November 28, 2005

Dear Sasha Beilinson,

the matters here drafted have already been in my thoughts for a long time. I presented this subject very roughly in a letter to M. Saito on July 1, 2001 (and more recently mentioned in letters to S. Bloch, P. Deligne, B. Kahn and V. Srinivas). Since then, the optimistic strength – which is necessary to show these thoughts up – is growing constantly and I really think that we now can start to grasp the general theory of "formal Hodge structures" pointing to that of "sharp" cohomologies (singular, De Rham, crystalline, etc.) and even up to "sharp" motives.

Recall the following abelian categories:

MHS here denotes Deligne's category of (graded polarizable, if needed) $\mathbb{Z}$-mixed Hodge structures $(H_{\mathbb{Z}}, W_*, F^*_{Hodge})$,

VSP is the category of diagrams V given by

$$\cdots = V_k = V_{k-1} \to \cdots \to V_{k-h} \to 0 \to 0 \cdots$$

composable linear mappings of finite dimensional $\mathbb{C}$-vector spaces and

EHS is Bloch-Srinivas category of enriched mixed Hodge structures (E, V) given by commutative diagrams

$$\begin{array}{ccccc} H_{\mathbb{C}} & \twoheadrightarrow & H_{\mathbb{C}}/F^{k-1}_{Hodge} & \twoheadrightarrow \cdots \twoheadrightarrow & H_{\mathbb{C}}/F^{k-h}_{Hodge} \\ \uparrow & & \uparrow & & \uparrow \\ E & \longrightarrow & V_{k-1} & \to \cdots \longrightarrow & V_{k-h} \\ \uparrow & & & & \\ H_{\mathbb{C}} & & & & \end{array}$$

for $(H_{\mathbb{Z}}, W_*, F^*_{Hodge})$ with $0 = F^k_{Hodge} \subseteq \cdots \subseteq F^{k-h-1}_{Hodge} = H_{\mathbb{C}}$ and a splitting $E = H_{\mathbb{C}} \times E_{\dagger}$.

We here consider a larger one:

FHS the category of *formal Hodge structures* which is defined in the following way.

An abelian formal group[1] $H = H_{\mathbb{Z}} \times H^0$ over $\mathbb{C}$ such that $H_{\mathbb{Z}} = H_{\text{ét}}$ is the underlying group of $(H_{\mathbb{Z}}, W_*, F^*_{Hodge}) \in$ MHS, an object V of VSP and a suboject $V^0 \subseteq V$, an augmentation

[1]Note that in order to skip psychological side effects due to "formal" groups we just have to remember that a (affine, commutative) formal group F over a field k of zero characteristic is given by a canonical splitting $F_{\text{ét}} \times F^0$ where $F_{\text{ét}}$ is étale over k and F^0 is "infinitesimal" which means that is given by a finite number of copies of $\widehat{\mathbb{G}}_a$, the connected formal additive k-group. Further, we have that Lie (F^0) is a finite dimensional k-vector space, $F^0 \cong \widehat{\text{Lie}\,(F^0)} \hookrightarrow \text{Lie}\,(F^0)$ and there is a natural equivalence of connected formal groups with k-vector spaces.

map $v : H \to V$ and a $\mathbb{C}$-isomorphism $\sigma : H_{\mathbb{C}}/F_{Hodge} \xrightarrow{\simeq} V/V^0$ such that

$$\begin{array}{ccc} H_{\mathbb{Z}} & \xrightarrow{v_{\mathbb{Z}}} & V \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle pr} \\ H_{\mathbb{C}}/F_{Hodge} & \xrightarrow{\sigma} & V/V^0 \end{array}$$

commutes, *i.e.*, for $1 \leq i \leq h$, we have $\sigma_i : H_{\mathbb{C}}/F^{k-i}_{Hodge} \xrightarrow{\simeq} V_{k-i}/V^0_{k-i}$ which is an isomorphism (filtered by weights), the induced map $v^i_{\mathbb{Z}} : H_{\mathbb{Z}} \hookrightarrow H \to V_{k-1} \to \cdots \to V_{k-i}$, the canonical map $c_i : H_{\mathbb{Z}} \to H_{\mathbb{C}}/F^{k-i}_{Hodge}$ and the projection $\mathrm{pr}_i : V_{k-i} \twoheadrightarrow V_{k-i}/V^0_{k-i}$ in such a way that

