarrow Y_\star$ an acyclic fibration in $\mathcal{S}\mathbf{Set}$. Let $h : R_\star \times \Delta^1 \longrightarrow X_\star$ and $k : S_\star \times \Delta^1 \longrightarrow Y_\star$ be homotopies such that the following commutes*

$$\begin{array}{ccc} R_\star \times \Delta^1 & \xrightarrow{\ h\ } & X_\star \\ {\scriptstyle i\times id_{\Delta^1}}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ S_\star \times \Delta^1 & \xrightarrow{\ k\ } & Y_\star \end{array} \tag{2.2}$$

and let θ_0, $\theta_1 : S_\star \to X_\star$ be maps such that the following diagrams commute

$$\begin{array}{ccc} R_\star & \xrightarrow{\ id\times d^e\ } & R_\star \times \Delta^1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h} \\ S_\star & \xrightarrow{\ \theta_e\ } & X_\star \\ {\scriptstyle id\times d^e}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ S_\star \times \Delta^1 & \xrightarrow{\ k\ } & Y_\star \end{array} \qquad \text{for } e = 0, 1.$$

Then, there is a homotopy $\ell : S_\star \times \Delta^1 \longrightarrow X_\star$ making all the previous diagrams commutative.

Proof. Applying exponential law from Theorem 1.3.1 to (2.2) we get a commutative diagram

$$\begin{array}{ccc} R_\star & \longrightarrow & \mathbf{Hom}(\Delta^1, X)_\star \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f_\star} \\ S_\star & \longrightarrow & \mathbf{Hom}(\Delta^1, Y)_\star \end{array}$$

We can see $(\theta_1, \theta_2) \in \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(S_\star, X_\star) \times \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(S_\star, X_\star)$. Note that $\partial\Delta^1 \cong \Delta^0 \times \Delta^0$ and using the exponential law we get a map $\varphi \in \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(S_\star, \mathbf{Hom}(\partial\Delta^1, X)_\star)$ since

$$\begin{aligned} \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(S_\star, \mathbf{Hom}(\partial\Delta^1, X)_\star) &\cong \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(\partial\Delta^1 \times S_\star, X_\star) \\ &\cong \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(S_\star, X_\star) \times \mathrm{Hom}_{\mathcal{S}\mathbf{Set}}(S_\star, X_\star). \end{aligned}$$

Then, calling $j : \partial\Delta^1 \longrightarrow \Delta^1$ the inclusion in $\mathcal{S}\mathbf{Set}$ we get a commutative diagram

$$\begin{array}{ccc} R_\star & \longrightarrow & \mathbf{Hom}(\Delta^1, X)_\star \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle (j^\star, p_\star)} \\ S_\star & \longrightarrow & \mathbf{Hom}(\partial\Delta^1, X)_\star \times_{\mathbf{Hom}(\Delta^1, Y)_\star} \mathbf{Hom}(\Delta^1, Y)_\star \end{array}$$

where the map on the right is an acyclic fibration of simplicial sets by Proposition 1.3.2. Hence, there is a map $S_\star \longrightarrow \mathbf{Hom}(\Delta^1, X)_\star$ making the diagram commutative. This map corresponds to a map $\ell : S_\star \times \Delta^1 \longrightarrow X_\star$ via the exponential law, which is the solution to our original lifting problem. □

Remark. Note that we are just rewriting Proposition 1.3.2 using the exponential law. Therefore, we can extend this result for any other category for which an analogous version of Proposition 1.3.2 holds (i.e., for any closed simplicial model category). In particular, this can be done for simplicial rings using the model structure coming from simplicial sets.[3] Thus, we get the following result:

Proposition 2.2.7. *Let $i : R_\star \longrightarrow S_\star$ be a cofibration and $p : X_\star \longrightarrow Y_\star$ an acyclic fibration in $\mathcal{S}\mathbf{Alg}_k$. Let $h : R_\star \otimes \Delta^1 \longrightarrow X_\star$ and $k : S_\star \otimes \Delta^1 \longrightarrow Y_\star$ be homotopies such that the following commutes*

$$\begin{array}{ccc} R_\star \otimes \Delta^1 & \xrightarrow{h} & X_\star \\ {\scriptstyle i \times id_{\Delta^1}}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ S_\star \otimes \Delta^1 & \xrightarrow{k} & Y_\star \end{array}$$

and let $\theta_0, \theta_1 : S_\star \to X_\star$ be maps such that the following diagrams commute

$$\begin{array}{ccc} R_\star & \xrightarrow{id \times d^e} & R_\star \otimes \Delta^1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h} \\ S_\star & \xrightarrow{\theta_e} & X_\star \\ {\scriptstyle id \times d^e}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ S_\star \otimes \Delta^1 & \xrightarrow{k} & Y_\star \end{array} \qquad \textit{for } e = 0, 1.$$

Then, there is a homotopy $\ell : S_\star \otimes \Delta^1 \longrightarrow X_\star$ making all the previous diagrams commutative.

For any k-algebra R, $\mathbf{Alg}_k/R$ is the category of k-algebras over R, i.e., the category whose objects are k-algebras P equipped with an algebra map $P \to R$, and whose morphisms are maps $P \to Q$ such that the diagram

$$\begin{array}{ccc} P & \longrightarrow & Q \\ & \searrow \quad \swarrow & \\ & R & \end{array}$$

is commutative. Analogously for any $A_\star, B_\star \in \mathcal{S}\mathbf{Alg}_k$, we let $\mathbf{Alg}_A/B$ be the category of simplicial k-algebras under $A_\star$ over $B_\star$, i.e., the category whose objects are simplicial k-algebras $T_\star$ equipped with structure maps $A_\star \to T_\star$, $T_\star \to B_\star$, and whose morphisms are maps $T_\star \to T'_\star$ compatible with the structure maps. As a direct consequence of the last proposition we have the following two corollaries.

[3]Quillen [12] II.4. Theorem 4.

Corollary 2.2.8. *Let $i : A_\star \to B_\star$ be a cofibration and $p : X_\star \to Y_\star$ an acyclic fibration in $\mathcal{S}\mathbf{Alg}_k$. Then given a commutative square*

$$\begin{array}{ccc} A_\star & \xrightarrow{u} & X_\star \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B_\star & \xrightarrow{v} & Y_\star \end{array}$$

any two choices for the dotted arrow are homotopic in $\mathbf{Alg}_A/Y$.

Proof. Let us call $\theta_0, \theta_1 : B_\star \longrightarrow X_\star$ any two such lifts. Then last proposition for h and k the constant maps at u, v respectively gives a map $\ell : B_\star \otimes \Delta^1 \longrightarrow Y_\star$ such that

$$\begin{array}{ccccc} B_\star & \xrightarrow{\mathrm{id}\times d^1} & B_\star \otimes \Delta^1 & \xleftarrow{\mathrm{id}\times d^0} & B_\star \\ & {\scriptstyle \theta_1}\searrow & \downarrow{\scriptstyle \ell} & \swarrow{\scriptstyle \theta_0} & \\ & & X_\star & & \end{array}$$

commutes, so it is in fact a homotopy from θ_1 to θ_2. □

Corollary 2.2.9. *The cofibrant-acyclic fibrant factorization M5 (i) of a map $u : A_\star \longrightarrow B_\star$ in $\mathcal{S}\mathbf{Alg}_k$ is unique up to simplicial homotopy in the category $\mathbf{Alg}_A/B$.*

Proof. If there are two such factorizations

$$A_\star \xrightarrow{i} T_\star \xrightarrow{p} B_\star, \qquad A_\star \xrightarrow{i'} T'_\star \xrightarrow{p'} B_\star$$

for a given map $u : A_\star \to B_\star$ in $\mathcal{S}\mathbf{Alg}_k$, the following commutative diagrams

$$\begin{array}{ccc} A_\star & \xrightarrow{i'} & T'_\star \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p'} \\ T_\star & \xrightarrow{p} & B_\star \end{array} \qquad \begin{array}{ccc} A_\star & \xrightarrow{i} & T_\star \\ {\scriptstyle i'}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ T'_\star & \xrightarrow{p'} & B_\star \end{array}$$

provide maps $\varphi : T_\star \to T'_\star$, $\psi : T'_\star \to T_\star$. Then, both maps $\psi\varphi$ and $\mathrm{id}_{T_\star}$ solve the lifting problem in the following left diagram, and both $\varphi\psi$ and $\mathrm{id}_{T'_\star}$ are lifts in the following right diagram

$$\begin{array}{ccc} A_\star & \xrightarrow{i} & T_\star \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ T_\star & \xrightarrow{p} & B_\star \end{array} \qquad \begin{array}{ccc} A_\star & \xrightarrow{i'} & T'_\star \\ {\scriptstyle i'}\downarrow & \nearrow & \downarrow{\scriptstyle p'} \\ T'_\star & \xrightarrow{p'} & B_\star \end{array}$$

Thus, by Corollary 2.2.8, $\varphi\psi$ and $\psi\varphi$ are homotopic to $\mathrm{id}_{T'_\star}$ and $\mathrm{id}_{T_\star}$ respectively in $\mathbf{Alg}_A/B$. □

Definition. Let $R \in \mathbf{Alg}_k$. A *simplicial resolution of* R is an acyclic fibration $E_\star \longrightarrow cR$ in the model category $\mathcal{S}\mathbf{Alg}_k$.

