SHARP COHOMOLOGY

Sharp cohomology theories: a road map

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September 17, 2011
What is $H^1$ in algebraic geometry?
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$H^1$ is just an avatar of Pic!
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$H^1$ of proper smooth schemes

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- $H^1_{\text{dR}}(X) = \text{Lie Pic}^{!}(X)$ in zero characteristic
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- $H^1_{\text{dR}}(X) = \text{Lie Pic}^{\flat}(X)$ in zero characteristic
- $H^1_{\text{crys}}(X) = \text{Lie Pic}^{\text{crys}}(X)$ in positive characteristics
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We can see that

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with the additional information given by the image of

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Actually, this subspace is clearly independent from the chosen compactification!
For any scheme $X$ we get a smooth hypercovering $\tilde{X}$ and we can see that

$$\text{Pic}^+(X) := \text{Pic}^+(\tilde{X}) \sim H^1(X) = H^1(\tilde{X})$$

That is $T_{Hodge}, T_{dR}, T_{\ell}, T_{\text{crys}}$ applied to the 1-motive $\text{Pic}^+(X)$ yield $H^1(X, \mathbb{Z}(1)), H^1(X_{\text{ét}}, \mathbb{Z}_{\ell}(1)), H^1_{dR}(X), H^1_{\text{crys}}(X)$. 
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- For $X$ proper we discover that $\text{Pic}^+(X)$ is just the semiabelian quotient of $\text{Pic}^0(X)$. Thus we may regard $H^1(X)$ as a quotient of a refined $H^1_\#(X)$.
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- For $X$ smooth just consider the formal completion at zero $\text{Inf}^0_{\text{Y}}(\overline{X})$ of

$$\ker H^1(\overline{X}, \mathcal{O}_{\overline{X}}) \rightarrow H^1(X, \mathcal{O}_X)$$

Thus $\text{Div}^0_{\text{Y}}(\overline{X}) \times \text{Inf}^0_{\text{Y}}(\overline{X})$ is a formal group and

$$\text{Pic}^+_a(X) := [\text{Div}^0_{\text{Y}}(\overline{X}) \times \text{Inf}^0_{\text{Y}}(\overline{X}) \rightarrow \text{Pic}^0(\overline{X})] \leadsto H^1_\#(X)$$

so that $H^1(X)$ is a subobject of $H^1_\#(X)$.
The **sharp** (singular, de Rham, etc.) cohomology

\[
(X, Z) \mapsto H_\#^*(X, Z)
\]

is at least a contravariant functor from pairs \((X, Z)\) with \(Z \subseteq X\) closed to **formal** groups (formal Hodge structures, etc.) which is provided with a long exact sequence of the triples.
The sharp (singular, de Rham, etc.) cohomology 

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is at least a contravariant functor from pairs $(X, Z)$ with $Z \subseteq X$ closed to formal groups (formal Hodge structures, etc.) which is provided with a long exact sequence of the triples. That is:

1. $f^* : H^*_\#(X, Z) \to H^*_\#(X', Z')$ for a morphism $f : X' \to X$ such that $f|_{Z'} : Z' \to Z$ and
2. $H^*_\#(X, Y) \to H^*_\#(X, Z) \to H^*_\#(Y, Z) \to H^*_{\#}+1(X, Y)$ exact for $Z \subseteq Y \subseteq X$ closed in $X$
The sharp (singular, de Rham, etc.) cohomology

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By the way, ordinary singular cohomology Hodge structure

\[H^\ast(X, Z) = H^\#{\ast}(X, Z)_{\text{ét}}\]

is the étale structure associated to the formal Hodge structure.
A formal Hodge structure of level $\leq n$ is given by

- $H := H_\mathbb{Z} \times H^0$ a formal group over $\mathbb{C}$ such that $H_\mathbb{Z}$ is the underlying group of a level $\leq n$ mixed Hodge structure $H_{\text{ét}} = (H_\mathbb{Z}, W_*, F^*_{\text{Hodge}})$,

- $V := V_n \to \cdots \to V_1$ a diagram given by composable linear mappings of finite dimensional $\mathbb{C}$-vector spaces,

