GROTHENDIECK-SERRE DUALITY

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1. Preliminaries

1.1. Koszul Complex I.

Definition 1.1.1. (1) Let A be a commutative ring and E be a A-module. Then, for any A-morphism $u : E \to A$, we can define the Koszul Complex $K.(u) \in C^{\leqslant 0}(A)$ as follows:

$$K_n(u) = \wedge^n E, n \geqslant 0;$$

$$d_n : K_n(u) \to K_{n-1}(u), \text{with } d_n(x_1 \wedge ... \wedge x_n) = \sum (-1)^{i-1} u(x_i) x_1 \wedge ... \wedge \hat{x_i} \wedge ... \wedge x_n.$$

Clearly, $d^2 = 0, d_1 = u$ and $d(a \wedge b) = da \wedge b + (-1)^p a \wedge db (a \in \wedge^p E, b \in \wedge^q E)$.

(2) A, E, u as above, M is an A-module, define $K.(u, M) = K.(u) \otimes_A M$, with $d(x \otimes m) = dx \otimes m$.

Remarks 1.1.2. If $E = E_1 \oplus E_2, u = u_1 + u_2 : E \to A$, where $u_i : E_i \to A$, then $K.(u, E) = K.(u_1) \otimes K.(u_2), \wedge^n E = \oplus_{p+q=n} \wedge^p E_1 \otimes \wedge^q E_2, d = d_1 \otimes 1 + (-1)^* 1 \otimes d_2$.

Dually, we also have

[**Definition 1.1.1.'**] (1') For any morphism $v : A \to F$, we can define a complex $K^{\cdot}(v) \in C^{\geqslant 0}(A)$ called the Koszul Complex, too, as follows:

$$K^n(v) = \wedge^n F;\ d : K^n(v) \to K^{n+1}(v),\ d(x) = v \wedge x.$$

Here we identify the morphism v with $v(1) \in F$.

(2') Similar definition for $K^{\cdot}(v, N)$ with A, F, v as above and N an A-module.

[**Remark 1.1.2.'**] In this case, we also have: for $F = F_1 \oplus F_2$, $v = (v_1, v_2)$, $K^{\cdot}(v) = K^{\cdot}(v_1) \otimes K^{\cdot}(v_2)$.

Lemma 1.1.3. *For $f = (f_1, ..., f_r) \in A^r$, we have two Koszul complexes $K.(f)$ and $K^{\cdot}(f)$.*

$$K.(f) : 0 \to A \to A^r (= \wedge^{r-1} A^r) \to ... \to (\wedge^1 A^r =) A^r \xrightarrow{f} A \to 0, f(a_1, ..., a_r) = \sum f_i a_i.$$

$$K^{\cdot}(f) : 0 \to A \xrightarrow{f} A^r (= \wedge^1 A^r) \to ... \to (\wedge^{r-1} A^r =) A^r \to A \to 0,\ f(a) = (f_1 a, ..., f_r a).$$

Then $K^{\cdot}(f)$ can be viewed as the naive dual of $K.(f)$. Furthermore, we have a canonical isomorphism between the two: $K^{\cdot}(f)[r] \simeq K.(f)$.

Proof. The first part is immediate. For the second part, the isomorphism is defined as follows:

Let $\{e_1, ..., e_r\}$ be a basis of A^r. For any $I = \{i_1 < ... < i_p\} \subset \{1, ..., r\}$, let $e_I = e_{i_1} \wedge ... \wedge e_{i_p}$, then $e_I \mapsto \varepsilon(J, I) e_J$, where $J = \{j_1 < ... < j_{r-p}\}$ is the complement of I in $\{1, ..., r\}$ and $\varepsilon(J, I) =$sign$(j_1, ..., j_{r-p}, i_1, ..., i_p)$. □

Lemma 1.1.4. *Let $L \in C(A)$ and $x \in A$, $K.(x) = (0 \to A \xrightarrow{x} A \to 0)$. Then $K.(x) \otimes L \simeq Cone(L \xrightarrow{x} L)$.*

Proof. This is simply a computation. ☐

Now we discuss a little about this lemma. For this, we get a distinguished triangle $L \xrightarrow{x} L \to K.(x) \otimes L \to$, so we get a long exact sequence:

$$... \to H^q(L) \xrightarrow{x} H^q(L) \to H^q(K.(x) \otimes L) \to H^{q+1}(L) \xrightarrow{x} ...$$

Rewrite it into short exact sequences, and we can get:

$$0 \to H^0K.(x, H^q(L)) \to H^q(K.(x) \otimes L) \to H^{-1}K.(x, H^{q+1}(L)) \to 0. \ (*)$$

Theorem 1.1.5 (Serre)**.** *Let A be a noetherian ring, M an A-module of finite type and $f = (f_1, ..., f_r) \in A^r$ with $f_i \in$rad(A), then the following conditions are equivalent:*

(1) *f is M-regular.*
(2) *$K.(f, M) \to M/(f_1, ..., f_r)M$ is quasi-isomorphism.*
(3) *$H^{-1}K.(f, M) = 0$.*

Proof. (1)⇒(2) Use induction on r. For $r = 1$, the statement is just the definition.

Assume the statement is true for $m \leqslant r - 1$, then let

$$L = K.(f_1, ..., f_{r-1}, M) = K.(f_1, ..., f_{r-1}) \otimes M,$$

then $K.(f_r) \otimes L \simeq K.(f_1, ..., f_r, M)$. Hence we have the exact sequence

$$0 \to H^0K.(f_r, H^q(L)) \to H^qK.(f_1, ..., f_r, M) \to H^{-1}K.(f_r, H^{q+1}(L)) \to 0.$$

We are left to show that $H^qK.(f_1, ..., f_r, M) = 0$ for all $q < 0$. For $q \leqslant -2$, it follows from the above sequence and the inductive hypothesis. For $q = -1$, it is true since $\mathrm{Ker}(f_r : M/(f_1, ..., f_{r-1})M \to M/(f_1, ..., f_{r-1})M) = 0$ by definition of M-regular.

(2)⇒(3) trivial.