Remark. Let $R \in \mathbf{Alg}_k$. A simplicial object $P_\star$ in the category $\mathbf{Alg}_k/R$ is just a factorization

$$ck \longrightarrow P_\star \longrightarrow cR \tag{2.3}$$

of the map $ck \longrightarrow cR$. Hence, a simplicial cofibrant k-algebra resolution of R is just a simplicial object in $\mathbf{Alg}_k/R$, $P_\star$, such that the factorization (2.3) is a cofibrant-acyclic fibrant factorization in $\mathcal{S}\mathbf{Alg}_k$.

For the rest of this section, we fix $\perp$ to be the cotriple on $\mathbf{Alg}_k$ constructed in Example 2.1.2, so that $\perp R$ is the polynomial algebra con the underlying set of the k-algebra R. Let $j : k \hookrightarrow k[R] = \perp R$ be the structure map, and

$$i : ck \longrightarrow \perp_\star R, \qquad p : \perp_\star R \longrightarrow cR$$

be the maps given in degree $n \geqslant 0$ by the composition

$$i_n = s_{n-1}s_{n-2}\dots s_0\, j = s_0^n\, j : k \longrightarrow \perp_n R, \qquad p_n = \varepsilon d_0 d_1 \dots d_{n-1} = \varepsilon d_0^n : \perp_n R \longrightarrow R.$$

Definition. A map of simplicial rings $A_\star \to B_\star$ is *free* if for all $q \geqslant 0$ there are subsets $C_q \subseteq B_q$ such that

(i) $\eta^\star C_q \subseteq C_p$ for every surjective map $\eta : \mathbf{p} \to \mathbf{q}$ in Δ,

(ii) B_q is a free A_q - algebra with generators C_q.

Example 2.2.10. The map $i : ck \longrightarrow \perp_\star R$ is a free map, where

$$C_q = \{e_r \mid r \in \perp_{q-1} R\} \subseteq k[\perp_{q-1} R] = \perp_q R, \quad \text{for } q \geqslant 0,$$

with $\perp_{-1} R = R$. From the explicit description of the unit η in Example 2.1.2, we see that $\delta : \perp \longrightarrow \perp^2$ is given by

$$\begin{array}{rccc} \delta_A : & k[A] & \longrightarrow & k[k[A]] \\ & e_a & \longmapsto & e_{e_a}\,. \end{array}$$

for any k-algebra A. Given any surjective map $\eta : \mathbf{p} \to \mathbf{q}$ in Δ, we can write it as a compostion $\eta = s^{j_1} \cdots s^{j_t}$ of codegeneracy maps. But for $0 \leqslant j \leqslant h$, the map

$$\begin{array}{rccc} \delta\perp^{h-j} : & \perp^{h-j+1} R & \longrightarrow & \perp^{h-j+2} R, \\ & e_r & \longmapsto & e_{e_r} \end{array} \qquad \text{for any } r \in \perp^{h-j} R,$$

sends $\delta\perp^{h-j} C_{h-j} \subseteq C_{h-j+1}$. Recall that for $\varphi : R \longrightarrow S$ a map in $\mathbf{Alg}_k$, $\perp(\varphi)$ is given by

$$\begin{array}{rccc} \perp(\varphi) : & k[R] & \longrightarrow & k[S], \\ & e_r & \longmapsto & e_{\varphi(r)} \end{array} \qquad \text{for any } r \in R.$$

Hence, the map

$$s_j = \perp^j \delta \perp^{h-j} : \perp_h R \longrightarrow \perp_{h+1} R$$

sends basis elements in $\perp_h R$ to basis elements in $\perp_{h+1}$. Therefore

$$\eta^\star C_q = (s_{j_t} \cdots s_{j_1}) C_q \subseteq (s_{j_t} \cdots s_{j_2}) C_{q+1} \subseteq \dots \subseteq C_p.$$

Proposition 2.2.11. *Any free map of simplicial rings is a cofibration.*

Proof. See Goerss and Jardine [3] VII. Example 1.14. □

Proposition 2.2.12. *The composition*

$$ck \xrightarrow{\ i\ } \perp_\star R \xrightarrow{\ p\ } cR$$

is a cofibrant-acyclic fibrant factorization of the canonical map $ck \longrightarrow cR$. Therefore, $\perp_\star R$ is a cofibrant simplicial k-algebra resolution of R.

Proof. The previous example and the last proposition show that i is a cofibration. On the other hand, the map p is an acyclic fibration if the induced map $U(p) : U(\perp_\star R) \to U(cR)$ is an acyclic fibration. Take the map $g : U(cR) \longrightarrow U(\perp_\star R)$ given by $g_n : R \to U(\perp_n R)$ with $g_n(r) = s_0^n(e_r)$ for $r \in R$ and $n \geqslant 0$. Then, g is a section for $U(p)$, i.e., $U(p)\,g = \mathrm{id}$, and thus p is an acyclic fibration. The only thing we are left to show is that the composition $p\,i$ is the structure map $ck \longrightarrow cR$, but this is clear since the composition $\varepsilon\, j$ is precisely the structure map $k \longrightarrow R$. □

Corollary 2.2.13. *For any k-algebra R there is always a unique cofibrant simplicial k-algebra resolution of R up to simplicial homotopy.*

Proof. Direct consequence of Proposition 2.2.12 and Corollary 2.2.9. □

We denote by $\Omega_{R/k}$ the R-module of Kähler differentials of R over k, and by $\mathrm{Der}_k(R,M)$ the R-module of k-derivations $R \longrightarrow M$ for an R-module M, so that

$$\mathrm{Der}_k(R,M) \cong \mathrm{Hom}_R(\Omega_{R/k}, M).$$

Let $R \in \mathbf{Alg}_k$ and $M \in \mathbf{Mod}_R$. If we apply the functor $\mathrm{Der}_k(\cdot, M) : \mathbf{Alg}_k \to \mathbf{Mod}_R$ to a simplicial k-algebra $P_\star$, with $\varepsilon : P_\star \to \mathrm{c}R$, we obtain a cosimplicial R-module $\mathrm{Der}_k(P_\star, M)$, given by

$$\mathbf{n} \mapsto \mathrm{Der}_k(P_n, M).$$

Note that M is a P_n - module via the map ε_n. Moreover, if $P_\star$ and $Q_\star$ are two simplicial cofibrant k-algebra resolutions of R, then the simiplicial homotopy equivalence $P_\star \simeq Q_\star$ from Corollary 2.2.9 induces a simplicial homotopy equivalence

$$\mathrm{Der}_k(P_\star, M) \simeq \mathrm{Der}_k(Q_\star, M).$$

Definition (André-Quillen)**.** The *cohomology* of the k-algebra R with values in the R-module M is the sequence of R-modules

$$D^n(R/k, M) = \pi^n \mathrm{Der}_k(P_\star, M), \quad \text{for } n \geqslant 0$$

where $P_\star$ is a simplicial cofibrant k-algebra resolution of R.

Although this definition seems more general, using Proposition 2.2.12 we realize that it is just a cotriple cohomology.

Theorem 2.2.14. *The cohomology of R with coefficients in an R-module M is the cotriple cohomology of R with values in* $Der_k(\cdot, M)$*, i.e.,*

$$D^n(R/k, M) = H^n(R, Der_k(\cdot, M)) = \pi^n Der_k(\perp_\star R, M), \quad n \geqslant 0.$$

In order to better understand this cohomology, and to see how homology can be defined, we introduce the cotangent complex. We consider the functor

$$\begin{array}{rccc} L: & \mathbf{Alg}_k/R & \longrightarrow & \mathbf{Mod}_R \\ & P & \longmapsto & R \otimes_P \Omega_{P/k} \end{array}.$$

As before, if $P_\star$ and $Q_\star$ are two simplicial cofibrant k-algebra resolutions of R, then we also have a homotopy equivalence $L(P_\star) \simeq L(Q_\star)$.

Definition. The *cotangent complex* $\mathbb{L}_{R/k}$ of the k-algebra R is the simplicial R-module $L(P_\star)$ given by

$$\mathbf{n} \mapsto R \otimes_{P_n} \Omega_{P_n/k} \; ,$$

where $P_\star$ is a simplicial cofibrant k-algebra resolution of R.

Proposition 2.2.15. *For all $n \geqslant 0$,*

$$D^n(R/k, M) \cong \pi^n Hom_R(\mathbb{L}_{R/k}, M).$$

Proof. Using the cotriple resolution $\perp_\star R \to R$, we just need to show that

$$\mathrm{Der}_k(\perp_\star R, M) \cong \mathrm{Hom}_R(\mathbb{L}_{R/k}, M).$$

Let $n \geqslant 0$. On the one hand, since $\perp_n R = k[\perp^n R]$, then $\Omega_{\perp_n R/k}$ is just $(\perp_n R)^{\perp^n R}$, the free $(\perp_n R)$-module with basis $\{dx : x \in \perp^n R\}$. On the other hand,

$$\mathrm{Hom}_{k[R]}(k[R], M) \cong \mathrm{Hom}_R(R, M) \Longrightarrow \mathrm{Hom}_{\perp_n R}(\perp_n R, M) \cong \mathrm{Hom}_R(R, M),$$

and therefore

$$\begin{aligned} \mathrm{Der}_k(\perp_n R, M) &\cong \mathrm{Hom}_{\perp_n R}((\perp_n R)^{\perp^n R}, M) \cong \mathrm{Hom}_R(R, M)^{\perp^n R} \\ &\cong \mathrm{Hom}_R(R \otimes_{\perp_n R} \perp_n R, M)^{\perp^n R} \cong \mathrm{Hom}_R(R \otimes_{\perp_n R} (\perp_n R)^{\perp^n R}, M). \end{aligned}$$

□

This result motivates the following definition for homology.

