THE INTERSECTION FORMULA

FEDERICO BINDA

ABSTRACT. Notes for a seminar in commutative algebra about Serre's "Tor" formula. The goal is to present and describe the minimum background material in order to understand Serre's definition of intersection multiplicity.

CONTENTS

1. THE SAMUEL MULTIPLICITY OF AN IDEAL

Following [8], in this section we review some classical results on Samuel's multiplicity. Let $B = \bigoplus_{i\geq 0} B_i$ be a graded ring, where B_0 is Artinian and B is generated, as B_0-algebra, by r elements of degree 1. Recall that for a graded module $M = \bigoplus_{i\geq 0} M_i$, finitely generated over B, we define the Hilbert function H_M of M by $H_M(i) = \text{length}(M_i)$. This function is polynomial-like and we call the associated polynomial the Hilbert polynomial of M. We call the degree of the Hilbert function the degree of the associated Hilbert polynomial.

Let A be a local Noetherian ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated module of dimension d. Let $\mathfrak{q} \subseteq \mathfrak{m}$ be an ideal of A such that $M/\mathfrak{q}M$ has finite length, i.e. $\text{Supp}(M/\mathfrak{q}M)(= \text{Supp}(M \otimes A/\mathfrak{q}) = \text{Supp}(M) \cap \text{Supp}(A/\mathfrak{q})) = \{\mathfrak{m}\}$, so that $\mathfrak{m}$ is the only prime ideal in the support of M containing $\mathfrak{q}$.

In our setting, we have the graded ring $G_\mathfrak{q}(A) = \bigoplus_{i\geq 0} \mathfrak{q}^i/\mathfrak{q}^{i+1}$, called the associated graded ring of $\mathfrak{q}$, and the associated graded module $G_\mathfrak{q}(M) = \bigoplus_{i\geq 0} \mathfrak{q}^i M/\mathfrak{q}^{i+1}M$. So the Hilbert function of M is $H_M(i) = \text{length}(\mathfrak{q}^i M/\mathfrak{q}^{i+1}M)$. Since

$$\text{length}(\mathfrak{q}^i M/\mathfrak{q}^{i+1}M) = \text{length}(M/\mathfrak{q}^{i+1}M) - \text{length}(M/\mathfrak{q}^i M)$$

we conclude that there exists a polynomial, called the Hilbert-Samuel polynomial (or, when this will not create confusion, simply the Hilbert polynomial) of M with respect to $\mathfrak{q}$, denoted $P_\mathfrak{q}(M)$.

1.1. **Definition.** The *Samuel multiplicity* of $\mathfrak{q}$ on M, denoted $e(\mathfrak{q}, M)$, is $d!$ times the coefficient of the term of degree d in the polynomial $P_{\mathfrak{q}}(M)$. The multiplicity of $\mathfrak{q}$ in A or simply the multiplicity of $\mathfrak{q}$ is $e(\mathfrak{q}, A)$. The multiplicity of A is the multiplicity of the maximal ideal $\mathfrak{m}$ of A.

1.2. **Remark.** $e(\mathfrak{q}, M)$ is an integer > 0.

1.3. **Proposition.** *Let M be a module of dimension d and let $\mathfrak{q}$ be an ideal of A such that $M/\mathfrak{q}M$ has finite length. Then*

$$e(\mathfrak{q}, M) = d! \lim_{n\to\infty} \frac{\mathrm{length}(M/\mathfrak{q}^n M)}{n^d}$$

Proof. Let $P_{\mathfrak{q}}(M)(n) = a_d n^d + a_{d-1}n^{d-1} + \ldots + a_0$ be the Hilbert polynomial of M. By definition of Samuel multiplicity, the leading term of this polynomial is $e(\mathfrak{q}, M)/d!$. Moreover, for $n \gg 1$, $P_{\mathfrak{q}}(M)(n) = \mathrm{length}(M/\mathfrak{q}^n M)$. So we have:

$$\begin{aligned} d! \lim_{n\to\infty} \frac{\mathrm{length}(M/\mathfrak{q}^n M)}{n^d} &= d! \lim_{n\to\infty} \frac{P_{\mathfrak{q}}(M)(n)}{n^d} \\ = d! \lim_{n\to\infty} \frac{a_d n^d + a_{d-1}n^{d-1} + \ldots + a_0}{n^d} &= d! a_d = e(\mathfrak{q}, M). \end{aligned}$$

□

We recall some facts about regular local rings. Let $k = A/\mathfrak{m}$ be the residue field of A and let $d = \dim A$. We say that A is *regular* if it satisfies one (and then all) of the following equivalent conditions:

i) the maximal ideal $\mathfrak{m}$ is generated by d elements;
ii) $\dim A = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$;
iii) the k-algebra $G(A) = \bigoplus_{i\geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is isomorphic to $k[X_1, \ldots, X_d]$.