(3)⇒(1) Also use induction on r. Again, the case $r = 1$ is trivial. For $r \geqslant 2$, again let $L = K.(f_1, ..., f_{r-1}, M)$. First show that $(f_1, ..., f_{r-1})$ is M-regular. By (*), we have an inclusion

$$H^q(L)/f_rH^q(L) \hookrightarrow H^qK.(f_1, ..., f_r, M)$$

When $q = -1, H^qK.(f_1, ..., f_r, M) = 0$, hence $H^{-1}(L) = f_rH^{-1}(L)$. Since A is noetherian and M is of finite type, $H^{-1}(L)$ is finitely generated over A, so $H^{-1}(L) = 0$ since $f_r \in$rad(A). Now $(f_1, ..., f_{r-1})$ is M-regular by induction. Furthermore, condition (3) implies that

$$\mathrm{Ker}(f_r : M/(f_1, ..., f_{r-1})M \to M/(f_1, ..., f_{r-1})M) = 0.$$

Hence $f = (f_1, ..., f_r)$ is M-regular ☐

Corollary 1.1.6. *Assume $f = (f_1, ..., f_r) \in A^r$ is regular and $B = A/(f_1, ..., f_r)A$. Then $Ext^q_A(B, A) = \begin{cases} 0 & q \neq r \\ B & q = r \end{cases}$.*

Proof. Since $K.(f) \to B$ is a quasi-isomorphism,

$$RHom_A(B, A) = Hom_A(K.(f), A) = K^{\cdot}(f) \simeq K.(f)[-r],$$

hence

$$Ext^q_A(B, A) = \begin{cases} H^{q-r}K.(f) = 0 & q \neq r \\ H^0 K.(f) = B & q = r \end{cases}.$$

□

1.2. **Koszul Complex II.** In this section, we generalize the discussion in section 1 to ringed spaces. In this section, we abuse the notations $\mathcal{O}_Y$ and $i_*\mathcal{O}_Y$ when $i : Y \hookrightarrow X$ is a closed immersion.

Definition 1.2.1. Let $(X, \mathcal{O}_X)$ be a ringed space and $E \in Mod(X)$, then for any morphism $u : E \to \mathcal{O}_X$, define the Koszul complex $\mathcal{K}.(u)$ by

$$(... \to \wedge^n E \xrightarrow{d} \wedge^{n-1} E \to ... \to E \xrightarrow{u} \mathcal{O}_X \to 0),$$

where d is the right interior product by u.

Definition 1.2.2. For $i : Y \hookrightarrow X$ a closed immersion defined by the ideal sheaf $I \subset \mathcal{O}_X$, we say that i is regular of codimension r if $\forall x \in Y$, $\exists U \subset X$ an open neighbourhood of x and an $\mathcal{O}_U$-module E locally free of rank r and an $\mathcal{O}_U$-linear map $u : E \to \mathcal{O}_U$, s.t. $H^q\mathcal{K}.(u) = 0 (q < 0)$ and $I|_U = u(E) \subset \mathcal{O}_U$. In other words, this is equivalent to say that $\exists$ locally a squence $f = (f_1, ..., f_r) \in \mathcal{O}_U^r$ s.t. $I|_U = (f_1, ..., f_r)$ and $\mathcal{K}.(f) \to \mathcal{O}_U/I\mathcal{O}_U$ is a resolution.

Remarks 1.2.3. If X is locally noetherian, then i is regular iff $\forall x \in Y$, $\exists x \in U$ open, s.t. I is defined by a sequence $f = (f_1, ..., f_r)$ of sections of $\mathcal{O}_X$ s.t. $f_x = ((f_1)_x, ..., (f_r)_x) \in \mathfrak{m}_x^r$ is regular.

Proposition 1.2.4. *If $i : Y \hookrightarrow X$ is a regular immersion of codimension r, $\mathcal{I}$ be the ideal sheaf of i, $N_{Y/X} = \mathcal{I}/\mathcal{I}^2$, then*

(1) $\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) = \begin{cases} 0 & q \neq r \\ \omega_{Y/X} & q = r \end{cases}$, *where $\omega_{Y/X}$ is a line bundle on Y. In other words, $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) \simeq \omega_{Y/X}[-r]$.*
(2) *$N_{Y/X}$ is locally free of rank r.*
(3) $\omega_{Y/X} \simeq (\wedge^r N_{Y/X})^\vee$.
(4) *For $F \in D^+(X)$, there exists a functorial isomorphism*

$$\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_Y, F) \simeq \mathcal{T}or^{\mathcal{O}_X}_{r-q}(\mathcal{O}_Y, F) \otimes \omega_{Y/X}.$$

Proof. (1) For any U =Spec$A$ open in X, $U \cap Y$ =Spec$B$, with $B = A/(f_1, ..., f_r)$ where $f = (f_1, ..., f_r)$ is regular, by [Cor], we have

$$Ext^q_{\Gamma(U,\mathcal{O}_X)}(\Gamma(U \cap Y, \mathcal{O}_Y), \Gamma(U, \mathcal{O}_X)) = Ext^q_A(B, A) = \begin{cases} 0 & q \neq r \\ B & q = r \end{cases}.$$

The conclusion then holds immediately.

(2) From the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

we get

$$N_{Y/X} = \mathcal{I}/\mathcal{I}^2 \simeq \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y).$$

Since i is regular, locally one has a Koszul complex $\mathcal{K}.(f_1, ..., f_r) = (0 \to \mathcal{O}_X \to ... \to \mathcal{O}_X^r \to \mathcal{O}_X)$, which is a resolution of $\mathcal{O}_Y$. Hence locally

$$\mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) = \mathcal{H}^{-1}(\mathcal{K}.(f) \otimes_{\mathcal{O}_X} \mathcal{O}_Y) = \mathcal{O}_Y^r,$$

which proves (2). Also note that we can deduce

$$\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) = \wedge^q(\mathcal{O}_Y^r),$$

and

$$\wedge^* \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \xrightarrow{\sim} \mathcal{T}or_*^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y).$$

(4) It's enough to show that

$$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F) \simeq i_*(Li^*F \otimes^L_{\mathcal{O}_Y} \omega_{Y/X}[-r]) \qquad (*)$$

for each $F \in D^+(X)$, then apply H^q to both sides. By the first part of the following lemma, the RHS(right hand side) of the above formula is just

$$F \otimes^L_{\mathcal{O}_X} i_*\omega_{Y/X}[-r] = F \otimes^L_{\mathcal{O}_X} R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X).$$

By the second part of the following lemma, we see that this is canonically isomorphic to $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)$, which is exactly the LHS(left hand side).