- an augmentation map $\nu : H \to V$

- a subdiagram $V^0 \subset V$ such that $V/V^0 \cong H_\mathbb{C}/F^n_{\text{Hodge}}$ yielding a commutative diagram

\[
\begin{array}{cccccc}
H_\mathbb{Z} & \overset{c}{\longrightarrow} & H_\mathbb{C}/F^n_{\text{Hodge}} & \longrightarrow & \cdots & \longrightarrow & H_\mathbb{C}/F^1_{\text{Hodge}} \\
\downarrow & & \uparrow & & \uparrow & & \\
H & \overset{\nu}{\longrightarrow} & V_n & \longrightarrow & \cdots & \longrightarrow & V_1
\end{array}
\]
FHS is the abelian category obtained by taking $\text{Colim}_n \text{FHS}_n$ where $\text{FHS}_n$ are level $\leq n$ formal Hodge structures; note that we have a forgetful (faithful exact) functor $(H, V) \rightsquigarrow H_{\mathbb{Z}} \times H^0 \times \hat{V}^0$ from FHS to $\text{Fgrp}_{\mathbb{C}}$ formal groups.
**Formal Hodge structures**

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**FHS}_{\text{ét}}$ is the full subcategory of FHS of étale structures, i.e., for $(H, V) \in \text{FHS}_n$ let

$$(H, V)_{\text{ét}} := (H_\mathbb{Z}, V/V^0) \cong (H_\mathbb{Z}, H_\mathbb{C}/F_{Hodge})$$

and say that $(H, V)$ is étale if $(H, V)_{\text{ét}} = (H, V)$
**Formal Hodge structures**

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(H, V)_{\text{ét}} := (H_{\mathbb{Z}}, V / V^0) \cong (H_{\mathbb{Z}}, H^_/ F_{\text{Hodge}})
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and say that \((H, V)\) is \(\text{étale}\) if \((H, V)_{\text{ét}} = (H, V)\).

**FHS^0** are the **connected** structures, i.e., \((H, V)_{\text{ét}} = 0\).
Formal Hodge structures

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$\text{FHS}^0$ are the connected structures, i.e., $(H, V)_{\text{ét}} = 0$.

Denote $(H, V)_{\times} := (H, V/V^0)$. We have a canonical extension

$$0 \to (H, V)_{\text{ét}} \to (H, V)_{\times} \to (H^0, 0) \to 0$$
In this framework, for \((X, Z)\) with \(\dim X = n\) over \(\mathbb{C}\) we may seek for \(H^\#_\ast(X, Z) = (H^\ast(X, Z) \times H^0, V) \in \text{FHS}_n\) along with the canonical extension

\[
0 \rightarrow H^\ast(X, Z) \rightarrow H^\#_\ast(X, Z)/V^0 \rightarrow H^0 \rightarrow 0
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In this framework, for $(X, Z)$ with $\dim X = n$ over $\mathbb{C}$ we may seek for $H_\#^*(X, Z) = (H^*(X, Z) \times H^0, V) \in \text{FHS}_n$ along with the canonical extension

$$0 \to H^*(X, Z) \to H_\#^*(X, Z)/V^0 \to H^0 \to 0$$

For example, we have that $H_\#^1(X), H_\#^{1-dR}(X)$, etc. is sitting in an extension

$$0 \to H^1(X) \to H_\#^1(X)/V(\text{Pic}) \to V(\text{Alb})^\vee \to 0$$

where

- $V(\text{Pic}) :=$ the Lie algebra $V^0$ of the vector group given by the maximal additive subgroup of $\text{Pic}^0$
- $V(\text{Alb})^\vee :=$ the connected formal group $H^0 = \text{Inf}$ whose Lie algebra is just dual of the maximal additive subgroup of Faltings-Wüstholz Alb
Note that we also have that \((H^0, V)\) is a connected structure associated to any \((H, V)\) but it is not a substructure, in general.
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\(\text{FHS}^s\) are the special structures, i.e., say that \((H, V)\) is special if \((H^0, V^0) := (H, V)^0\) is a substructure of \((H, V)\) or, equivalently, \((H, V)_{\text{ét}}\) is a quotient of \((H, V)\), so that we have an extension

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in this case.