1.4. **Proposition.** *Let A be a regular local ring of dimension d with maximal ideal $\mathfrak{m}$ and residue field k.*

i) The associated graded ring $G(A) = \bigoplus_{i\geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is isomorphic to $k[X_1, \ldots, X_d]$ a polynomial ring over k in d variables.
ii) A is an integral domain.
iii) The multiplicity of A is one.

Proof. By assumption, there exist d elements, say $x_1, \ldots, x_d$, such that $\mathfrak{m} = (x_1, \ldots, x_d)$. Consider the epimorphism of graded rings $\varphi\colon k[X_1, \ldots, X_d] \to G(A)$ defined by $X_i \mapsto \overline{x_i}$ mod $\mathfrak{m}^2$. Assume, by contradiction, that the map is not an isomorphism and let $N = \operatorname{Ker}\varphi$. The degree of the Hilbert function of the graded ring $G(A)$ is $d-1$, since it is one less the degree of the Hilbert-Samuel polynomial of $\mathfrak{m}$, which is equal to the dimension of A. Moreover, we know that the degree of the Hilbert function of $k[X_1, \ldots, X_d]$ is also $d-1$. Since $k[X_1, \ldots, X_d]$ is an integral domain, (0) is prime and so the degree of the Hilbert function of a proper quotient of $k[X_1, \ldots, X_d]$ is at most $d-2$. If $N \neq 0$, then $k[X_1, \ldots, X_d]/N \simeq G(A)$, so that $G(A)$ is a proper quotient of $k[X_1, \ldots, X_d]$, which is a contradiction. This proves i).

For ii), we can prove, in general, the following lemma:

1.5. **Lemma.** *Let A be a Noetherian ring and let $I \subset \mathfrak{r}(A)$ (the Jacobson radical of A). Let $G_I(A)$ be the graded ring associated to the I-adic filtration of A. If $G_I(A)$ is a domain, then so is A.*

Proof. let $a, b \neq 0$ in A. By the Artin-Rees lemma, $\cap I^n = 0$. Hence there exist $m, n \in \mathbb{N}$ such that $a \in I^m$, $a \notin I^{m+1}$, $b \in I^n$, $b \notin I^{n+1}$. Hence $\bar{a}, \bar{b} \neq 0$ in $G_I(A)$. Since it is a domain, $\bar{a}\bar{b} = \overline{ab} \neq 0$, so $ab \neq 0$ in A. □

Since we have shown in i) that $G(A)$ is an integral domain, by the previous lemma this implies that A is an integral domain. From statement i), it also follows that the Hilbert-Samuel polynomial of $\mathfrak{m}$ is equal to the Hilbert-Samuel polynomial of a polynomial ring in d variables. Hence length$(A/\mathfrak{m}^i)$ is equal to the dimension of the k-vector space of the homogeneous polynomials of degree i in d variables, so that

$$\text{length}(A/\mathfrak{m}^i) = \binom{i+d-1}{d}.$$

This polynomial is of degree d and leading coefficient $1/d!$. Hence the Samuel multiplicity of A is $e(\mathfrak{m}, A) = 1$. □

The following proposition describe the behaviour of the Hilbert-Samuel polynomials with respect to short exact sequences.