Definition (André Quillen)**.** The *homology* of R with values in an R-module M is the sequence of R-modules

$$D_n(R/k, M) = \pi_n(\mathbb{L}_{R/k} \otimes_R M), \quad n \geqslant 0.$$

When $M = R$ we write $D_\star(R/k)$ for $D_\star(R/k, R)$.

The same way we did for cohomology, we can also see this homology as a cotriple homology for some specific setting. We can extend the cotriple $\perp$ in $\mathbf{Alg}_k$ to a cotriple in the category $\mathbf{Alg}_k/R$, with $\perp(P) = \perp(P, u : P \to R) = (k[P], \tilde{u} : k[P] \to R)$ where $\tilde{u}$ sends any $p \in P$ to $u(p) \in R$. By abuse of notation we also call $\perp$ the induced cotriple in $\mathbf{Alg}_k/R$. Hence, we have:

Theorem 2.2.16. *The homology of R with values in an R-module M is the cotriple homology of R with coefficients in $L(\,\cdot\,) \otimes_R M$, i.e.,*

$$D_n(R/k, M) = H_n(R, L(\,\cdot\,) \otimes_R M) = \pi_n(\mathbb{L}_{R/k} \otimes_R M), \quad n \geqslant 0.$$

Example 2.2.17. If R is a polynomial k-algebra, then the trivial resolution $P_\star \longrightarrow R$, with $P_n = R$, $\partial_n = \mathrm{id}_R$ for all $n \geqslant 0$ is a simplicial polynomial resolution of R. Hence, for any R-module M and $i \neq 0$,

$$D^i(R/k, M) = D_i(R/k, M) = 0.$$

2.3 Computations in low degrees

For $R \in \mathbf{Alg}_k$, an extension of R by an R-module M is an exact sequence

$$0 \longrightarrow M \xrightarrow{i} E \xrightarrow{u} R \longrightarrow 0$$

where u is a map in $\mathbf{Alg}_k$ such that $\ker(u)^2 = 0$, i induces an isomorphism of R-modules $M \cong \ker(u)$, and the R-module structure of $\ker(u)$ is induced by u. Two extensions (E,i,u), (E',i',u') of R by M are equivalent if there is an isomorphism of k-algebras $f : E \to E'$ such that the following diagram

$$\begin{array}{ccccccc} & & & E & & & \\ 0 \longrightarrow M & \overset{i}{\nearrow} & & \downarrow f & & \overset{u}{\searrow} & R \longrightarrow 0 \\ & \underset{i'}{\searrow} & & E' & & \underset{u'}{\nearrow} & \end{array}$$

is commutative. An extension (E,i,u) is trivial if there is a k-algebra homomorphism $s : R \to E$ such that $us = \mathrm{id}_R$. In this case we say that the short exact sequence splits. We obtain a trivial extension of R by M by giving the module $M \oplus R$ a k-algebra structure via

$$(m_1,r_1)(m_2,r_2) = (r_1m_2 + r_2m_1, r_1r_2), \quad m_1,m_2 \in M,\, r_1,r_2 \in R.$$

We get

$$0 \longrightarrow M \xrightarrow{i} M \oplus R \xrightarrow{u} R \longrightarrow 0$$

where the maps are given by $i(m) = (m,0)$ and $u(m,r) = r$ for $m \in M$, $r \in R$. The structure map on $M \oplus R$ is given by $r \mapsto (0,r)$. We denote by $\mathrm{Exalcomm}_k(R,M)$ the set of isomorphism classes of extensions of R by M, and $M \ltimes R$ the equivalence class of the trivial extension we just defined. Notice that any trivial extension will be in the equivalence class $M \ltimes R$.

Lemma 2.3.1. *Let $R \in \mathbf{Alg}_k$, $M \in \mathbf{Mod}_R$ and $E \in \mathbf{Alg}_k/R$ with $w : E \to R$. We see M as an E-module via the map w. There is a bijection*

$$\begin{array}{rccc} v_{(-)}: & Der_k(E,M) & \xrightarrow{\sim} & Hom_{\mathbf{Alg}_k/R}(E, M \ltimes R) \\ & D & \longmapsto & (e \mapsto (D(e), w(e))). \end{array}$$

which is natural in E.

Proof. The map is well defined. For any $D \in \mathrm{Der}_k(E,M)$, v_D is a k-homomorphism since the Leibniz rule satisfied by D is compatible with the product defined in $M \ltimes R$. Moreover, $u(D(e),w(e)) = w(e)$, so there is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{v_D} & M \ltimes R \\ & {\scriptstyle w}\searrow \quad \swarrow{\scriptstyle u} & \\ & R & \end{array}$$

and v_D is in fact a map in $\mathbf{Alg}_k/R$. Let $D, D' \in \mathrm{Der}_k(E,M)$ such that $v_D = v_{D'}$ and call $\mathrm{pr}_M : M \ltimes R \to M$ the projection to the M factor. Then $D = \mathrm{pr}_M v_D = \mathrm{pr}_M v_{D'} = D'$.
On the other hand, if $v : E \to M \ltimes R$ is a map in $\mathbf{Alg}_k/R$, then $w = uv$. Take $D = \mathrm{pr}_M v$. D is a k-homomorphism, and for $e, f \in E$,

$$v(ef) = v(e)v(f) = (D(e),w(e))(D(f),w(f)) = (w(f)D(e) + w(e)D(f), w(e)w(f)),$$

so $D(ef) = w(e)D(f) + w(f)D(e)$, which means that D is in fact a k-derivation from E to M. Naturality is clear recalling that for any map $\varphi : E \longrightarrow S$ in $\mathbf{Alg}_k/R$ the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & S \\ & {\scriptstyle w'}\searrow \quad \swarrow{\scriptstyle w} & \\ & R & \end{array}$$

commutes and hence $w\varphi = w'$. □

Corollary 2.3.2. *There is a pair of adjoint functors*

$$L : \mathbf{Alg}_k/R \rightleftarrows \mathbf{Mod}_R : T$$

where $T(M) = M \ltimes R$*, for any* $M \in \mathbf{Mod}_R$.

Proof. Direct consequence of the Lemma. For any $M \in \mathbf{Mod}_R$ and $E \in \mathbf{Alg}_k/R$,

$$\mathrm{Hom}_{\mathbf{Alg}_k/R}(E, M \ltimes R)) \cong \mathrm{Der}_k(E,M) \cong \mathrm{Hom}_E(\Omega_{E/k}, M) \cong \mathrm{Hom}_{\mathbf{Mod}_R}(\Omega_{E/k} \otimes_E R, M).$$

□

Proposition 2.3.3. *Let* $R \in \mathbf{Alg}_k$*,* $M \in \mathbf{Mod}_R$*, then*

(a) $D^0(R/k,M) \cong \mathit{Der}_k(R,M)$, *and* $D_0(R/k,M) \cong \Omega_{R/k} \otimes_R M$,

(b) $D^1(R/k,M) \cong \mathit{Exalcomm}_k(R,M)$.

Proof. (a) Recall that ker $\varepsilon = \mathrm{im}\,(d_0 - d_1)$. Hence elements in $D^0(R/k,M)$ are just derivations $D \in \mathrm{Der}_k(\perp R, M)$ such that $D(d_0 - d_1) = 0$, and therefore there is a factorization

$$\begin{array}{ccc} \perp R & \xrightarrow{D} & M \\ \downarrow & & \uparrow{\scriptstyle D'} \\ \perp R/\mathrm{im}(d_0 - d_1) & \xrightarrow[\varepsilon]{\sim} & R \end{array}$$

where $D' \in \mathrm{Der}_k(R,M)$. For homology, by the last corollary, the functor L is left exact, so if we apply it to the exact sequence

$$\perp^2 R \xrightarrow{d_0 - d_1} \perp R \xrightarrow{\varepsilon} R \longrightarrow 0$$

we get $\Omega_{R/k} \cong D_0(R/k,R)$. Now, since $\cdot \otimes_R M$ is right exact, it follows

$$\Omega_{R/k} \otimes_R M \cong D_0(R/k,M).$$

(b) The forgetful functor

$$(\perp^2 R)\text{ -mod} \longrightarrow \mathbf{Set}$$

has left adjoint

$$\mathbf{Set} \longrightarrow (\perp^2 R)\text{ -mod},$$

where every set X is sent to $(\perp^2 R)^X$, the free $(\perp^2 R)$-module on X. Since $\perp^2 R = k[\perp R]$, then $\Omega_{\perp^2 R/k}$ is just the free module on the underlying set of $\perp R$, $(\perp^2 R)^{\perp R}$. This way

$$\mathrm{Hom}_{\mathbf{Set}}(\perp R, M) \cong \mathrm{Hom}_{\perp^2 R}((\perp^2 R)^{\perp R}, M) \cong \mathrm{Hom}_{\perp^2 R}(\Omega_{\perp^2 R/k}, M) \cong \mathrm{Der}_k(\perp^2 R, M),$$

where the composite isomorphism sends a map of sets $\varphi : \perp R \to M$ to a map $\perp^2 R \to M$, that sends the elements $e_b \mapsto \varphi(b)$ for all $b \in \perp R$ and is extended k-linearly. For every k-extension E of R by M we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & E & \xrightarrow{u} & R & \longrightarrow & 0 \\ & & & & & \nwarrow^{\theta} & \uparrow{\scriptstyle \varepsilon} & & \\ & & & & & & \perp R & & \end{array} \tag{2.4}$$