(3) We set $F = \mathcal{O}_Y$ in (4), then we get

$$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \mathcal{O}_Y \otimes_{\mathcal{O}_X} i_*\omega_{Y/X}[-r].$$

Note that $\otimes^L_{\mathcal{O}_X} i_*\omega_{Y/X}[-r] \simeq \otimes_{\mathcal{O}_X} i_*\omega_{Y/X}[-r]$ since $\omega_{Y/X}[-r]$ is locally free.

Apply H^0 to the above formula, and note that $H^0(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)) = \mathcal{O}_Y$, we get

$$\mathcal{O}_Y \simeq \mathcal{T}or_r^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \otimes \omega_{Y/X}.$$

In (2), we have already seen that $\mathcal{T}or_r^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \wedge^r N_{Y/X}$, thus $\omega_{Y/X} \simeq (\wedge^r N_{Y/X})^\vee$. □

Lemma 1.2.5. *Let $i : Y \hookrightarrow X$ be a regular closed immersion, then one has*

$$F \otimes^L_{\mathcal{O}_X} i_*G \simeq i_*(Li^*F \otimes^L_{\mathcal{O}_Y} G)$$

and

$$F \otimes^L_{\mathcal{O}_X} R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)$$

for $F \in D^+(X)$ and $G \in D^b(Y)$.

Proof. For the first part, first note that i^* is of finite cohomological dimension since $Li^* = \mathcal{O}_Y \otimes^L_{\mathcal{O}_X}$ and $\mathcal{O}_Y$ admits a Koszul resolution. Hence Li^* makes sense on $D^+(X)$.

We have a natural map between $Li^*(F \otimes^L_{\mathcal{O}_X} i_*G)$ and $Li^*F \otimes^L_{\mathcal{O}_Y} G$ defined by

$$Li^*(F \otimes^L_{\mathcal{O}_X} i_*G) \to Li^*F \otimes^L_{\mathcal{O}_Y} Li^*(i_*G) \to Li^*F \otimes^L_{\mathcal{O}_Y} G,$$

where the last map is given by the natural map $Li^*i_*G \to G$. This gives the desired map $F \otimes^L_{\mathcal{O}_X} i_*G \to i_*(Li^*F \otimes^L_{\mathcal{O}_Y} G)$ by the adjiontness of Li^* and i^*.

To show this is an isomorphism, by canonical truncations(note that i^* is of finite cohomological dimension), we may assum that $F \in D^b(X)$. Replacing F by its flat resolution, we can see that

$$F \otimes^L_{\mathcal{O}_X} i_*G = F \otimes_{\mathcal{O}_X} i_*G \simeq i_*(i^*F \otimes_{\mathcal{O}_Y} G) = i_*(Li^*F \otimes^L_{\mathcal{O}_Y} G).$$

For the second part, use the Koszul resolution $\mathcal{K}.(f) = (0 \to \mathcal{O}_X \to ... \to \mathcal{O}_X^r \to \mathcal{O}_X)$ of $\mathcal{O}_Y$. Note that this is a free resolution of $\mathcal{O}_Y$, then the conclusion follows immediately. □

Next, we consider the projective case $X = \mathbb{P}^r_Y$, with $f : X \to Y$ the projection. We will show using the tool of Koszul complex that $\Omega^r_{X/Y} \simeq \mathcal{O}_X(-r-1)$.

We know that there is a canonical exact sequence

$$0 \to \Omega^1_{X/Y} \xrightarrow{v} \mathcal{O}_X^{r+1}(-1) \xrightarrow{u} \mathcal{O}_X \to 0.$$

The Koszul complex of u is

$$0 \to \wedge^{r+1}(\mathcal{O}_X^{r+1})(-r-1) \to ... \to \mathcal{O}_X^{r+1}(-1) \to \mathcal{O}_X \to 0.$$

If we can prove that each sequence

$$0 \to \Omega^i_{X/Y} \xrightarrow{\wedge^i v} \wedge^i(\mathcal{O}_X^{r+1})(-i) \to ... \to \mathcal{O}_X^{r+1}(-1) \to \mathcal{O}_X \to 0, i \geqslant 0,$$

is exact, then in particular, let $i = r$ and compare it with the previous sequence, we have a canonical isomorphism $\Omega^r_{X/Y} \simeq \mathcal{O}_X(-r-1)$. We conclude it in the following lemma.

Lemma 1.2.6. *Let $(X, \mathcal{O}_X)$ be a ringed space,*

$$0 \to F \xrightarrow{v} E \xrightarrow{u} \mathcal{O}_X \to 0$$

be an exact sequence of locally free sheaves of finite ranks. Then the Koszul complex of u

$$\mathcal{K}.(u) = (0 \to \wedge^n E \xrightarrow{d_n} \wedge^{n-1}E \to ... \to E \xrightarrow{d_1=u} \mathcal{O}_X \to 0)$$

where n =rank$E$ is acyclic and each sequence

$$0 \to \wedge^i F \xrightarrow{\wedge^i v} \wedge^i E \xrightarrow{d} \wedge^{i-1}E \to ... \to E \xrightarrow{d} \mathcal{O}_X \to 0$$

is exact. Hence $\wedge^i v$ induces an isomorphism $\wedge^i F \to B^{-i-1}\mathcal{K}.(u)(i \geqslant 0)$. In particular, taking $i = n-1$, we get an isomorphism $\wedge^{n-1}F \to \wedge^n E$ s.t.

$$\begin{array}{ccc} \wedge^{n-1}F & \xrightarrow{\wedge^{n-1}v} & \wedge^{n-1}E \\ \downarrow & \nearrow_{d_n} & \\ \wedge^n E & & \end{array}$$

commutes, which coincides with the isomorphism $\wedge^{n-1}F \to \wedge^n E$ given by taking the highest exterior power of the original exact sequence and locally defined by $u(b)a \mapsto b \wedge (\wedge^{n-1}v)(a)$ for $a \in \wedge^{n-1}F(U), b \in E(U)$.