1.6. **Proposition.** *Let*

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of A-modules. Let $\mathfrak{q}$ be an ideal of A such that $M/\mathfrak{q}M$, $M'/\mathfrak{q}M'$ and $M''/\mathfrak{q}M''$ have finite length. Then, if $\dim M = d$ and if a_d, a'_d and a''_d are the coefficients of degree d of the Hilbert-Samuel polynomials of M, M' and M'' respectively, with respect to $\mathfrak{q}$, then

$$a_d = a'_d + a''_d.$$

Proof. See [8, p. 30]. □

Let A be a local ring of dimension r and $\mathfrak{q}$ be an ideal of A such that $M/\mathfrak{q}M$ has finite length. Let $e_r(\mathfrak{q}, M)$ denote $r!$ times the coefficient of degree r in the Hilbert-Samuel polynomial $P_{\mathfrak{q}}(M)$. By the previous proposition, we have that $e_r(\mathfrak{q}, -)$ is additive on short exact sequences and that is zero for any module M of dimension $d < r$.

The multiplicity of two ideals can be the same even when one is properly contained in the other. In fact, let A be the k-algebra $k[[X, Y]]$ where k is a field. It is well-known that A is a local ring: it can be identified with $k[X, Y]_{\mathfrak{m}}$ for $\mathfrak{m} = (X, Y) \subset k[X, Y]$; moreover, A is regular of dimension 2. Consider the two $\mathfrak{m}$-primary ideals $\mathfrak{a} = (X^2, Y^2)$ and $\mathfrak{b} = (X^2, Y^2, XY)$. Notice that, for all n, the ideal $\mathfrak{a}^n$ by all monomials X^iY^j with $i + j = 2n$ and both even, while $\mathfrak{b}^n$ is generated by all monomials of total degree $2n$. The quotient $\mathfrak{b}^n/\mathfrak{a}^n$ is then generated by all monomials X^iY^j with $i + j = 2n$, i and j odd. There are n such monomials and it's easy to see that $\mathfrak{b}^n/\mathfrak{a}^n$ has finite length equal to n. Hence, the Hilbert-Samuel polynomial of the module $\mathfrak{b}/\mathfrak{a}$ is of degree 1 (indeed $P(\mathfrak{b}/\mathfrak{a})(n) = n$). By the additivity on short exact sequences, we actually have that the Hilbert-Samuel polynomial of $\mathfrak{a}$ and $\mathfrak{b}$ differ by a polynomial of degree 1 and that the leading coefficients are equal. This is a consequence of a more general fact, but first we have to recall the following definition

1.7. **Definition.** Let I be an ideal of a ring A and B is an extension of A. We say that an element $x \in B$ is integral over I if x satisfies a monic equation of the form $x^n + a_{n-1}x^{n-1} + \ldots + a_0$ with $a_i \in I^i$ for all i.

Clearly for $I = A$ we get back the usual definition of integral element of B over A. If I and J are two ideals of A, we say that I is integral over J if every element of I is integral over J in the sense of the previous definition. The condition that an ideal I is integral over J is also referred to by saying that J is a reduction of I. There is a very nice relation between integral extensions and multiplicity, given by the following proposition:

1.8. **Proposition.** *Let $\mathfrak{p}$ and $\mathfrak{q}$ be $\mathfrak{m}$-primary ideals of a local ring $(A, \mathfrak{m})$ such that $\mathfrak{q} \subset \mathfrak{p}$. If $\mathfrak{p}$ is integral over $\mathfrak{q}$, then the multiplicity of $\mathfrak{p}$ is equal to the multiplicity of $\mathfrak{q}$.*

Proof. See [8, p. 36]. □

2. The Koszul Complex

In this section we follow [9] and again [8]. We need to recover some properties of the so-called *Koszul complex* of a sequence of elements of a ring A. Following Serre, we may begin with the "simple case". Let A be a commutative ring (which we may assume to be Noetherian, even if it is not strictly required for the time being) and let x be an element of A. We denote by $K(x)$ the following complex:

$$K_i(x) = 0 \quad \text{if } i \neq 0, 1;$$
$$K_1(x) = K_0(x) = A;$$

with the map $d\colon K_i(x) \to K_0(x)$ given by the multiplication by x. We identify $K_0(x)$ with A and we choose a basis of the free A-module $K_1(x)$ such that $d(e_x) = x$. The derivation d is thus defined by A-linearity $d(ae_x) = ax$ for all $a \in A$.