where i is an isomorphism to ker u, and θ makes the diagram commute and is obtained since $\perp R$ is a polynomial k-algebra, and therefore projective. Recall that $\varepsilon d_0 - \varepsilon d_1 = 0$, so

$$u(\theta d_1 - \theta d_0) = u\theta d_1 - u\theta d_0 = \varepsilon d_1 - \varepsilon d_0 = 0$$

and we can consider the k-module homomorphism

$$\theta(d_1 - d_0) : \perp^2 R \to \ker(u).$$

The $(\perp^2 R)$-module structure of $\ker(u)$ is given by $a \cdot \ell = b\ell$ where $b \in E$ is such that $u(b) = \varepsilon d_0(a)$, for any $a \in \perp^2 R$ and $\ell \in \ker(u)$. For very $a, b \in \perp^2 R$,

$$\begin{aligned} (d_1 - d_0)(ab) &= d_1(a)d_1(b) - d_0(a)d_0(b) + d_1(a)d_0(b) - d_1(a)d_0(b) \\ &= d_1(a)(d_1 - d_0)(b) + d_0(b)(d_1 - d_0)(a). \end{aligned}$$

and hence

$$\theta(d_1 - d_0)(ab) = b \cdot \theta(d_1 - d_0)(a) + a \cdot \theta(d_1 - d_0)(b)$$

since $u(\theta d_1(c)) = u(\theta d_0(c))$ for all $c \in \perp^2 R$. Therefore $\theta(d_1 - d_0)$ is a k-derivation, and using the induced isomorphism $i : M \xrightarrow{\sim} \ker(u)$ we obtain a k-derivation

$$D = i^{-1}\theta(d_1 - d_0) : \perp^2 R \to M.$$

We also have

$$\partial^2(D) = D\partial_2 = i^{-1}[\theta d_1 d_0 - \theta d_1 d_1 + \theta d_1 d_2 - \theta d_0 d_0 + \theta d_0 d_1 - \theta d_0 d_2] = i^{-1}(0) = 0,$$

so $D \in \ker(\partial^2)$. For any other lifting $\theta' : \perp R \to E$ with $u\theta = \varepsilon$, $\mathrm{im}(\theta - \theta') \in \ker u \cong M$, and for all $a, b \in \perp R$, we have

$$\begin{aligned} (\theta - \theta')(ab) &= \theta(a)\theta(b) - \theta'(a)\theta'(b) + \theta(a)\theta'(b) - \theta(a)\theta'(b) \\ &= a \cdot (\theta - \theta')(b) + b \cdot (\theta - \theta')(a) \end{aligned}$$

again since $u\theta(c) = u\theta'(c)$ for all $c \in \perp R$. Thus, θ' is of the form $\theta' = \theta + D'$ for some $D' \in \mathrm{Der}_k(\perp R, M)$. Hence, the class of D in $D^2(R/k, M)$ does not depend on the choice of the lift θ. We get a well-defined map

$$\begin{array}{rccc} \varphi : & \mathrm{Exalcomm}_k(R, M) & \longrightarrow & D^1(R/k, M) \\ & (E, u) & \longmapsto & [D] \end{array}$$

where $[D]$ denotes the class of D in $D^1(R/k, M)$. Conversely, let $D \in \mathrm{Der}_k(\perp^2 R, M)$, we define $f = (d_0 - d_1, D) : \perp^2 R \to \perp R \oplus M$, and $E = \mathrm{coker}(f)$ such that there is a k-extension of R by M given by

$$0 \longrightarrow M \xrightarrow{i} E \xrightarrow{u} R \longrightarrow 0$$

where $i(m) = [(0,m)]$, and $u([(y,m)]) = \varepsilon(y)$. The map u is well-defined since for any $h \in \perp^2 R$

$$u([d_0 - d_1(h), D(h)]) = \varepsilon(d_0 - d_1(h)) = 0.$$

Moreover, since $\ker\varepsilon = \mathrm{im}\,(d_0 - d_1)$, $\ker u = \{[0,m] \in E\}$. Clearly $(\ker u)^2 = 0$. If $D \in \ker(\partial^2)$, then i induces an isomorphism between M and $\ker u$

$$\begin{aligned} \ker(i) &= \{m \in M \mid D(b) = m, \text{ for some } b \in \perp^2 R \text{ such that } d_0 - d_1(b) = 0\} \\ &= \{m \in M \mid D(\partial^2(c)) = m, \text{ for some } c \in \perp^3 R\} = 0. \end{aligned}$$

This defines a map

$$\begin{array}{rccc} \Psi: & D^1(R/k, M) & \longrightarrow & \mathrm{Exalcomm}_k(R,M) \\ & [D] & \longmapsto & (\perp R \oplus M/\mathrm{im}(d_0 - d_1, D), i, u) \end{array}$$

which is inverse to φ. To see this, note that the map $\theta : \perp R \to \mathrm{coker}(f)$ given by $\theta(b) = [b, 0]$ for $b \in \perp R$ is well defined and makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & \mathrm{coker}(f) & \xrightarrow{u} & R & \longrightarrow & 0 \\ & & & & & \nwarrow^{\theta} & \uparrow{\scriptstyle\varepsilon} & & \\ & & & & & & \perp R & & \end{array}$$

commute. Then, $\varphi(\mathrm{coker}(f), i, u)$ is given by the class of the derivation

$$i^{-1}\theta(d_1 - d_0) : \perp^2 R \to M,$$

with

$$i^{-1}\theta(d_1 - d_0)(b) = i^{-1}[(d_1 - d_0)(b), 0] = i^{-1}[0, D(b)] = D(b),$$

so $[i^{-1}\theta(d_1 - d_0)] = [D]$, i.e., $\Phi\Psi = \mathrm{id}$. On the other hand, given an extension like (2.4), take $D = i^{-1}(\theta d_1 - \theta d_0)$ and consider the map

$$\begin{array}{rccc} (\theta + i): & \perp R \oplus M & \longrightarrow & E \\ & (b,m) & \longmapsto & \theta(b) + i(m). \end{array}$$

If $(b,m) \in \ker(\theta + i)$, then we get $\theta(b) = i(-m)$, so $\theta(b) \in \mathrm{im}(i) = \ker(u)$ which means that $b \in \ker(\varepsilon) = \mathrm{im}(d_0 - d_1)$. Then $b = (d_0 - d_1)(c)$ for some $c \in \perp^2 R$, and therefore $i(-m) = \theta(d_0 - d_1)(c)$, so $m = i^{-1}(\theta d_1 - \theta d_0)(c) = D(c)$. Thus, $\ker(\theta + i) \subseteq \mathrm{im}(d_0 - d_1, D)$. Moreover, for any $c \in \perp^2 R$ and $f = (d_0 - d_1, D) : \perp^2 R \to \perp R \oplus M$,

$$(\theta + i)(f(c)) = \theta((d_0 - d_1)(c)) + i(D(c)) = \theta((d_0 - d_1)(c)) + \theta((d_1 - d_0)(c)) = 0,$$

so $(\theta + i)f = 0$ and by the universal property of the cokernel we get an injective map

$$\begin{array}{rccc} h: & \perp R \oplus M/\mathrm{im}(d_0 - d_1, D) & \longrightarrow & E \\ & [b,m] & \longmapsto & \theta(b) + i(m). \end{array}$$

Finally,

$$\begin{array}{ccccccc} E & \overset{p}{\twoheadrightarrow} & R & \xrightarrow{\sim} & R/\ker(p) & \xrightarrow{\sim} & R/\mathrm{im}(i) \\ & \nwarrow^{\theta} & \uparrow{\scriptstyle\varepsilon} & & & & \\ & & \perp R & & & & \end{array}$$

means that there is a surjective map

$$\mathrm{im}(\theta) \twoheadrightarrow R/\mathrm{im}(i)$$

so $R \cong \mathrm{im}(\theta) + \mathrm{im}(i)$ and our map h is also surjective. So it is an isomorphism and makes the following diagram commutative

$$\begin{array}{ccccccc} & & & \perp R \oplus M/\mathrm{im}(d_0 - d_1, D) & & & \\ & & {\scriptstyle i'} \nearrow & \downarrow {\scriptstyle h} & \searrow {\scriptstyle u'} & & \\ 0 \longrightarrow & M & & & & R & \longrightarrow 0 \\ & & {\scriptstyle i} \searrow & E & \nearrow {\scriptstyle u} & & \end{array}$$

where $i'(m) = [(0,m)]$, and $u'[y,m] = \varepsilon(y)$ for $m \in M, y \in \perp R$. Therefore, the class of $\Psi(\Phi(E))$ is the same as the class of E in $\mathrm{Exalcomm}_k(R,M)$, and $\Psi\Phi = \mathrm{id}$. □

Let $E, R \in \mathbf{Alg}_k$, and let $w : E \longrightarrow R$ be a k-algebra map. Let $M \in \mathbf{Mod}_R$. For any k-extension of R by M

$$0 \longrightarrow M \xrightarrow{i} A \xrightarrow{u} R \longrightarrow 0$$

we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & A & \xrightarrow{u} & R & \longrightarrow & 0 \\ & & \cong \uparrow & & p_1 \uparrow & & w \uparrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & A \times_R E & \xrightarrow{p_2} & E & \longrightarrow & 0 \end{array}$$

where M' is just the kernel of $A \times_R E \longrightarrow E$ which is isomorphic to $\ker u = \mathrm{im}\, i \cong M$. Hence, the k-algebra $A \times_R E$ becomes an extension of E by M. Moreover, if B is another extension of R by M, which is equivalent to A, then $B \times_R E$ is also equivalent to $A \times_R E$. This defines a map

$$\begin{array}{rccc} w^1 : & \mathrm{Exalcomm}_k(R,M) & \longrightarrow & \mathrm{Exalcomm}_k(E,M) \\ & A & \longmapsto & A \times_R E. \end{array}$$