$$\begin{array}{ccc} H_{\mathbb{Z}} & \xrightarrow{v^i_{\mathbb{Z}}} & V_{k-i} \\ {\scriptstyle c_i}\downarrow & & \downarrow{\scriptstyle pr_i} \\ H_{\mathbb{C}}/F^{k-i}_{Hodge} & \xrightarrow{\sigma_i} & V_{k-i}/V^0_{k-i} \end{array}$$

commutes, compatibly as i varies $1 \leq i \leq h$. Call formal Hodge structure a pair (H, V) like that and a morphism a pair of compatible maps. The commutative diagram provided by (H, V) is

$$\begin{array}{ccccc} & & H_{\mathbb{C}}/F^{k-1}_{Hodge} & \twoheadrightarrow \cdots \twoheadrightarrow & H_{\mathbb{C}}/F^{k-h}_{Hodge} \\ & & \uparrow & & \uparrow \\ H & \longrightarrow & V_{k-1} & \to \cdots \longrightarrow & V_{k-h} \\ \uparrow & & & & \\ H_{\mathbb{Z}} & & & & \end{array}$$

Now define étale structures $\mathrm{FHS}_{\text{ét}}$, connected structures FHS^0 and special structures FHS^s as follows. Let $(H, V)_{\text{ét}} := (H_{\mathbb{Z}}, V/V^0) \cong (H_{\mathbb{Z}}, H_{\mathbb{C}}/F_{Hodge})$. Say that (H, V) is étale if $(H, V)_{\text{ét}} = (H, V)$ and connected if $(H, V)_{\text{ét}} = 0$. For example, here $(0, V^0)$ is a connected substructure of (H, V). Denote $(H, V)_{\times} := (H, V/V^0)$ and note that we have a *canonical* extension

$$0 \to (H, V)_{\text{ét}} \to (H, V)_{\times} \to (H^0, 0) \to 0$$

Here we also have that (H^0, V) is a connected structure associated to any (H, V) but it is not a substructure, in general. Say that (H, V) is special if $(H^0, V^0) := (H, V)^0$ is a substructure of (H, V) or, equivalently, $(H, V)_{\text{ét}}$ is a quotient of (H, V), so that we have an extension

$$0 \to (H, V)^0 \to (H, V) \to (H, V)_{\text{ét}} \to 0$$

in this case. We then have the following properties:

FHS^0 is equivalent to the category VSP *via* $(H, V) \mapsto \mathrm{Lie}\,(H^0) \to V_{k-1} \to \cdots \longrightarrow V_{k-h}$ (actually to the category of h-composable linear maps for a fixed $1 \leq h$ above)

$\mathrm{FHS}_{\text{ét}}$ is equivalent to the category MHS *via* $(H, V) \mapsto H_{\text{ét}}$

FHS^s is equivalent to the category EHS *via* $(H, V) \mapsto (H_{\mathbb{C}} \times \mathrm{Lie}\,(H^0), V)$ since $\mathrm{Hom}\,(H, V) = \mathrm{Hom}\,(H_{\mathbb{C}} \times \mathrm{Lie}\,(H^0), V)$ and, conversely, given (E, V) from the splitting $E = H_{\mathbb{C}} \times E_{\dagger}$ we get the formal group $H = H_{\mathbb{Z}} \times \widehat{E_{\dagger}}$ such that $\widehat{E_{\dagger}}$ is mapped to V^0_{k-1}.

The way they match together is as follows:

VSP is embedded as a Serre subcategory of FHS with a left inverse and quotient MHS

MHS fully embedded in FHS has also a left inverse "the étale forgetful functor"

EHS is the largest subcategory of FHS such that MHS into FHS has a left adjoint and VSP into FHS has a right adjoint (according with the presentation of an enriched or special structure as an extension of the étale part by the connected one).

We then have the following picture:

$$\begin{tikzcd} & \mathrm{FHS} & \\ & \mathrm{EHS} \arrow[u] & \\ \mathrm{MHS} \arrow[ur] & & \mathrm{VSP} \arrow[ul] \end{tikzcd}$$

along with the corresponding left inverses from FHS and restricting to left/right adjoints from EHS but I don't see any "reasonable" functor from FHS to EHS. The lovely advantage of formal Hodge structures is the following striking relation with Laumon & Deligne 1-motives:

$\mathrm{FHS}_1^{\mathrm{fr}}$ is equivalent to Laumon 1-motives, *i.e.*, Deligne's equivalence between 1-motives over $\mathbb{C}$ and mixed Hodge structures of level ≤ 1 *via* Hodge realization can be extended to an equivalence

$$\begin{array}{ccc} \text{Deligne's 1-motives} & \stackrel{\cong}{\to} & \mathrm{MHS}_1^{\mathrm{fr}} \\ \uparrow\downarrow & & \uparrow\downarrow \\ \text{Laumon's 1-motives} & \stackrel{\cong}{\to} & \mathrm{FHS}_1^{\mathrm{fr}} \end{array}$$