Proof. Without any loss, we may assume $E = \mathcal{O}_X \oplus F$ and u the projection since the three of them are all locally free of finite ranks. Then

$$d_i : \wedge^i F \oplus (\mathcal{O}_X \otimes \wedge^{i-1}F) = \wedge^{i-1}E \to \wedge^{i-1}E = \wedge^{i-1}F \oplus (\mathcal{O}_X \otimes \wedge^{i-1}F)$$

is induced by

$$(a, 1 \otimes b) \mapsto (b, 0).$$

Then it can be checked directly the exactness of the sequences. Take $i = n$, we get the acyclicality. The remainder is then obvious. □

2. Grothendieck-Serre Global Duality

In this chapter, for simplicity, we just discuss the locally noetherian case unless it's specially stated.

2.1. The Functor $f^!$.

Definition 2.1.1. (1) Let $i : Y \hookrightarrow X$ be a closed immersion. Given $F \in D^+(X)$, define

$$i^! F := R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, F)|_Y = i^{-1}R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, F) = i^*R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, F),$$

i.e. $i_*i^!F = R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)$. Clearly, $i^!F \in D^+(Y)$. Moreover, this gives a functor from $D^+(X)$ to $D^+(Y)$. We would prove it later.

(2) Let $f : X \to Y$ be a smooth morphism with relative dimension d, then $\omega_{X/Y} = \Omega^d_{X/Y}$ is a line bundle. Define a functor $f^! : D^+(X) \to D^+(Y)$ by

$$f^!F := f^*F \otimes^L_{\mathcal{O}_X} \omega_{X/Y}[d]$$

for an element $F \in D^+(X)$.

(3) Let

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ {\scriptstyle f}\downarrow & \nearrow_{g} & \\ Y & & \end{array}$$

be a commutative diagram with i a closed immersion and g smooth. We can define a functor $i^!g^!$ from $D^+(X)$ to $D^+(Y)$. The main goal of this part is to prove that the last functor is independent of the choice of i and g.

Lemma 2.1.2. *Suppose $i : Y \hookrightarrow X$ is a closed immersion, then*

(1) *$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)|_Y$ is an injective $\mathcal{O}_Y$-module if F is an injective $\mathcal{O}_X$-module.*
(2) *In this case, $i^!$ really is a functor.*

Proof. (1) For every $G \in \mathrm{Mod}(\mathcal{O}_Y)$, since

$$i_*G \otimes_{\mathcal{O}_X} i_*\mathcal{O}_Y \simeq i_*G \otimes_{i_*\mathcal{O}_Y} i_*\mathcal{O}_Y \simeq i_*G,$$

it's enough to prove that

$$Hom_{\mathcal{O}_X}(i_*G, \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, F)) \simeq Hom_{\mathcal{O}_Y}(G, i^{-1}\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, F)).$$

It's easy to see that $G \simeq i^{-1}i_*G$ by checking on stalks, so the RHS of the above formula can be rewritten as

$$Hom_{\mathcal{O}_Y}(i^{-1}i_*G, i^{-1}\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, F)) \simeq Hom_{\mathcal{O}_X}(i_*G, i_*i^{-1}\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, F)).$$

Taking $i_*i^{-1}\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, F) \simeq \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, F)$ into consideration (this is right since $i_*\mathcal{O}_Y$ is a locally finite presented $\mathcal{O}_X$-module, thus we can check on stalks), we get the conclusion.

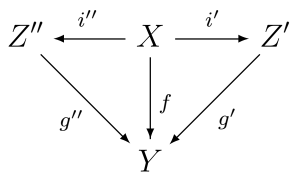
(2) For any $Z \xrightarrow{j} Y \xrightarrow{i} X$ a composition of closed immersions and $F \in D^+(X)$, replace F by its injective resolution, then

$$\begin{aligned} j^! i^! F &= \mathcal{H}om_{\mathcal{O}_Y}(j_* \mathcal{O}_Z, \mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{O}_Y, F)|_Y)|_Z \\ &\simeq \mathcal{H}om_{\mathcal{O}_X}((ij)_* \mathcal{O}_Z, \mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{O}_Y, F))|_Z \\ &\simeq \mathcal{H}om_{\mathcal{O}_X}(((ij)_* \mathcal{O}_Z) \otimes_{\mathcal{O}_X} (i_* \mathcal{O}_Y), F)|_Z \\ &\simeq \mathcal{H}om_{\mathcal{O}_X}((ij)_* \mathcal{O}_Z, F)|_Z \\ &= R\mathcal{H}om_{\mathcal{O}_X}((ij)_* \mathcal{O}_Z, F)|_Z \\ &= (ij)^! F. \end{aligned}$$

□

Now we come to the main theorem of this part. For the rest of the whole section, we again abuse the notations $\mathcal{O}_Y$ and $i_* \mathcal{O}_Y$ when $i : Y \hookrightarrow X$ is a closed immerstion.

Theorem 2.1.3. *Suppose we have a commutative diagram*



where i', i'' are closed immersions and g', g'' are smooth. Then there is a natural isomorphism

$$a(i', i'') : i'^! g'^! \simeq i''^! g''^!$$

satisfying the transitive formula:

$$a(i_2, i_3) \circ a(i_1, i_2) = a(i_1, i_3)$$

for any triple $(i_1, g_1), (i_2, g_2), (i_3, g_3)$.

We say that these $a(i', i'')$ form a transitive system. In order to prove this theorem, we still need some preparation.

Lemma 2.1.4. *Let X, Y be locally noetherian, and $f : X \to Y$ be a flat morphism. Then*

$$f^* R\mathcal{H}om(L, M) \xrightarrow{\sim} R\mathcal{H}om(f^* L, f^* M)$$

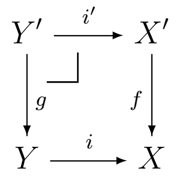
for $M \in D^+(Y)$ and $L \in D^b(Y)_{coh}$.