Let M be an A-module; we write $K(x, M)$ for the complex obtained by tensoring $K(x)$ with M, i.e. the tensor product complex $K(x) \otimes_A M$. Then we have:

$$K(x, M)_n = K_n(x) \otimes_A M = 0 \quad \text{if } n \neq 0, 1,$$
$$K(x, M)_1 = K_1(x) \otimes_A M \cong M,$$
$$K(x, M)_0 = K_0(x) \otimes_A M \cong M$$

and the derivation $d \otimes 1 = d$, $d\colon K(x, M)_1 \to K(x, M)_0$ given by

$$d(e_x \otimes m) = xm \quad \text{for all } m \in M.$$

We can compute the homology modules of this complex. They are:

$$H_0(K(x), M) = M/d(M) = M/xM,$$
$$H_1(K(x), M) = \mathrm{Ann}_M(x)/0 = \mathrm{Ann}_M(x) = \mathrm{Ker}(x_M\colon M \to M),$$
$$H_i(K(x), M) = 0 \quad \text{if } n \neq 0, 1.$$

We will denote $H_i(K(x), M)$ by $H_i(x, M)$.

The second step is to take a complex of A-modules L and consider the tensor product complex $K(x) \otimes_A L$. The homology modules of this complex are related to the homology modules of L in the following way:

2.1. **Proposition.** *For every $p \geq 0$, we have an exact sequence:*

$$0 \to H_0(x, H_p(L)) \to H_p(K(x) \otimes_A L) \to H_1(x, H_{p-1}(L)) \to 0.$$

Proof. By definition of tensor product of two complexes, for all n we have:

$$(K(x) \otimes_A L)_n = \bigoplus_{p+q=n} K(x)_p \otimes_A L_q.$$

Since $K(x)_i = 0$ for $i \neq 0, 1$, the above formula can be written as $(K(x) \otimes_A L)_p = (K(x)_0 \otimes_A L_p) \oplus (K_1(x) \otimes_A L_{p-1})$. If we consider $K_0(x)$ and $K_1(x)$ as A-modules, using the natural injection and projection, we obtain the following exact sequence

$$0 \to (K(x)_0 \otimes_A L)_p \to (K(x) \otimes_A L)_p \to (K_1(x) \otimes_A L)_{p-1} \to 0$$

so that we have an exact sequence of complexes. Using the fact that $K_0(x) = K_1(x) = A$, we can write that sequence in a fancy way

$$0 \to L \to K(x) \otimes_A L \to L[-1] \to 0,$$

where $L[-1]$ is the complex obtained from L by a shift of -1 (i.e. $L[-1]_n = L_{n-1}$) together with a sign change on the boundary map (this is a convention designed to simplify the notation). We have the corresponding long exact sequence of homology modules

$$\ldots \to K_1(x) \otimes_A H_p(L) \xrightarrow{d \otimes 1} K_0 \otimes_A H_p(L) \to H_p(K(x) \otimes_A L) \to$$
$$\to K_1(x) \otimes_A H_{p-1}(L) \xrightarrow{d \otimes 1} K_0 \otimes_A H_{p-1}(L) \to \ldots$$

where the boundary map $d \otimes 1$ is equal to scalar multiplication by x. Hence the above exact sequence splits into short exact sequences:

$$0 \to X_p \to H_p(K(x) \otimes_A L) \to Y_{p-1} \to 0$$

with

$$X_p = \mathrm{Coker}(K_1(x) \otimes_A H_p(L) \to K_0 \otimes_A H_p(L)) = H_0(x, H_p(L))$$
$$Y_p = \mathrm{Ker}(K_1(x) \otimes_A H_{p-1}(L) \to K_0 \otimes_A H_{p-1}(L)).$$

□

A complex of A-modules L is called an *acyclic complex* on M if $H_p(L) = 0$ for $p > 0$ and $H_0(L) = M$. In other words, we have an exact sequence

$$\ldots \to L_n \to \ldots \to L_1 \to L_0 \to M \to 0.$$

From the previous proposition we get the following

2.2. **Corollary.** *If L is an acyclic complex on M and if $x \in A$ is not a zero-divisor in M, then $K(x) \otimes_A L$ is an acyclic complex on M/xM.*