Under this equivalences Cartier duality corresponds to a canonical involution. Note that such (graded polarizable of level ≤ 1) special or enriched structures are not compatible with Cartier duality.[2]

The general plan is to associate to an algebraic $\mathbb{C}$-scheme X a "formal Hodge structure" called "sharp" cohomology $H_\sharp^*(X)$ which contains, in the underlying algebraic structure, a formal group which is an extension of ordinary singular cohomology (= its étale part, etc.). There will be "sharp" versions of De Rham and crystalline as well. Following Grothendieck strategy to construct a cohomology I will take care of the H^1 first *via* Laumon 1-motives and "sharp" realizations. It means that, we are requiring to visualize $H_\sharp^1(X)$, $H_{\sharp-DR}^1(X)$, $H_{\sharp-crys}^1(X)$ *via* a Laumon 1-motive $\mathrm{Pic}_a^+(X)$ (and, dually, the H_1 by its Cartier dual $\mathrm{Alb}_a^-(X)$). If X is proper $\mathrm{Pic}_a^+(X) := \mathrm{Pic}^0(X)$ and if X is smooth $\mathrm{Pic}_a^+(X)_{\text{ét}} = [\mathrm{Div}_Y^0(\overline{X}) \to \mathrm{Pic}^0(\overline{X})]$ where $(\overline{X}, Y)$ is a normal crossing compactification,

[2]To see this observe that a Laumon 1-motive $M = [F \stackrel{u}{\to} G]$ with u mapping F^0 to $V(G)$ (= the maximal additive subgroup of G) $\iff$ the realization $T_{\oint}(M) \in \mathrm{FHS}_1^{\mathrm{fr}}$ is special. Then note that the Cartier dual of $M = [\widehat{A} \to A]$ for an abelian variety A is the universal $\mathbb{G}_a$-extension $\mathrm{Pic}^{0,\natural}(A)$ of the dual $\mathrm{Pic}^0(A)$.

in such a way that the Cartier dual is an additive extension of the Serre's Albanese semi-abelian variety of X. In general, the largest 1-motivic part of $H_\sharp^{1+i}(X)$ should be given by applying

$$T_\oint : \text{Laumon's 1-motives} \longrightarrow \text{FHS}$$

to algebraically defined Laumon 1-motives $\mathrm{Pic}_a^+(X, i)$ for $i \geq 0$ (generalizing Deligne's conjecture on 1-motives, etc.).

Actually, the entire *cohomological* study of algebraic varieties should be restated in the context of "sharp" cohomologies (which agree with the classical cohomologies of projective non-singular varieties) taking care of "non-homotopy invariant theories" of schemes (as also remarked by S. Bloch and V. Voevodsky). There should be a "sharp" *cohomological* motive $M_\sharp(X)$ in a category $DM_\sharp$, related to Voevodsky category of motivic complexes, with a realisation in $D^b(\text{FHS})$. Your conjectural formalism for motivic complexes should be translated for $\sharp$-motivic complexes. For example, we also have the following picture:

$$\begin{array}{ccccc} & & D^b(\text{FHS}) & & \\ & & \uparrow & & \\ & & D^b(\text{EHS}) & & \\ & \nearrow & & \nwarrow & \\ D^b(\text{MHS}) & & & & D^b(\text{VSP}) \end{array}$$

and I suppose that a similar presentation that you gave of "absolute Hodge cohomology" *via* Deligne's mixed Hodge structures should be possible in $D^b(\text{FHS})$. I mean, for any algebraic $\mathbb{C}$-scheme X, there should be a canonical object $R\Gamma_\sharp(X) \in D^b(\text{FHS})$ whose underlying complex of abelian formal groups is the chain $\sharp$-singular complex and such that

$$H_\sharp^*(X) = H^*(R\Gamma_\sharp(X)) \in \text{FHS}$$

The $\sharp$-absolute formal Hodge cohomology should be given by

$$R\Gamma_{\sharp-\mathcal{H}}(X)(i) := RHom_{D^b(\text{FHS})}(\mathbf{1}, R\Gamma_\sharp(X)(i))$$

as you indicated for MHS.

That's all for the moment ... hoping to have puzzled you enough!

Kind Regards, Luca Barbieri Viale.