Proof. Replacing M by its injective resolution, then we get

$$f^* R\mathcal{H}om(L, M) = f^* \mathcal{H}om(L, M) \to \mathcal{H}om(f^* L, f^* M) \to R\mathcal{H}om(f^* L, f^* M),$$

which defines the map we want. To show it's an isomorphism, we may assume that Y is affine and noetherian, since the problem is local. Then there exists a quasi-isomorphism $L' \to L$ with each L'^i free of finite rank and $L'^i = 0$ when i is sufficiently large. Then it is clear. □

Lemma 2.1.5. *Consider a cartesian diagram*



where i is closed immersion and f is flat. Then we have $g^*i^! \simeq i'^! f^*$.

Proof. We have to show that for any $F \in D^+(X)$, there is a natural isomorphism $i'_* g^* i^! F \simeq i'_* i'^! f^* F$. But the LHS is

$$i'_* g^* i^! F = i'_*(g^* i^* R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)) = f^* R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F),$$

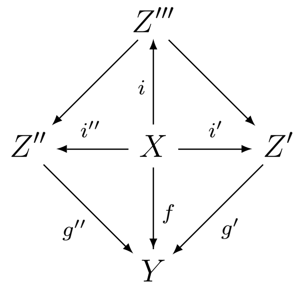
while the RHS is

$$i'_* i'^! f^* F = R\mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{O}_{Y'}, f^* F) = R\mathcal{H}om_{\mathcal{O}_{X'}}(f^* \mathcal{O}_Y, f^* F).$$

Then our conclusion follows from the previous lemma. □

Now we come back to the proof of the Theorem.

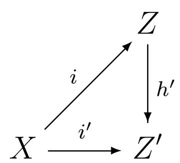
Proof of Theorem. Consider diagram , let $Z''' = Z' \times_Y Z''$, then we can complete the diagram as follows:



where i is the map determined by (i', i''). In general, i is not a closed immersion, but only an immersion, i.e. a composition of a closed immersion with an open immersion:

$$X \xrightarrow{\textit{closed}} Z \xrightarrow{\textit{open}} Z'''.$$

Thus we can replace Z''' by Z, and consider the diagram

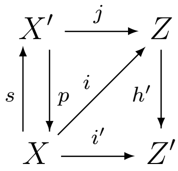


where i and i' are both closed immersions and h' is smooth. If we can show that $i'^! \simeq i^! h'^!$, then we have

$$i'^! g'^! \simeq i^! h'^! g'^! = i^! h''^! g''^! \simeq i''^! g''^!.$$

And This gives the desired functor isomorphism.

Let $X' = X \times_{Z'} Z$, then we get the following cartesian diagram:



where s is the section of X determined by $(1_X, i)$. Notice that in this case, p is smooth, s is a closed immersion since both i and j are, and $ps = 1_X$ is smooth, thus s is actually a regular closed immersion(we assume this).

Now suppose the relative dimension of h' is d, then $\forall F \in D^+(Z')$, we have

$$i^! h'^! F = s^! j^! h'^! F = s^! j^! (h'^* F \otimes \omega_{Z/Z'}[d]).$$

But

$$\begin{aligned} j^!(h'^* F \otimes \omega_{Z/Z'}[d]) &= R\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_{X'}, h'^* F \otimes \omega_{Z/Z'}[d])|_{X'} \\ &= (R\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_{X'}, h'^* F) \otimes \omega_{Z/Z'}[d])|_{X'} \\ &= R\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_{X'}, h'^* F)|_{X'} \otimes \omega_{X'/X}[d] \\ &= j^! h'^* F \otimes \omega_{X'/X}[d]. \end{aligned}$$

Hence

$$\begin{aligned} i^! h'^! F = s^! j^! h'^! F &= s^!(j^! h'^* F \otimes \omega_{X'/X}[d]) \\ &= s^!(p^* i'^! F \otimes \omega_{X'/X}[d]), \end{aligned}$$

where the third equality is according to Lemma 2.1.5. Hence it only remains to show that

$$s^!(p^* M \otimes \omega_{X'/X}[d]) = M$$

for any $M \in D^+(X)$. According to $(*)$ in the proof of Proposition 1.2.4.(4), the LHS is

$$Ls^*(p^* M) \otimes^L \omega_{X/X'}[-d] \otimes s^* \omega_{X'/X}[d] = M \otimes \omega_{X/X'} \otimes s^* \omega_{X'/X}$$

since $ps = 1_X$. But the canormal sheaf $N_{X/X'} \simeq s^* \Omega^1_{X'/X}$, and by Proposition 1.2.4.(3), it follows that

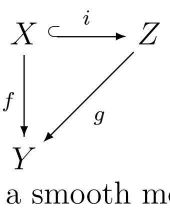
$$\omega_{X/X'} \simeq (\wedge^d N_{X/X'})^\vee = (s^* \omega_{X'/X})^\vee,$$

and hence

$$M \otimes \omega_{X/X'} \otimes s^* \omega_{X'/X} \simeq M$$

which completes the proof. □

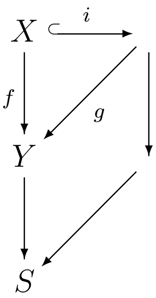
Definition 2.1.6. A morphism of schemes $f : X \to Y$ is smoothable if it can be decomposed as $f = gi$



where i is a closed immersion and g is a smooth morphism.

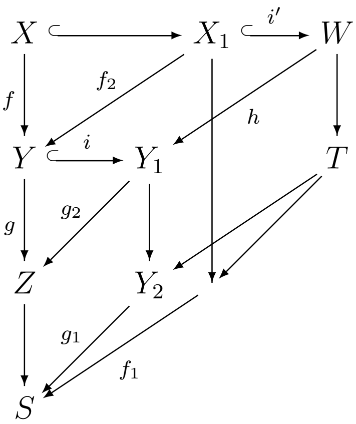
In this case, $i^! g^! : D^+(Y) \to D^+(X)$ depends only of f, and we denote it by $f^!$.