On the other hand, if $j : k \longrightarrow E$, $w : E \longrightarrow R$ are maps of rings, then for every E-extension A of R by an R-module M, we can see A and E as k-algebras via j and hence we get a k-extension of R by M. This defines a map

$$j^1 : \ \mathrm{Exalcomm}_E(R,M) \longrightarrow \mathrm{Exalcomm}_k(R,M).$$

Proposition 2.3.4. *Let k, E, R be commutative rings, $k \xrightarrow{j} E$, $E \xrightarrow{w} R$ two ring homomorphisms and let M be an R-module, which is also an E-module via w. There is an exact sequence*

$$0 \longrightarrow Der_E(R,M) \xrightarrow{j^0} Der_k(R,M) \xrightarrow{w^0} Der_k(E,M) \xrightarrow{\partial} Exalcomm_E(R,M) \xrightarrow{j^1} Exalcomm_k(R,M) \xrightarrow{w^1} Exalcomm_k(E,M)$$

where for any $D \in Der_k(E,M)$, $\partial(D)$ is the class of the E-extension of R by M defined by the structure map $v_D : E \longrightarrow M \ltimes R$.

Proof. Exactness at $\mathrm{Der}_E(R,M)$ and at $\mathrm{Der}_k(R,M)$ follow from the first fundamental exact sequence for differentials [4]. The kernel of ∂ is given by derivations $D \in \mathrm{Der}_k(E,M)$ such that the E-extension $\partial(D)$ is E-trivial, i.e., the extensions

$$0 \longrightarrow M \xrightarrow{\ i\ } \partial(D) \underset{s}{\overset{u}{\rightleftarrows}} R \longrightarrow 0$$

such that there is an E-homomorphism $s : R \longrightarrow \partial(D)$, $c \mapsto (\tilde{s}(c), c)$. Furthermore, this is a k-homomorphism, and $us = \mathrm{id}_R$, so using Lemma 2.3.1. for $E = R$ and $w = \mathrm{id}_R$, we see that such s is of the form $c \mapsto (c, D'(c))$ where $D' \in \mathrm{Der}_k(R,M)$. By the commutativity of the diagram

$$\begin{array}{ccc} & E & \\ {}^{v_D}\swarrow & & \searrow^{w} \\ M \ltimes R & \underset{s}{\overset{u}{\rightleftarrows}} & R \end{array}$$

we see that for every $e \in E$,

$$(D(e), w(e)) = v_D(e) = s(w(e)) = (D'(w(e)), w(e)),$$

and hence $D(e) = D'(w(e))$, so $D = D'w = w^0(D')$. This shows exactness at $\mathrm{Der}_k(E,M)$.
For any $D \in \mathrm{Der}_k(E,M)$, there is a k-homomorphism $R \to \partial(D)$ given by $c \mapsto (c, 0)$, which makes the sequence

$$0 \longrightarrow M \longrightarrow \partial(D) \longrightarrow R \longrightarrow 0$$

a split short exact sequence and therefore $\partial(D)$ is a trivial k-extension of R by M. Thus, $j^1\partial = 0$. The kernel of j^1 is given by the E-extensions of R by M that are k-trivial when seen as k-algebras via j, so we can assume they are E-algebras over the k-algebra $M \ltimes R$. Note that w is trivially a map of $\mathbb{Z}$-algebras, so by Lemma 2.3.1. any structure of E-algebra on $M \ltimes R$ is given by a map

$$\begin{array}{ccc} E & \longrightarrow & M \ltimes R \\ e & \longmapsto & (D(e), w(e)) \end{array}$$

where $D \in \mathrm{Der}_{\mathbb{Z}}(E,M)$. The structure of k-algebra can hence be seen as $t \mapsto (0, w(j(t)))$ where the diagram

$$\begin{array}{ccc} & k & \\ {}^{j}\swarrow & & \searrow \\ E & \longrightarrow & M \ltimes R \end{array}$$

must be commutative. Hence, $(0, w(j(t))) = (D(j(t)), w(j(t))$, so $D(j(t)) = D(0)$ for all $t \in k$. This means that D is a k-derivation, and the extension defined on $M \ltimes R$ is precisely $\partial(D)$. This shows exactness at $\mathrm{Exalcomm}_E(R,M)$.
The kernel of w^1 is given by the k-extensions A of R by M

$$0 \longrightarrow M \xrightarrow{\ i\ } A \xrightarrow{\ u\ } R \longrightarrow 0$$

such that the k-extension of E by M

$$0 \longrightarrow M \longrightarrow A \times_R E \underset{s}{\overset{p_2}{\rightleftarrows}} E \longrightarrow 0$$

[4] Matsumara [10] Theorem 25.1.

is trivial. This means that there is some k-homomorphism $s : E \longrightarrow A \times_R E$ with $p_2 s = \mathrm{id}_E$. But such an s induces a k-homomorphism $s' = p_1 s : E \longrightarrow A$ and a commutative diagram

$$\begin{array}{ccccc} & & E & & \\ & \overset{j}{\nearrow} & \Big\downarrow{\scriptstyle s'} & \overset{w}{\searrow} & \\ k & & & & R. \\ & \searrow & & \underset{u}{\nearrow} & \\ & & A & & \end{array}$$

Thus, A can be seen as an E-algebra via s', and the image of this E-algebra under j^1 is precisely our original k-algebra A. Therefore $\ker w^1 \subseteq \operatorname{im} j^1$. On the other hand, for every E-extension A of R by M, the structure map $E \longrightarrow A$ gives a k-homomorphism $s : E \longrightarrow A \times_R E$ with $s\,p_2 = \mathrm{id}_E$, so that the k-extension

$$0 \longrightarrow M \longrightarrow A \times_R E \underset{s}{\overset{p_2}{\rightleftarrows}} E \longrightarrow 0$$

is trivial. This means that $w^1\, j^1 = 0$, and hence the sequence is also exact at $\mathrm{Exalcomm}_k(R, M)$.
$\square$

Chapter 3

(Co-) Homology for Universal Algebras

We begin this chapter by pointing out some important properties of the cotangent complex and its direct implications to the cohomology of commutative rings. Then, we will extend these definitions to more general categories, and we will see how this cohomology is related to other cohomology theories.

3.1 The cotangent complex

Let k, R be commutative rings (with 1). In the previous chapter we saw that any map of simplicial rings of the form $\mathrm{c}k \to \mathrm{c}R$ had a cofibrant factorization (unique up to simplicial homotopy). Moreover, this factorization came through a simplicial ring $\perp_\star R$ which is a free k-algebra in each degree. From now on, we will omit the star ($_\star$) notation for simplicial objects in order to simplify notation.

Remark. Any map of simplicial rings $A \longrightarrow B$ admits also a cofibrant-acyclic fibrant factorization where the first map is free [1], which by Corollary 2.2.9 is also unique up to simplicial homotopy. We call this a free A-algebra resolution of B. Note how this just extends what happens for constant simplicial rings to arbitrary simplicial rings.

Definition. A simplicial module P over a simplicial ring A is *free* if for all $q \geqslant 0$ there are subsets $C_q \subseteq P_q$ such that

(i) $\eta^\star C_q \subseteq C_p$ for every surjective map $\eta : \mathbf{p} \to \mathbf{q}$ in Δ,

(ii) P_q is a free A_q-module with basis C_q.

Proposition 3.1.1. *Let $R \in \mathbf{Alg}_k$. The cotangent complex $\mathbb{L}_{R/k}$ is a free simplicial R-module.*

Proof. Let $\mathrm{c}k \longrightarrow \perp_\star R \longrightarrow \mathrm{c}R$ be the free cotriple k-algebra resolution of R. For each $\perp_n R$ we call C_n the set of generators $\{e_r \mid r \in \perp_{n-1} R\} \subseteq \perp_n R$. Then, the sets

$$C'_n = \{db \otimes 1 \mid b \in C_n\} \subseteq \Omega_{\perp_n R/k} \otimes_{\perp_n R} R = (\mathbb{L}_{R/k})_n$$

are an R-basis for the free modules $(\mathbb{L}_{R/k})_n$. Moreover, for a surjective map $\eta : \mathbf{p} \to \mathbf{q}$ in Δ, we saw in Example 2.2.10 that $\eta^\star C_q \subseteq C_p$, and therefore $\eta^\star C'_q \subseteq C'_p$. So $\mathbb{L}_{R/k}$ is a free simplicial R-module. □

[1] See the proof of Quillen [14] II.4. Proposition 3.

This proposition shows that $\mathbb{L}_{R/k}$ is a projective resolution for R, and this provides an alternative way to look at homology and cohomology:

$$D_n(R/k,N) \cong \mathrm{Tor}_n^R(\mathbb{L}_{R/k},N), \qquad D^n(R/k,N) \cong \mathrm{Ext}_R^n(\mathbb{L}_{R/k},N),$$

for any R-module N. Using the long exact sequences for the derived functors Tor and Ext we get the following long exact sequences.