Proof. Since L is acyclic, we get $H_0(x, H_p(L)) = H_1(x, H_{p-1}(L)) = 0$ for all $p > 1$, so that, by the previous proposition, $H_p(K(x) \otimes_A L) = 0$ for $p > 1$. For $p = 1$ we have $H_0(x, H_1(L)) = 0$, hence $H_1(K(x) \otimes_A L) \cong H_1(K(x) \otimes_A H_0(L)) = H_1(K(x) \otimes_A M) = \mathrm{Ker}(x_M) = 0$ since x is not a zero divisor for M. Similarly, for $p = 0$, we get $H_0(K(x) \otimes_A L) \cong H_0(x, M) = M/xM$. □

Now we consider the general case. Let $(x_1, \ldots, x_r)$ be r elements of A. We denote by $K(x_1, \ldots, x_r)$ the tensor product complex

$$K(x_1, \ldots, x_r) = K(x_1) \otimes_A K(x_2) \otimes_A \cdots \otimes_A K(x_r).$$

One can show[1] that $K_p(x_1, \ldots, x_n)$ is the free module generated by the elements $e_{i_1} \otimes \ldots \otimes e_{i_p}$, $i_1 < i_2 < \ldots < i_p$, where $e_i = e_{x_i}$ is the generator of the free A-module $K_1(x_i)$ mentioned above. In particular, it is isomorphic to $\bigwedge^p(A^r)$, the p-th exterior product of A^r. Let M be an A-module. We write $K(x_1, \ldots, x_r; M) = K(\mathbf{x}, M)$ for the tensor product complex

$$K(x_1, \ldots, x_r) \otimes_A M,$$

where $\mathbf{x} = (x_1, \ldots, x_r)$. For all p, the module $K_p(x_1, \ldots, x_r; M)$ is a direct sum of modules $e_{i_1} \otimes_A \cdots \otimes_A e_{i_p} \otimes_A M$, $i_1 < i_2 < \ldots < i_p$ with derivations $d\colon K_p(\mathbf{x}, M) \to K_{p-1}(\mathbf{x}, M)$ (see [6], Appendix C)

$$d(e_{i_1} \otimes \ldots \otimes e_{i_p} \otimes m) = \sum_{k=1}^{p} (-1)^{k+1} x_{i_k} e_{i_1} \otimes \ldots \otimes \hat{e}_{ik} \otimes \ldots \otimes e_{i_p} \otimes m.$$

We denote by $H_p(\mathbf{x}, M)$ the p-th homology module of the Koszul complex $K(\mathbf{x}, M)$. An easy computation shows that

$$H_0(\mathbf{x}, M) = M/(x_1, \ldots, x_r)M$$

$$H_1(\mathbf{x}, M) = \bigcap_{i=1}^{r} \mathrm{Ker}((x_i)_M\colon M \to M) = \{m \in M \mid x_i m = 0 \text{ for all } i\}.$$

One of the main uses of the Koszul complex is as a criterion for a sequence of elements to be regular for a module M, in the sense of the following

2.3. **Definition.** Let M be a non-zero module over a local ring A and let $x_1, \ldots, x_r$ be a sequence of elements in the maximal ideal $\mathfrak{m}$. We say that $x_1, \ldots, x_r$ form a regular sequence on M if, for every i, $1 \le i \le r$, x_i is not a zero-divisor in $M/(x_1, \ldots, x_{i-1})M$.

The following propositions describe the case where the homology modules of the complex are zero for $p > 0$ (see [9, IV.A.2] for proofs).

2.4. **Proposition.** *Under the preceding hypotheses, if for all i, $1 \le i \le r$, x_i is not a zero-divisor in $M/(x_1, \ldots, x_{i-1})M$, then $H_p(\mathbf{x}, M) = 0$ for $p > 0$.*

2.5. **Proposition.** *If A is Noetherian, M is finitely generated and the x_i belong to the radical* $\mathrm{Nil}(A)$*, then the following are equivalent:*

(1) *$H_p(\mathbf{x}, M) = 0$ for $p \ge 1$.*
(2) *$H_1(\mathbf{x}, M) = 0$.*
(3) *For every i, $1 \le i \le r$, x_i is not a zero-divisor in $M/(x_1, \ldots, x_{i-1})M$.*

2.6. **Corollary.** *Condition 3 does not depend on the order of the sequence $(x_1, \ldots, x_r)$.*

[1] direct computation in lower degrees, then by induction.