This is the largest subcategory of FHS such that \(\text{MHS} = \text{FHS}_{\text{ét}}\) into \(\text{FHS}^s\) has a left adjoint and \(\text{FHS}^0 = \text{VSP}\) into \(\text{FHS}^s\) has a right adjoint.
Deligne’s Hodge realization for 1-motives with torsion can be further extended to an equivalence with graded polarizable (twisted) formal Hodge structures of level $\leq 1$

$$T_\mathcal{f} : \text{Laumon 1-motives} \xrightarrow{\sim} \text{FHS}_1^p$$

where $T_\mathcal{f}([F \xrightarrow{u} G]) := (T_\mathcal{f}(F), \text{Lie}(G))$ where $T_\mathcal{f}(F)_{\text{ét}}$ is the underlying abelian group to $T_{\text{Hodge}}([F \xrightarrow{u} G]_{\text{ét}})$ and $T_\mathcal{f}(F)^0 = F^0$. 
Deligne’s Hodge realization for 1-motives with torsion can be further extended to an equivalence with graded polarizable (twisted) formal Hodge structures of level $\leq 1$

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where $T_\mathfrak{f}([F \xrightarrow{u} G]) := (T_\mathfrak{f}(F), \text{Lie}(G))$ where $T_\mathfrak{f}(F)_{\text{ét}}$ is the underlying abelian group to $T_{\text{Hodge}}([F \xrightarrow{u} G]_{\text{ét}})$ and $T_\mathfrak{f}(F)^0 = F^0$. Thus get a diagram

\[
\begin{array}{ccc}
\text{Deligne 1-motives} & \xrightarrow{T_{\text{Hodge}}} & \text{MHS}_1^p \\
\uparrow \downarrow & & \uparrow \downarrow \\
\text{Laumon 1-motives} & \xrightarrow{T_\mathfrak{f}} & \text{FHS}_1^p
\end{array}
\]

and

$$H^1_{\#}(X) := T_\mathfrak{f}(\text{Pic}^+(X))$$
For a Laumon 1-motive $M = [F \to G]$ set $M_\times := [F \to G/V(G)]$ where $V(G)$ is the maximal additive subgroup of $G$. 

This is a sharp version of the de Rham comparison theorem.
For a Laumon 1-motive $M = [F \to G]$ set $M_\times := [F \to G/ V(G)]$ where $V(G)$ is the maximal additive subgroup of $G$. We get the sharp universal $\mathbb{G}_a$-extension $M^\# := [F \to G^\#]$ by the pull-back of the universal $\mathbb{G}_a$-extension $M^\#_\times$ along $M \to M_\times$. 

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$$T^\#_a(M) = \text{Lie}(G^\#) \quad H^1_{dR}(X) := T^\#_a(\text{Pic}^+_a(X))$$
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We get the sharp universal \( \mathbb{G}_a \)-extension \( M^\# := [F \to G^\#] \) by the pull-back of the universal \( \mathbb{G}_a \)-extension \( M^\#_\times \) along \( M \to M_\times \).

The sharp de Rham realization

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T^\#(M) = \text{Lie}(G^\#) \quad \quad H^1_{\#-dR}(X) := T^\#(\text{Pic}_a^+(X))
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For \((H, V) \in \text{FHS}_1\), similarly, we get the sharp envelope \((H, V)^\# \in \text{FHS}_1\). Note that if \((H, V)\) is étale, \textit{i.e.}, \(H^0 = V^0 = 0\), we get \((H, V)^\# \cong (H_\mathbb{Z}, H_\mathbb{C}/F_{\text{Hodge}}^0)^\# = (H_\mathbb{Z}, H_\mathbb{C})\).
For a Laumon 1-motive $M = [F \to G]$ set $M_\times := [F \to G/V(G)]$ where $V(G)$ is the maximal additive subgroup of $G$. We get the sharp universal $\mathbb{G}_a$-extension $M^\#: = [F \to G^\#]$ by the pull-back of the universal $\mathbb{G}_a$-extension $M^\#_\times$ along $M \to M_\times$. 