Corollary 3.1.2. *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, then there are long exact sequences*

$$0 \to D^0(R/k,M') \longrightarrow D^0(R/k,M) \longrightarrow D^0(R/k,M'') \longrightarrow D^1(R/k,M') \to \dots$$

and

$$\dots \to D_1(R/k,M'') \longrightarrow D_0(R/k,M') \longrightarrow D_0(R/k,M) \longrightarrow D_0(R/k,M'') \to 0.$$

Corollary 3.1.3. *There is a universal coefficient spectral sequence*

$$E^2_{pq} = Tor_p^R(D_q(R/k),M) \Longrightarrow D_{p+q}(R/k,M).$$

Proof. This is just the base-change for Tor [2] once we realize that

$$D_\star(R/k) = D_\star(R/k,R) = \pi_\star(\mathbb{L}_{R/k}).$$

□

Definition. Let $\mathcal{C}$ be a model category. The *homotopy category* $\mathbf{Ho}(\mathcal{C})$ is the category obtained from $\mathcal{C}$ by formally inverting the weak equivalences.

Remark. In the case $\mathcal{C}$ is the category of simplicial R-modules, then $\mathbf{Ho}(\mathcal{C})$ is equivalent to the full subcategory of the derived category of R-modules consisting of the bounded above cochain complexes. We denote this category by $\mathcal{D}\mathbf{Mod}_R$. If S is a simplicial ring, we denote by $\mathcal{S}\mathbf{Mod}_S$ the category of simplicial S-modules.

Let S be a simplicial ring and A and B are two simplicial S-modules. We define the hypertor simplicial S-module $\mathrm{Tor}_p^S(A,B)$ by

$$\mathbf{n} \mapsto \mathrm{Tor}_p^{S_n}(A_n,B_n), \text{ for all } n \geqslant 0.$$

In particular, if

$$S \longrightarrow P \to A, \quad S \longrightarrow Q \to B$$

are two cofibrant-acyclic fibrant factorizations, then the homology of $P \otimes_S Q$ is independent of the choice of the factorization. We denote by $A \otimes_S^{\mathcal{L}} B$ the total derived tensor product of A and B, so that in $\mathcal{D}\mathbf{Ab}$ (the full subcategory of the the derived category of abelian groups consisting of the bounded above cochain complexes) there is an isomorphism $A \otimes_S^{\mathcal{L}} B \cong P \otimes_S Q$. Recall that it defines a functor

$$\otimes_S^{\mathcal{L}} : \mathbf{Ho}(\mathcal{S}\mathbf{Mod}_S) \times \mathbf{Ho}(\mathcal{S}\mathbf{Mod}_S) \longrightarrow \mathcal{D}\mathbf{Ab}.$$

[2] Weibel [17] Theorem 5.5.6.

Proposition 3.1.4. *There is a spectral sequence*

$$E^2_{pq} = H_p(Tor^S_q(A,B)) \Longrightarrow H_{p+q}(A \otimes^{\mathcal{L}}_S B)$$

where A, B are two simplicial S-modules. Moreover, the edge morphism

$$H_n(A \otimes^{\mathcal{L}}_S B) \longrightarrow H_n(A \otimes_S B)$$

is induced by the canonical map $A \otimes^{\mathcal{L}}_S B \longrightarrow A \otimes_S B$.

Proof. See Quillen [12] II. Theorem 6. □

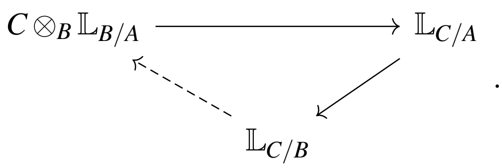
Corollary 3.1.5. *If* $Tor^S_q(A,B) = 0$ *for* $q > 0$, *then* $A \otimes^{\mathcal{L}}_S A \cong A \otimes_S B$ *in* $\mathcal{D}\mathbf{Ab}$.

Proof. Direct consequence of the previous proposition since $\mathrm{Tor}^S_q(A,B) = 0$ for $q > 0$ means that the spectral sequence collapses to the first row, and hence the canonical map

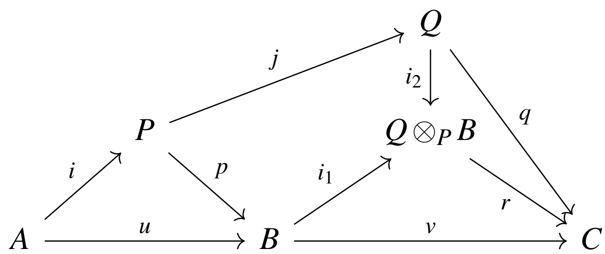
$$A \otimes^{\mathcal{L}}_S B \longrightarrow A \otimes_S B$$

is a weak equivalence. □

Theorem 3.1.6. *If* $A \to B \to C$ *are morphisms of rings, then there is a canonical distinguished triangle in the derived category* $\mathcal{C}\mathbf{Mod}_R$



Proof. Let P be a free A-algebra resolution of B and let Q be a free P-algebra resolution of C, so that there is a commutative diagram



By the first fundamental exact sequence applied to $A \to P \to Q$ we get a split (since Q_n a free P_n-module for every n) short exact sequence

$$0 \longrightarrow Q \otimes_P \Omega_{P/A} \longrightarrow \Omega_{Q/A} \longrightarrow \Omega_{Q/P} \longrightarrow 0$$

and we obtain the exact sequence

$$0 \longrightarrow C \otimes_B (B \otimes_P \Omega_{P/A}) \longrightarrow C \otimes_Q \Omega_{Q/A} \longrightarrow C \otimes_Q \Omega_{Q/P} \longrightarrow 0$$

where the first term is precisely $C \otimes_B \mathbb{L}_{B/A}$ and the second one is $\mathbb{L}_{C/A}$. As for the last one we have

$$C \otimes_{Q \otimes_P B} \Omega_{Q \otimes_P B/B} \cong C \otimes_{Q \otimes_P B} (\Omega_{Q/P} \otimes_P B) \cong C \otimes_{Q \otimes_P B} (\Omega_{Q/P} \otimes_Q (Q \otimes_P B)) \cong C \otimes_Q \Omega_{Q/P}.$$

Recall that $\cdot \otimes_P^L Q$ defines a functor in the derived category of P-modules, where p is an isomorphism. Hence $p \otimes_P^L \mathrm{id}_Q : P \otimes_P^L Q \to B \otimes_P^L Q$ is also a weak equivalence. Note that $\mathrm{Tor}_q^P(P,Q) = \mathrm{Tor}_q^P(B,Q) = 0$ for all $q > 0$, so

$$p \otimes_P^L \mathrm{id}_Q \cong i_2 : P \otimes_P Q \to B \otimes_P Q,$$

which shows that i_2 is a weak equivalence. Then, r is a weak equivalence by commutativity of the diagram since q is a weak equivalence too. The map r is also surjective in every degree since q is, so it is an acyclic fibration. Recall that cofibrations are the maps that have the RLP with respect to all acyclic fibrations. For any such acyclic fibration $X \to Y$ in $\mathcal{S}\mathbf{Alg}_A$, and a commutative digram

$$\begin{array}{ccccc} P & \xrightarrow{p} & B & \longrightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i_1} & & \downarrow \\ Q & \xrightarrow{i_2} & Q \otimes_P B & \longrightarrow & Y \end{array}$$

there is a lifting $Q \to X$ making the diagram commute since j is a cofibration. This map induces a map $Q \otimes_P B \to X$. Hence, i_1 is also a cofibration. Note that we have showed the more general fact that cofibrations are closed under cobase change. Therefore, $Q \otimes_P B$ is a cofibrant simplicial B-algebra resolution of C, so

$$\mathbb{L}_{C/B} \cong C \otimes_{Q \otimes_P B} \Omega_{Q \otimes_P B/B}.$$

The cofibration sequence in the derived category of C-modules associated to this exact sequence gives the desired triangle. □

As a direct consequence of this theorem we can extend the exact sequence we obtained in Proposition 2.3.4.

Corollary 3.1.7. *Let k, E, R be commutative rings, $k \to E$, $E \to R$ two ring homomorphisms and let M be an R-module. There is a long exact sequence*

$$0 \longrightarrow Der_E(R,M) \longrightarrow Der_k(R,M) \longrightarrow Der_k(E,M) \longrightarrow$$
$$Exalcomm_E(R,M) \longrightarrow Exalcomm_k(R,M) \longrightarrow Exalcomm_k(E,M) \longrightarrow$$
$$D^2(R/E,M) \longrightarrow D^2(R/k,M) \longrightarrow D^2(E/k,M) \dashrightarrow$$
$$\dashrightarrow D^n(R/E,M) \longrightarrow D^n(R/k,M) \longrightarrow D^n(E/k,M) \to \dots$$

Theorem 3.1.8 (Flat base change)**.** *If B and C are A-algebras such that $Tor_q^A(B,C) = 0$ for $q > 0$. Call $D = B \otimes_A C$, then there are isomorphisms in the derived category of D-modules*

$$\mathbb{L}_{D/C} \cong \mathbb{L}_{B/A} \otimes_A C,$$
$$\mathbb{L}_{D/A} \cong (\mathbb{L}_{B/A} \otimes_A C) \oplus (B \otimes_A \mathbb{L}_{C/A}).$$