Definition 2.1.7. A morphism of S-schemes $f : X \to Y$ is S-smoothable if there exists a commutative diagram



with i a closed immersion and g a smooth morphism s.t. the parallelogram is Cartesian.

Let $f : X \to Y$ and $g : Y \to Z$ be S-smoothable morphisms. Then there exists a commutative diagram



with f_1, g_1 smooth, $X \to X_1, i : Y \to Y_1$ closed immersions s.t. all the parallelograms are Cartesian (thus f_2, g_2, h are smooth, i' is a closed immersion.) It follows that $X \to X_1 \xrightarrow{i'} W$ is a closed immersion, and the morphism $W \xrightarrow{h} Y_1 \xrightarrow{g_2} Z$ is the base change of the smooth morphism $T \to Y_2 \xrightarrow{g_1} S$. Hence gf is S-smoothable. By Lemma, $f_2^! i^! \simeq i'^! h^!$, and thus $(gf)^! \simeq f^! g^!$.

2.2. **Trace Map.** Now define a natural transformation of functors $\mathrm{Tr}_f : Rf_* f^! \to id$ in certain cases.

(1) Let $i : Y \to X$ be a closed immersion. For $E \in D^+(X)$, define Tr_i to be the morphism

$$i_* i^! E \simeq R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, E) \to R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, E) \simeq E$$

induced by $\mathcal{O}_X \to i_*\mathcal{O}_Y$.

(2) Let $X = \mathbb{P}^r_Y$, $f : X \to Y$ be the projection. Define $\mathrm{Tr}_f : Rf_*\omega[r] \to \mathcal{O}_Y$, where $\omega = \omega_{X/Y} = \Omega^r_{X/Y}$ as follows: we have a morphism $c : \mathcal{O}_X \to \omega[r]$ in $D(X)$ defined

by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega & \longrightarrow & 0 & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \wedge^r(\mathcal{O}_x^{r+1})(-r) & \longrightarrow & \ldots & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0
\end{array} .$$

Since $Rf_*(\mathcal{O}_X^q(-i)) = 0$ for $1 \leqslant i \leqslant r$ and for all q, $Rf_*(0 \to \wedge^r(\mathcal{O}_X^{r+1})(-r) \to \ldots \to \mathcal{O}_X^{r+1}(-1) \to 0) = 0$, Rf_*c is an isomorphism. We define Tr_f to be the inverse of the composition of isomorphisms

$$\mathcal{O}_Y \xrightarrow{\sim} Rf_*\mathcal{O}_X \xrightarrow{Rf_*c} Rf_*\omega[r],$$

where the first morphism is the canonical map $\mathcal{O}_Y \to f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X$, which is an isomorphism.

When Y is affine, the image of c under the morphism

$$\mathrm{Hom}_{D(X)}(\mathcal{O}_X, \omega[r]) \simeq H^r(X, \omega) \simeq H^0(Y, Rf_*\omega[r]) \xrightarrow{H^0(Y, \mathrm{Tr}_f)} H^0(Y, \mathcal{O}_Y)$$

is 1.

For $E \in D^+(Y)$, define Tr_f by

$$Rf_*f^!E = Rf_*(f^*E \otimes \omega[r]) \simeq E \otimes^L Rf_*\omega[r] \xrightarrow{E \otimes^L \mathrm{Tr}_f} E,$$

where the isomorphism in the middle is the projection formula.

(3) The general case. Let $f : X \to Y$ be a morphism which can be factorize as

$$\begin{array}{ccc}
X & \xhookrightarrow{i} & \mathbb{P}^r_Y \\
{\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\
Y & &
\end{array}$$

where i is a closed immersion and g is the projection. This is the case when, e.g. f is projective and Y has an ample line bundle. Define $\mathrm{Tr}_f := \mathrm{Tr}_g(Rg_* \mathrm{Tr}_i\, g^!)$. More specifically, for $E \in D^+(Y)$, define Tr_f by the composition

$$Rf_*f^!E \simeq Rg_*i_*i^!g^!E \xrightarrow{Rg_* \mathrm{Tr}_i(g^!E)} Rg_*g^!E \xrightarrow{\mathrm{Tr}_g} E.$$

This does not depend on the embedding, and is compatible with composition and flat base change.(Assume it).

2.3. **The Duality Theorem.** Let $f : X \to Y$ be a projective morphism with Y noetherian, $\dim Y < \infty$, Y having ample line bundle. Then the condition (3) above holds and so $\dim X < \infty$. Hence, f_* has finite cohomological dimension. It follows that Rf_* extends to a functor $D(X) \to D(Y)$(sending $D^-(X) \to D^-(Y)$ and $D^b(X) \to D^b(Y)$).

For $E, F \in Mod(X)$, define a canonical morphism

$$f_*\mathcal{H}om(E, F) \to \mathcal{H}om(f_*E, f_*F)$$

as follows. For $U \subset Y$ open, an element in $\Gamma(U, f_*\mathcal{H}om(E,F))$ is a morphism $E|_{f^{-1}(U)} \to F|_{f^{-1}(U)}$. It induces homomorphisms $\Gamma(f^{-1}(V), E|_{f^{-1}(U)}) \to \Gamma(f^{-1}(V), F|_{f^{-1}(U)})$ for all $V \subset U$ open, which determine a morphism $f_*E|_U \to f_*F|_U$, that is, an element in $\Gamma(U, \mathcal{H}om(f_*E, f_*F))$.