The sharp de Rham realization

$$T_\#(M) = \text{Lie}(G^\#) \quad H^{1}_{\#-\text{dR}}(X) := T_\#(\operatorname{Pic}^+_{\text{a}}(X))$$

For $(H, V) \in \text{FHS}_1$, similarly, we get the sharp envelope $(H, V)^\# \in \text{FHS}_1$. Note that if $(H, V)$ is étale, i.e., $H^0 = V^0 = 0$, we get $(H, V)^\# \cong (H_{\mathbb{Z}}, H_{\mathbb{C}}/F^0_{\text{Hodge}})^\# = (H_{\mathbb{Z}}, H_{\mathbb{C}})$. Actually

$$T_\#(M)^\# \cong T_\#(M^\#) = (T_\#(F), T_\#(M))$$

This is a sharp version of the de Rham comparison theorem.
Under $T_\sharp$ Cartier duality corresponds to a canonical involution

\[ T_\sharp(M)^\vee \cong T_\sharp(M^\vee) \]
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$$T_\$^\#(M) ^\vee \cong T_\$^\#(M ^\vee)$$

Also for sharp de Rham

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Under $T_{\#}$ Cartier duality corresponds to a canonical involution

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However, note that special structures are not compatible with Cartier duality.
Under $T_\flat$ Cartier duality corresponds to a canonical involution

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However, note that special structures are not compatible with Cartier duality. For a Laumon 1-motive $M = [F \xrightarrow{u} G]$ we have: the realization $T_\flat(M) \in \text{FHS}_1$ is special $\iff$ $u$ is mapping $F^0$ to $V(G)$ ( = the maximal additive subgroup of $G$).
Under $T\flat$ Cartier duality corresponds to a canonical involution

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Also for sharp de Rham

$$T\#(M)^\vee \cong T\#(M^\vee)$$

However, note that special structures are not compatible with Cartier duality. For a Laumon 1-motive $M = [F \overset{u}{\rightarrow} G]$ we have: the realization $T\flat(M) \in \text{FHS}_1$ is special $\iff$ $u$ is mapping $F^0$ to $V(G)$ (= the maximal additive subgroup of $G$).

Example: the Cartier dual of $M = [\hat{A} \rightarrow A]$ for an abelian variety $A$ is the universal $\mathbb{G}_a$-extension $\text{Pic}^{0,\#}(A)$ of the dual $\text{Pic}^0(A)$. 
For $k \hookrightarrow \mathbb{C}$ let $D(Sch_k)^{op}$ be the following graph: objects are triples $(X, Y, i)$ where $X \in Sch_k$ and $Y \subseteq X$ is closed and $i$ is an integer, the arrows are as follows

a) $f^{op}: (X', Y', i) \rightarrow (X, Y, i)$ for any morphism $f : X \rightarrow X'$ such that $f|_Y : Y \rightarrow Y'$ and

b) $\delta^{op}: (Y, Z, i - 1) \rightarrow (X, Y, i)$ for any $Z \subseteq Y \subseteq X$ closed in $X$. 

We get a canonical representation $H^\ast\# : D(Sch_k)^{op} \rightarrow Fgrp$ given by $(X, Y, i) \mapsto H^i\#(X, Y)$, forgetting the (formal) Hodge structure of the singular sharp cohomology of the pair $(X_{an}, Y_{an})$, i.e., by the contravariant functoriality and the long exact sequence of the triple.
For \( k \hookrightarrow \mathbb{C} \) let \( D(Sch_k)^{op} \) be the following graph: objects are triples \((X, Y, i)\) where \( X \in Sch_k \) and \( Y \subseteq X \) is closed and \( i \) is an integer, the arrows are as follows