Proof. Let P be a cofibrant A-algebra resolution of B, which induces a morphism of simplicial C-algebras $P \otimes_A C \longrightarrow B \otimes_A C$. We let $q\,i$ be the cofibrant factorization of this map such that there is a commutative diagram

$$\begin{array}{ccccc} P & \longrightarrow & P\otimes_A C & \xrightarrow{\ i\ } & Q \\ \downarrow & & \downarrow & & \downarrow q \\ B & \longrightarrow & B\otimes_A C & \xrightarrow{\ =\ } & B\otimes_A C \end{array}$$

where Q is a simplicial C-algebra. The map $C \longrightarrow P\otimes_A C$ is a cofibration since $A \longrightarrow P$ is a cofibration and we showed in the proof of Theorem 3.1.6 that cofibrations are closed under cobase change. Since i is also a cofibration, it follows that $C \longrightarrow Q$ is a cofibration. Moreover, the commutative diagram of solid arrows

$$\begin{array}{ccc} C & \longrightarrow & Q \\ \downarrow & \nearrow & \downarrow q \\ P\otimes_A C & \longrightarrow & B\otimes_A C \end{array}$$

shows that there is a map $P\otimes_A C \longrightarrow Q$ since q is an acyclic fibration and the map on the left is a cofibration of simplicial C-algebras. We get a map of simplicial D-modules

$$\Omega_{P\otimes_A C/C}\otimes_{P\otimes_A C} D \longrightarrow \Omega_{Q/C}\otimes_Q D$$

where the second term is precisely $\mathbb{L}_{D/C}$. As for the first one, we have

$$\begin{aligned} \Omega_{P\otimes_A C/C}\otimes_{P\otimes_A C} D &\cong (\Omega_{P/A}\otimes_A C)\otimes_{P\otimes_A C}(B\otimes_A C) \\ &\cong (\Omega_{P/A}\otimes_P (P\otimes_A C))\otimes_{P\otimes_A C}(B\otimes_A C) \\ &\cong (\Omega_{P/A}\otimes_P B)\otimes_A C \cong \mathbb{L}_{B/A}\otimes_A C. \end{aligned}$$

Note that $P \longrightarrow B$ is a weak equivalence, so the derived morphism $P\otimes^{\mathcal{L}}_A C \longrightarrow B\otimes^{\mathcal{L}}_A C$ is also a weak equivalence. $\mathrm{Tor}^A_p(P,C) = \mathrm{Tor}^A_p(B,C) = 0$ for all $p > 0$, so this maps is isomorphic to $P\otimes_A C \longrightarrow B\otimes_A C$. So

$$C \longrightarrow P\otimes_A C \longrightarrow D$$

is a cofibrant factorization and we have

$$\mathbb{L}_{D/C} \cong \Omega_{P\otimes_A C/C}\otimes_{P\otimes_A C} D \cong \mathbb{L}_{B/A}\otimes_A C.$$

To see the second isomorphism let R be a cofibrant A-algebra resolution of C. Since $\mathrm{Tor}^A_q(B,C)$ vanishes for $q > 0$, by Corollary 3.1.5, the map $P\otimes_A R \longrightarrow B\otimes_A C$ is a weak equivalence. Since it is also surjective it is an acyclic fibration. On the other hand, $A \longrightarrow P$ is a cofibration, so by cobase change, $R \longrightarrow R\otimes_A P$ is also a cofibration. $A \longrightarrow Q$ is a cofibration too, so the composition $A \longrightarrow P\otimes_A R$ is a cofibration. Hence,

$$A \longrightarrow P\otimes_A R \longrightarrow D$$

is a cofibrant factorization, and we have

$$\begin{aligned} \mathbb{L}_{D/A} &\cong \Omega_{P\otimes_A R/A}\otimes_{P\otimes_A R} D \cong (R\otimes_A \Omega_{P/A} \oplus P\otimes_A \Omega_{R/A})\otimes_{P\otimes_A R} D \\ &\cong (\Omega_{P/A}\otimes_P B)\otimes_A C \oplus (\Omega_{R/A}\otimes_R C)\otimes_A B \cong (\mathbb{L}_{B/A}\otimes_A C)\oplus(\mathbb{L}_{C/A}\otimes_A B). \end{aligned}$$

□

Corollary 3.1.9. *Let E, R be k-algebras and let M be a $E\otimes_k R$-module. If $Tor_q^k(E,R)=0$ for $q>0$, then there are isomorphisms*

$$D^q(E\otimes_k R/R,M)\cong D^q(E/k,M),$$
$$D^q(E\otimes_k R/E,M)\cong D^q(E/k,M)\oplus D^q(R/k,M).$$

3.2 Homology and cohomology for universal algebras

We fix a category $\mathcal{C}$, and we denote by $U:\mathbf{Ab}\to\mathbf{Set}$ the forgetful functor. For simplicity, we will write h_A for the contravariant representable functor $\mathrm{Hom}_{\mathcal{C}}(\cdot,A)$, where $A\in\mathcal{C}$.

Definition. An *abelian functor* on $\mathcal{C}$ is a functor $F:\mathcal{C}^{\mathrm{op}}\longrightarrow\mathbf{Ab}$ to the category of abelian groups. An *abelian object* is an object $A\in\mathcal{C}$ such that $h_A=U\,F$ for some abelian functor F on $\mathcal{C}$. We denote by $\mathcal{C}_{\mathrm{ab}}$ the subcategory of abelian objects in the category $\mathcal{C}$.

For an abelian object $A\in\mathcal{C}_{\mathrm{ab}}$, the abelian functor F induces natural transformations

$$h_A\times h_A\xrightarrow{m}h_A,\quad h_A\xrightarrow{J}h_A,\quad \mathbb{1}\xrightarrow{e}h_A$$

where for any $B\in\mathcal{C}$, $m(B)$ is the abelian group operation on $F(B)$, $J(B)$ is the opposite map on $F(B)$ and $e(B)$ sends the singleton to the neutral element in $F(B)$. Therefore, the following diagrams are commutative:

$$\begin{array}{ccc} h_A\times h_A\times h_A & \xrightarrow{m\times\mathrm{id}} & h_A\times h_A \\ \downarrow{\scriptstyle \mathrm{id}\times m} & & \downarrow{\scriptstyle m} \\ h_A\times h_A & \xrightarrow{m} & h_A \end{array}
\qquad
\begin{array}{ccccc} & \overset{\mathrm{id}\times J}{\nearrow} & h_A\times h_A & \overset{m}{\searrow} & \\ h_A & \xrightarrow{\pi} & \mathbb{1} & \xrightarrow{e} & h_A \\ & \underset{J\times\mathrm{id}}{\searrow} & h_A\times h_A & \underset{m}{\nearrow} & \end{array}$$

$$\begin{array}{ccccc} h_A\times\mathbb{1} & \xrightarrow{\mathrm{id}\times e} & h_A\times h_A & \overset{m}{\searrow} & \\ \uparrow{\scriptstyle \mathrm{id}\times\pi} & & & & \\ h_A & & \xrightarrow{\mathrm{id}} & & h_A \\ \downarrow{\scriptstyle \pi\times\mathrm{id}} & & & & \\ \mathbb{1}\times h_A & \xrightarrow{e\times\mathrm{id}} & h_A\times h_A & \underset{m}{\nearrow} & \end{array}$$

where $\pi:h_A\to\mathbb{1}$ send any $B\in\mathcal{C}$ to the singleton. Now, if we let $C=\mathbf{Alg}_k/R$, with $A\overset{u}{\to}R\in\mathcal{C}$. Then $\mathbb{1}\cong h_R$ and by Yoneda Lemma we get maps

$$A\times A\xrightarrow{\mu}A,\quad R\xrightarrow{\varepsilon}A,\quad A\xrightarrow{\iota}A$$

in $\mathbf{Alg}_k/R$, called *multiplication*, *unit* and *inverse map* respectively, such that

$$\begin{array}{ccc} A\times A\times A & \xrightarrow{\mu\times\mathrm{id}} & A\times A \\ \downarrow{\scriptstyle \mathrm{id}\times\mu} & & \downarrow{\scriptstyle \mu} \\ A\times A & \xrightarrow{\mu} & A \end{array} \qquad \begin{array}{ccccc} & & A\times A & & \\ & \nearrow^{\mathrm{id}\times\iota} & & \searrow^{\mu} & \\ A & \xrightarrow{u} & R & \xrightarrow{\varepsilon} & A \\ & \searrow_{\iota\times\mathrm{id}} & & \nearrow_{\mu} & \\ & & A\times A & & \end{array}$$

$$\begin{array}{ccccc} & A\times R & \xrightarrow{\mathrm{id}\times\varepsilon} & A\times A & \\ \nearrow^{\mathrm{id}\times u} & & & & \searrow^{\mu} \\ A & & \xrightarrow{\qquad\mathrm{id}\qquad} & & A \\ \searrow_{u\times\mathrm{id}} & & & & \nearrow_{\mu} \\ & R\times A & \xrightarrow{\varepsilon\times\mathrm{id}} & A\times A & \end{array}$$

Example 3.2.1. Still in $\mathbf{Alg}_k/R$, let $M\in\mathbf{Mod}_R$. In Lemma 2.3.1 we saw that

$$h_{M\ltimes R}\cong \mathrm{Der}_k(\cdot,M),$$

where $\mathrm{Der}_k(E,M)$ has an abelian group structure for all $E\in\mathbf{Alg}_k/R$. So $M\ltimes R$ is an abelian object, taking $\mathrm{Der}_k(\cdot,M):\mathbf{Alg}_k/R\to\mathbf{Ab}$ as abelian functor. The maps are given by

$$\begin{array}{ccc} (M\ltimes R)\times(M\ltimes R) & \xrightarrow{\mu} & M\ltimes R \\ ((m,r),(m',r)) & \longmapsto & (m+m',r) \end{array}, \quad \begin{array}{ccc} R & \xrightarrow{\varepsilon} & M\ltimes R \\ r & \longmapsto & (0,r) \end{array}, \quad \begin{array}{ccc} M\ltimes R & \xrightarrow{\iota} & M\ltimes R \\ (m,r) & \longmapsto & (-m,r). \end{array}$$