For $E, F \in C(X)$, we get a morphism of complexes

$$f_*\mathcal{H}om^\bullet(E,F) \to \mathcal{H}om^\bullet(f_*E, f_*F).$$

For $E \in D(X), F \in D^+(X)$, take quasi-isomorphisms $F \to F', E \to E'$ with $F' \in C^+(X)$, F'^i injective, E'^i f_*-acyclic for all i. Then $R\mathcal{H}om(E,F) \simeq \mathcal{H}om^\bullet(E',F')$. Observe that $\mathcal{H}om^i(E',F')$ is flasque for all i. In fact, for any $L, M \in Mod(X)$ with M injective, we have $\mathcal{H}om(L,M)$ is flasque. For an open embedding $j : U \hookrightarrow X$, any morphism $L|_U \to M|_U$ can be extended to L as below since M is injective:

$$\begin{array}{ccccc} 0 & \longrightarrow & j_!j^*L & \longrightarrow & L \\ & & \downarrow & \swarrow & \\ & & M & & \end{array}$$

We define a morphism

$$Rf_*R\mathcal{H}om(E,F) \to R\mathcal{H}om(Rf_*E, Rf_*F)$$

by composition of canonical morphisms

$$\begin{aligned} Rf_*R\mathcal{H}om(E,F) &\simeq f_*\mathcal{H}om^\bullet(E',F') \to \mathcal{H}om^\bullet(f_*E', f_*F') \\ &\to R\mathcal{H}om^\bullet(f_*E', f_*F') \simeq R\mathcal{H}om(Rf_*E, Rf_*F) \end{aligned}$$

For $L \in D(X), M \in D^+(Y)$, define $\theta_f(L,M)$(sometimes abbreviated θ_f) to be the composition

$$Rf_*R\mathcal{H}om(L, f^!M) \to R\mathcal{H}om(Rf_*L, Rf_*f^!M) \xrightarrow{R\mathcal{H}om(Rf_*L, \mathrm{Tr}_f)} R\mathcal{H}om(Rf_*L, M),$$

where the first map is the canonical map defined above.

Theorem 2.3.1 (Grothendieck). *For $L \in D^-(X)_{coh}$, $M \in D^+(Y)_{coh}$, the morphism θ_f is an isomorphism.*

Proof. $f : X \to Y$ can be factorized as

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & P = \mathbb{P}^r_Y \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ Y & & \end{array}$$

where i is a closed immersion and g is the projection. Then it is easily seen that $\theta_f(L,M) = \theta_g(Ri_*L, M) \circ (Rg_*\theta_i(L, g^!M))$, with $Ri_*L \in D^-(P)_{coh}$ and $g^!M \in D^+(P)_{coh}$, so it is enough to check that θ_i, θ_g are isomorphisms.

Let $L \in D^-(X)_{coh}, M \in D^+(P)_{coh}$. To show θ_i is an isomorphism, we may assume, by canonical truncation($\tau_{\leqslant}$), induction and "way out functor", that $L \in Coh(X)$. We may assume P affine. Then we can write

$$L \simeq (... \to L^{-1} \to L^0),$$

with L^i free of finite type. Using naive truncation $\sigma_{\geqslant}$, we may assume $L = \mathcal{O}_X$. Then θ_i is nothing but the canonical isomorphism

$$i_* R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, i^! M) = i_* i^! M \to R\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_X, M).$$

Therefore, we may assume that $f : X = \mathbb{P}^r_Y \to Y$ is the projection. Using $\tau_{\leqslant}$, we may assume L is concentrated in degree 0, that is, $L \in Coh(X)$. Then there is an exact sequence

$$\dots \to \mathcal{O}_X(-n_1)^{m_1} \to \mathcal{O}_X(-n_0)^{m_0} \to L \to 0$$

with all $n_i > r+1$. Using $\sigma_{\geqslant}$, we may assume $L = \omega(-d)$ with $d \geqslant 0$(where $\omega = \Omega^r_{X/Y} \simeq \mathcal{O}_X(-r-1)$).

Then we have isomorphisms

$$\begin{aligned} Rf_* R\mathcal{H}om(L, f^! M) &= Rf_* R\mathcal{H}om(\omega(-d), f^* M \otimes \omega)[r] \simeq Rf_*(f^* M)(d)[r] \\ &\simeq M \otimes^L Rf_* \mathcal{O}_X(d)[r] \simeq M \otimes^L f_* \mathcal{O}_X(d)[r], \end{aligned}$$

where the last but second isomorphism is the projection formula, and isomorphisms

$$\begin{aligned} R\mathcal{H}om(Rf_* L, M) &= R\mathcal{H}om(Rf_* \omega(-d), M) \\ &\simeq \mathcal{H}om^\bullet(R^r f_* \omega(-d)[-r], M) \\ &\simeq M \otimes \mathcal{H}om(R^r f_* \omega(-d), \mathcal{O}_Y)[r], \end{aligned}$$

where we have used the fact that $R^r f_* \omega(-d)$ is a locally free sheaf of finite type. We have to check

$$\theta_f : f_* \mathcal{O}_X(d) \to \mathcal{H}om(R^r f_* \omega(-d), \mathcal{O}_Y)$$

is an isomorphism, that is, the pairing

$$f_* \mathcal{O}_X(d) \otimes R^r f_* \omega(-d) \to \mathcal{O}_Y$$

is perfect. For $V = \mathrm{Spec}(A) \subset Y$, the pairing

$$\Gamma(V, f_* \mathcal{O}_X(d)) \times \Gamma(V, R^r f_* \omega(-d)) \to \Gamma(V, \mathcal{O}_Y)$$

is given by

$$(t^a, \frac{1}{t^b t_0 \dots t_r}) \mapsto \begin{cases} 0, & \text{if } a \neq b \\ 1, & \text{if } a = b \end{cases}$$

where $\sum a_i = \sum b_i = d$, and thus is a perfect pairing. □

Applying $R\Gamma$ to θ_f, we get an isomorphism

$$R\,\mathrm{Hom}(L, f^! M) \xrightarrow{\sim} R\,\mathrm{Hom}(Rf_* L, M)$$

in $D(\mathcal{A}b)$. Applying H^i, we get $\mathrm{Ext}^i(L, f^! M) \xrightarrow{\sim} \mathrm{Ext}^i(Rf_* L, M)$.

2.4. **Some Discussions.** In this part, suppose $Y = \mathrm{Spec}(k)$, $f : X \to Y$ projective.

Definition 2.4.1. $K_X := f^! \mathcal{O}_Y \in D^+(X)$ is called a dualizing complex on X.