2.7. **Remark.** Notice that we are not assuming the local hypothesis in proposition 2.5. The proof of the same result in the local case can be found in [8, p. 60]. However, the exactness of the Koszul complex does not imply that the sequence is regular for nonlocal rings: a regular sequence is defined as in the local case except that the condition that the x_i are in the maximal ideal is replaced by the condition that $M/(x_1, \dots, x_r)M \neq 0$. In order to obtain a similar result we have to require (as Serre does) that the x_i belong to the radical. To see this, we can make the following example.

Let $A = k[X, Y, Z]$ and consider the sequence $XY, X-1, XZ$. This sequence is regular for A (in the sense specified above). Indeed, $A/(XY, X-1, XZ) \neq 0$; $X-1$ is not a zero divisor in $A/(XY)$ (thanks to the constant term -1). $A/(XY, X-1) \cong k[Y, Z]/(Y)$ and $XZ = Z$ in this quotient (since $X = 1$). Hence, again, we have that XZ is not a zero divisor in $A/(XY, X-1)$. Now consider the permutation $XY, XZ, X-1$ of the sequence: this is not a regular sequence (XZ is a zero divisor in $A/(XY)$). However, the Koszul complex $K(XY, XZ, X-1; A)$ is exact.

The map $M \mapsto K(\mathbf{x}, M)$ is clearly functorial for a fixed sequence of elements $x_1, \dots, x_r$. Notice that for all p, the module $K_p(x_1, \dots, x_n)$ is free, being isomorphic to $\bigwedge^p(A^r)$, hence flat as A-module. In particular, this implies that the (covariant) functor $M \mapsto K(\mathbf{x}, M)$ is exact. So, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, we obtain an exact sequence of complexes

$$0 \to K(\mathbf{x}, M') \to K(\mathbf{x}, M) \to K(\mathbf{x}, M'') \to 0$$

and the corresponding exact sequence of homology modules:[2]

$$\begin{aligned} 0 \to H_r(\mathbf{x}, M') \to H_r(\mathbf{x}, M) \to H_r(\mathbf{x}, M'') \to H_{r-1}(\mathbf{x}, M') \to \dots \\ \dots \to H_0(\mathbf{x}, M') \to H_0(\mathbf{x}, M) \to H_0(\mathbf{x}, M'') \to 0 \end{aligned}$$

Moreover, $H_0(\mathbf{x}, M) = M/(\mathbf{x})M$ is isomorphic (naturally in M) to $A/(\mathbf{x}) \otimes_A M$. It is possible (see [3], Chap. III, Proposition 5.2 and Corollary 5.3) to extend this isomorphism of functors (in a unique way) to a natural transformation

$$\psi_i \colon H_i(x_1, \dots, x_r; M) \to \operatorname{Tor}_i^A(A/(x_1, \dots, x_r), M).$$

Suppose that conditions a), b) and c) of proposition 2.5 are satisfied for $M = A$ (in particular, this is equivalent of requiring that for $1 \leq i \leq r$, x_i is not a zero-divisor in $A/(x_1, \dots, x_{i-1})$). Then $K(x_1, \dots, x_r)$ is an A-free (hence projective) resolution of $H_0(x_1, \dots, x_r; A) = A/(x_1, \dots, x_r)$ and so the map ψ_i is an isomorphism for every i and every M, i.e. the functors $\operatorname{Tor}_i^A(A/(x_1, \dots, x_r), -)$ and $H_i(x_1, \dots, x_r; -)$ are isomorphic. Similarly we have natural maps

$$\varphi^i \colon \operatorname{Ext}_A^i(A/(x_1, \dots, x_r), M) \to H^i(\operatorname{Hom}_A(K(\mathbf{x}; M)) \cong H_{r-i}(\mathbf{x}; M)$$

that, in the same assumptions, turns out to be an isomorphism of functors.

Finally, we summarize some useful properties of the Koszul complex in the following proposition:

2.8. **Proposition.** *Let A be a local Noetherian ring and assume that the ideal $\mathbf{x} = (x_1, \dots, x_r)$ is contained in the maximal ideal $\mathfrak{m}$ of A. Let M be a finitely generated A-module. Then:*

[2] Notice that we clearly have $H_{r+1}(x_1, \dots, x_r; M) = 0$ for all A-modules M, since $\bigwedge^{r+1}(A^r) = 0$.