Moreover, let $P\xrightarrow{v}R\in(\mathbf{Alg}_k/R)_{\mathrm{ab}}$, and set $M=\ker v$. We denote by μ_P, ε_P, ι_P the multiplication, unit and inverse maps in P. Then M can be seen as an R-module via ε_P and we can define a map in the category $\mathbf{Alg}_k/R$ given by

$$\begin{array}{cccc} \varphi: & M\ltimes R & \longrightarrow & P \\ & (m,r) & \longmapsto & m+\varepsilon_P(r). \end{array}$$

To see that it is well defined, note that for any $n,m\in M$, we have

$$n=\mu_P(\mathrm{id}\times\varepsilon_P v)(n)=\mu_P(n,0), \qquad m=\mu_P(\varepsilon_P v\times\mathrm{id})(m)=\mu_P(0,m)$$

so $nm=\mu_P((n,0)(0,m))=0$, which means that M has zero multiplication. Thus, the map is well defined. If $(r,m)\in\ker\varphi$, then $m=\varepsilon_P(-r)$. Since ε_P is a map in $\mathbf{Alg}_k/R$, $\mathrm{id}_R=v\,\varepsilon_P$, and we get

$$0=v(m)=v(\varepsilon_P(-r))=-r.$$

So φ is injective. On the other hand, v surjective, so $\mathrm{im}(\varepsilon_P v)=\mathrm{im}\ \varepsilon_P$. Note that $M\subseteq\ker(\varepsilon_P v)$, so there is a surjective map

$$P/M \twoheadrightarrow \mathrm{im}(\varepsilon_P v)=\mathrm{im}\ \varepsilon_P$$

and therefore $P\cong\mathrm{im}\ \varepsilon_P+M$, which makes the map φ surjective. This shows that actually, any abelian object in $\mathbf{Alg}_k/R$ is of the form $M\ltimes R$ for some R-module M. Moreover, there is an equivalence of categories

$$\begin{array}{rccc} \cdot \ltimes R: & \mathbf{Mod}_R & \longrightarrow & (\mathbf{Alg}_k/R)_{\mathrm{ab}} \\ & M & \longmapsto & M \ltimes R \\ & \ker \nu & \longleftarrow & P(\xrightarrow{\nu} R). \end{array}$$

In particular, $(\mathbf{Alg}_k/R)_{\mathrm{ab}}$ is an abelian category. The pair of adjoints from Corollary 2.3.2 can be now seen as

$$\mathbf{Alg}_k/R \underset{T}{\overset{L}{\rightleftarrows}} (\mathbf{Alg}_k/R)_{\mathrm{ab}},$$

where T is the natural faithful functor, and L is sometimes called *abelianization functor*. Recall that we defined the cotangent complex $\mathbb{L}_{R/k}$ to be the simplicial R-module $L(P_\star)$, where $P_\star$ is a simplicial cofibrant k-algebra resolution of R.

For now on, we assume that $\mathcal{C}$ is closed under finite limits, that $\mathcal{SC}$, $\mathcal{SC}_{\mathrm{ab}}$ are both model categories, and that the abelianization functor $\mathrm{Ab}: \mathcal{C} \longrightarrow \mathcal{C}_{\mathrm{ab}}$ is left adjoint to the natural faithful functor $\mathcal{C}_{\mathrm{ab}} \longrightarrow \mathcal{C}$. For any object $X \in \mathcal{C}$, we denote by $\mathcal{C}/X$ the category over X.

An object P of $\mathcal{C}$ we now be called projective if for any effective epimorphism $p: X \longrightarrow Y$, the induced map

$$\mathrm{Hom}_{\mathcal{C}}(P,p): \mathrm{Hom}_{\mathcal{C}}(P,X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P,Y)$$

is surjective. We assume that $\mathcal{C}$ has enough projective objects, meaning that for ever $X \in \mathcal{C}$ there is an effective epimorphism $P \longrightarrow X$ with $P \in \mathcal{C}$ projective. In particular, this holds for the category of universal algebras defined by a set of operations and relations. The initial object in $\mathcal{CS}$ is denoted by ϕ. Analogously to what we did for $\mathbf{Alg}_k/R$ we have:

Definition. Let $X \in \mathcal{C}$. A *simplicial resolution of* X is an acyclic fibration $P \longrightarrow \mathrm{c}X$. A simplicial object $Q \in \mathcal{SC}$ is *cofibrant* if the map $\phi \longrightarrow Q$ is a cofibration.

Proposition 3.2.2. *For any $X \in \mathcal{C}$ there is always a unique cofibrant simplicial resolution of X up to homotopy equivalence, which depend functorially on X up to homotopy.*

Proof. See Quillen [12], IV, Proposition 3. □

Let $X \in \mathcal{C}$, and let M be an abelian object in $\mathcal{C}/X$. Then, for any P cofibrant simplicial resolution of X, $\mathrm{Hom}_{\mathcal{C}/X}(P,M)$ is a cosimplicial abelian group, whose cohomotopy is independent of the choice of P by the last proposition.

Definition. The *cohomology groups of X with values in M* are

$$D^q(X,M) = \pi^q(\mathrm{Hom}_{\mathcal{C}/X}(P,M)).$$

Moreover, we can define $\mathcal{L}\mathrm{Ab}(X) = \mathrm{Ab}(P)$ as a simplicial object in $(\mathcal{C}/X)_{\mathrm{ab}}$. Viewing it as a chain complex (via the Dold-Kan correspondence), we see that it is independent of the choice of P up to homotopy equivalence, so it is an object in the derived category of $(\mathcal{C}/X)_{\mathrm{ab}}$. Moreover, we can rewrite the cohomology groups of X with values in M as

$$D^q(X,M) = \pi^q(\mathrm{Hom}_{(\mathcal{C}/X)_{\mathrm{ab}}}(\mathcal{L}\mathrm{Ab}(X),M)).$$

Hence, we can think of $\mathcal{L}\mathrm{Ab}(X)$ as the analogous to the complex chains of a space X. Finally, we get the following definition for homology.

Definition. The qth *homology object of X* is

$$D_q(X) = \pi_q(\mathcal{L}\text{Ab}(X)).$$

Remark. This is is just a generalization of the homology of a k-algebra R with values in the R-module R:

$$D_n(R/k, R) = \pi_n(\mathbb{L}_{R/k} \otimes_R R) = \pi_n(\mathbb{L}_{R/k}), \quad n \geqslant 0.$$

The computations we did for homology and cohomology of commutative rings can be also extended. For example, there is a universal coefficient spectral sequence

$$E_2^{pq} = \text{Ext}^p_{(\mathcal{C}/X)_{\text{ab}}}(D_q(X), M) \Longrightarrow D^{p+q}(X, M),$$

and for degree 0 we also have

$$D^0(X, M) = \text{Hom}_{\mathcal{C}/X}(X, M).$$

Example 3.2.3. This extended cohomology can again be seen as a cotriple cohomology. For any category $\mathcal{A}$, recall that a pair of adjoint functors

$$F : \mathcal{A} \rightleftarrows \mathcal{C} : U$$

defines a cotriple $\perp = FU : \mathcal{C} \longrightarrow \mathcal{C}$. We also get an augmented simplicial object $\perp_\star X \longrightarrow X$, which is in fact a simplicial object in $\mathcal{C}/X$. For any $M \in \mathcal{C}_{\text{ab}}$ we consider the representable functor $h_M = \text{Hom}_{\mathcal{C}/X}(\cdot, M) : \mathcal{C}/X \longrightarrow \mathbf{Ab}$ and define the cotriple cohomology groups

$$H^n(X; h_M) = \pi^n(h_M \perp_\star X), \quad n \geqslant 0.$$

Quillen showed [3] that if $\perp Y \longrightarrow Y$ is an effective epimorphism for all $Y \in \mathcal{C}$ and $F(B)$ is projective for all $B \in \mathcal{A}$, then

$$D^q(X, M) \cong H^q(X; h_M), \quad q \geqslant 0.$$

Remark. The cohomology groups X with values in M are a special case of the more general cohomology constructed using Grothendieck topologies. Broadly speaking, we can define a Grothendieck topology on $\mathcal{C}$ as follows: for any object $Y \in \mathcal{C}$, the set the covering of Y to be the family consisting of a single map $U \longrightarrow Y$ which is an effective epimorphism (the existence of such a map is provided by the enough projectives condition on $\mathcal{C}$). Effective epimorphisms are stable under composition and base change, so this defines a pretopology on $\mathcal{C}$. The associated topology on $\mathcal{C}$ induces a Grothendieck topology on $\mathcal{C}/X$. Now, the representable functor $h_M : \mathcal{C}/M \longrightarrow \mathbf{Ab}$ is a sheaf of abelian groups for the induced topology on $\mathcal{C}/X$ and we obtain sheaf cohomology groups

$$H^q(\mathcal{C}/X, h_M).$$

Quillen showed [3] that this cohomology also the cohomology of X with values in M, i.e.,

$$D^q(X, M) \cong H^q(\mathcal{C}/X, h_M), \quad q \geqslant 0.$$

[3] Quillen [12] II. 5. Theorem 5.

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