We have an immediate corollary of the last paragraph of the previous section:

Corollary 2.4.2. *Let X/k be projective, $L \in D^-(X)_{coh}$. Then there is a perfect pairing of finite dimensional k-spaces between $H^j(X, L)$ and $\mathrm{Ext}^{-j}(L, K_X)$.*

Proof. By the last paragraph of the previous section, $\mathrm{Ext}^i(L, K_X) \simeq \mathrm{Ext}^i(R\Gamma(X, L), k) = \mathrm{Hom}(H^{-i}(X, L), k)$. Hence the corollary follow. □

Next, we first consider the case when X/k is smooth.

Corollary 2.4.3 (Serre)**.** *Let X/k be projective, smooth, purely of dimension d. Then $K_X = \omega_X[d]$. Hence there is a perfect pairing between $H^j(X, L)$ and $\mathrm{Ext}^{d-j}(L, \omega_X)$. In particular, for L locally free of finite type, $H^j(X, L)$ is dual to $H^{d-j}(X, L^\vee \otimes \omega_X)$, where $L^\vee = \mathcal{H}om(L, \mathcal{O}_X)$.*

Proof. We only need to prove that last assertion. For that, $R\mathcal{H}om(L, \omega_X) = L^\vee \otimes \omega_X$, so $\mathrm{Ext}^n(L, \omega_X) = H^n R\Gamma(X, R\mathcal{H}om(L, \omega_X)) = H^n(X, L^\vee \otimes \omega_X)$. □

In fact, the perfect paring is given by the natural pairing followed by Tr:

$$H^j(X, L) \otimes H^{d-j}(X, L^\vee \otimes \omega_X) \to H^d(X, \omega_X) \xrightarrow{\mathrm{Tr}} k.$$

When $d = 1$, we get "Roch's half" of the Riemann-Roch theorem, which claims that for L a line bundle, $H^1(X, L)$ is dual to $H^0(X, L^\vee \otimes \omega_X)$.

Corollary 2.4.4. *Let X/k be projective, smooth, purely of dimension d. Then $H^j(X, \Omega^i_X)$ is dual to $H^{d-j}(X, \Omega^{d-i}_X)$.*

This is somehow related to Hodge theory.

Then, we discuss K_X in general.

Proposition 2.4.5. *Let X/k be projective with $\dim X = n$. Then $K_X \in D^{[-n,0]}(X)_{coh}$.*

Proof. We have

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & P = \mathbb{P}^N_Y \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ \mathrm{Spec}(k) & & \end{array}$$

with i a closed immersion. $i_* K_X = R\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P)[N]$, so it's enough to show $\mathcal{E}xt^{i+N}_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P) = 0$ for $i \notin [-n, 0]$, that is,

$$\mathcal{E}^j = \mathcal{E}xt^j_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P) = 0$$

for $j < N - n$ or $j > N$. This holds for $j > N$ since for all $x \in X$, $\mathcal{E}xt^j_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P)_x = \mathrm{Ext}^j_{\mathcal{O}_{P,x}}(\mathcal{O}_{X,x}, \omega_{P,x})$, where $\omega_{P,x} \simeq \mathcal{O}_{P,x}$ is regular of dimension $\leqslant N$. Note that for $q >> 0$, $\mathcal{E}^j(q)$ is generated by global sections. It then suffices to show for a fixed $j < N - n$, $\Gamma(P, \mathcal{E}^j(q)) = 0$ for $q >> 0$. This is right since $\mathcal{O}_x, \omega_P \in Coh(P)$ implies that $\Gamma(P, \mathcal{E}xt^j(\mathcal{O}_X, \omega_P)(q)) = \mathrm{Ext}^j_P(\mathcal{O}_X, \omega_P(q))$ (we assume this), which is dual to $H^{N-j}(P, \mathcal{O}_X(-q)) = H^{N-j}(X, \mathcal{O}_X(-q)) = 0$ since $N - j > n = \dim X$. □

Let A be a local ring with residue field k, M be an A-module. A is called Cohen-Macaulay if its depth is equal to $\dim A$. A scheme X is called Cohen-Macaulay if all its local rings are Cohen-Macaulay.

Proposition 2.4.6. *Let X/k be projective. Suppose X is Cohen-Macaulay and all irreducible components have dimension n. Then $K_X \in D^{[-n,-n]}(X)$, and so $K_X \simeq \omega_X^\circ[n]$ with $\omega_X^\circ = H^{-n}(K_X)[n]$.*

Proof. By the proof of the previous proposition, we only need to show that $\forall j > N-n$, $x \in X$,

$$\mathrm{Ext}^j_{\mathcal{O}_{P,x}}(\mathcal{O}_{X,x}, \omega_{P,x}) = 0,$$

which follows from the equation

$$\begin{aligned} \operatorname{proj\,dim} \mathcal{O}_{P,x} &= \dim \mathcal{O}_{P,x} - \operatorname{depth}_{\mathcal{O}_{P,x}} \mathcal{O}_{X,x} \\ &= \dim \mathcal{O}_{P,x} - \operatorname{depth}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x} \\ &= \dim \mathcal{O}_{P,x} - \dim \mathcal{O}_{X,x} = N - n. \end{aligned}$$

□

The sheaf ω_X° in the proposition is called the dualizing sheaf for X.

X is Cohen-Macaulay if, e.g., there is a regular k-immersion i of X into a projective space over k.

$$\begin{array}{ccc} X & \xhookrightarrow{i} & P = \mathbb{P}^N_k \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ \mathrm{Spec}(k) = S & & \end{array}$$

In this case, we even have ω_X° is a line bundle. Indeed,

$$\begin{aligned} f^!\mathcal{O}_Y &\simeq i^! g^! \mathcal{O}_Y \\ &= i^! \omega_P[N] \\ &\simeq i^*\omega_P \otimes \omega_{X/P}[-(N-n)][N] \qquad \text{by } (*) \text{ in the proof of Proposition 1.2.4.(4)} \\ &= i^*\omega_P \otimes \omega_{X/P}[n], \end{aligned}$$

and hence $\omega_X^\circ = i^*\omega_P \otimes \omega_{X/P}$.