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H^*_\# : D(Sch_k)^{op} \to Fgrp
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given by \((X, Y, i) \rightsquigarrow H^i_\#(X, Y)\), forgetting the (formal) Hodge structure of the singular sharp cohomology of the pair \((X_{an}, Y_{an})\), i.e., by the contravariant functoriality and the long exact sequence of the triple.
Given a representation \( T : D \to \mathcal{A} \) of any (small) graph \( D \) into a suitable abelian category \( \mathcal{A} \) there exists an abelian category \( \mathcal{C}(T) \), a forgetful (faithful, exact) functor \( F_T : \mathcal{C}(T) \to \mathcal{A} \) and \( \tilde{T} : D \to \mathcal{C}(T) \) such that \( F_T \circ \tilde{T} = T \) universally, i.e., \( \mathcal{C}(T) \) is initial (up to isomorphisms of functors) with respect to all these factorizations of the representation \( T \).
Sharp motives via Nori’s theorem

Given a representation $T : D \to \mathcal{A}$ of any (small) graph $D$ into a suitable abelian category $\mathcal{A}$ there exists an abelian category $\mathcal{C}(T)$, a forgetful (faithful, exact) functor $F_T : \mathcal{C}(T) \to \mathcal{A}$ and $\tilde{T} : D \to \mathcal{C}(T)$ such that $F_T \circ \tilde{T} = T$ universally, i.e., $\mathcal{C}(T)$ is initial (up to isomorphisms of functors) with respect to all these factorizations of the representation $T$.

For $T = H^*$ and $\mathcal{A} =$ finitely generated abelian groups call effective cohomological mixed motives the resulting abelian categories

$$\text{ECM} := \mathcal{C}(H^*)$$
Given a representation $T : D \to \mathcal{A}$ of any (small) graph $D$ into a suitable abelian category $\mathcal{A}$ there exists an abelian category $\mathcal{C}(T)$, a forgetful (faithful, exact) functor $F_T : \mathcal{C}(T) \to \mathcal{A}$ and $\tilde{T} : D \to \mathcal{C}(T)$ such that $F_T \circ \tilde{T} = T$ universally, i.e., $\mathcal{C}(T)$ is initial (up to isomorphisms of functors) with respect to all these factorizations of the representation $T$.

For $T = H^*$ and $\mathcal{A} = \text{finitely generated abelian groups}$ call effective cohomological mixed motives the resulting abelian categories

$$\text{ECM} := \mathcal{C}(H^*)$$

For $T = H^*_\#$ and $\mathcal{A} = \text{Fgrp}$ call effective cohomological sharp mixed motives the resulting abelian categories

$$\text{ECM}^\# := \mathcal{C}(H^*_\#)$$

Note that this is just a speculation!
Scholium

Existence of the sharp cohomology $H^*_\#$ functor such that

$$H^*_\#(X, Y) \to H^*_\#(X, Z) \to H^*_\#(Y, Z) \to H^*_{\#+1}(X, Y)$$

is exact for $Z \subseteq Y \subseteq X$ closed in $X$
Existence of the sharp cohomology $H^*_\#$ functor such that

$$H^*_\#(X, Y) \to H^*_\#(X, Z) \to H^*_\#(Y, Z) \to H^*_{\#+1}(X, Y)$$

is exact for $Z \subseteq Y \subseteq X$ closed in $X \implies$ existence of sharp motives with realisations

$$\begin{array}{c}
\text{ECM}^\# & \xrightarrow{R^\#} & \text{FHS}^p \\
\uparrow & & \uparrow \\
\text{ECM} & \xrightarrow{R_{Hodge}} & \text{MHS}^p
\end{array}$$
Existence of the sharp cohomology $H^\#_\ast$ functor such that

$$H^\#_\ast(X, Y) \rightarrow H^\#_\ast(X, Z) \rightarrow H^\#_\ast(Y, Z) \rightarrow H^\#_{\ast+1}(X, Y)$$

is exact for $Z \subseteq Y \subseteq X$ closed in $X \implies$ existence of sharp motives with realisations

$$\matrix{ ECM^\# & \xrightarrow{R^\#} & FHS^p \\ \uparrow & & \uparrow \\ ECM & \xrightarrow{R_{Hodge}} & MHS^p }$$

such that

$$\matrix{ Laumon 1\text{-}motives & \rightarrow & ECM^\# \\ \uparrow & & \uparrow & & \uparrow \\ Deligne 1\text{-}motives & \rightarrow & ECM }$$
Thanks!

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