(1) *The annihilator of* $H_i(x_1, \dots, x_r; M)$ *contains* $(x_1, \dots, x_r)$ *and* $\operatorname{Ann} M$ *for all* i.
(2) *If* S *is a multiplicative subset of* A*, then* $K(\mathbf{x}, S^{-1}M) = S^{-1}K(\mathbf{x}, M)$*. Therefore* $H_i(\mathbf{x}, S^{-1}M) = S^{-1}H_i(\mathbf{x}, M)$.

Proof. See [9, p. 56] and [2, chap. 17]. □

3. Filtration of a Koszul complex

The Koszul complex on a set of elements has many properties that play an important role for intersection multiplicities.

3.1. **Definition.** Let F be a *bounded* complex of modules such that, for all i, $H_i(F)$ has finite length. Then we can define the *Euler-Poincaré characteristic of* F to be

$$\chi(F) = \sum_i (-1)^i \text{length}(H_i(F)).$$

The Euler-Poincaré characteristic of a complex is additive on short exact sequences, i.e. if

$$0 \to E \to F \to G \to 0$$

is an exact sequence of bounded complexes with homology of finite length, then $\chi(F) = \chi(G) + \chi(E)$.

Let A be a local Noetherian ring and assume that the ideal $\mathbf{x} = (x_1, \dots, x_r)$ is contained in the maximal ideal $\mathfrak{m}$ of A. Let M be a finitely generated A-module such that $M/\mathbf{x}M$ has finite length. The homology modules of the Koszul complex $H_p(\mathbf{x}, M)$ are the finitely generated and, by prop. 2.8, are all annihilated by $\mathbf{x} + \operatorname{Ann} M$. Hence[3] they have finite length. We can then define the Euler-Poincaré characteristic

$$\chi(\mathbf{x}, M) = \sum_{p=0}^{r} (-1)^p \text{length}(H_p(\mathbf{x}, M)).$$

Since $M/\mathbf{x}M$ has finite length by assumption, $\dim_A(M) \leq r$ (using Chevalley definition of dimension), hence the Hilbert-Samuel polynomial $P_{\mathbf{x}}(M)$ has degree $\leq r$. In section 1 we defined the Samuel multiplicity of an ideal to be $r!$ times the term of degree r of the Hilbert-Samuel polynomial. The main result of this section is to show that the Samuel multiplicity of the ideal generated by $x_1, \dots, x_r$ is equal to the Euler-Poincaré characteristic of the Koszul complex $K(\mathbf{x}, M)$, i.e.

3.2. **Theorem.** *Under the previous assumptions*

$$\chi(\mathbf{x}, M) = e_r(\mathbf{x}, M).$$

Proof. This is the proof given by Serre in [9]. A (slightly) different approach can be found in [8].

Let $K = K(\mathbf{x}, M)$. We can write K as the direct sum of its components $K_p = K_p(\mathbf{x}, M) \cong \bigwedge^p(A^r)$. For every $i \in \mathbb{Z}$, define the submodule $F^i(K_p)$ of K_p by $F^iK_p = \mathbf{x}^{i-p}K_p$ where $\mathbf{x}^j = A$ if $j \leq 0$. Denote by F^iK the complex obtained considering F^iK_p (that is actually a subcomplex of the Koszul complex K). By definition we have that

[3] We know that $\mathfrak{p} \in \operatorname{Supp}(H_p(\mathbf{x}, M))$ if and only if $\mathfrak{p} \supseteq \operatorname{Ann} H_p(\mathbf{x}, M) \supseteq \mathbf{x} + \operatorname{Ann} M$. Hence $\mathfrak{p} \supset \operatorname{Ann} M$, so $\mathfrak{p} \in \operatorname{Supp}(M)$ and $\mathfrak{p} \supset \mathbf{x}$. So $\mathfrak{p} \in \operatorname{Supp}(M/\mathbf{x}M) = \{\mathfrak{m}\}$.

$F^0K = K$ (since $F^0K_p = K_p$). Moreover, for every fixed p we have that $\mathbf{x}^{i-p}K_p = F^iK_p \supset F^{i+1}K_p = \mathbf{x}^{i+1-p}K_p$ since $\mathbf{x}^{i-p} \supset \mathbf{x}^{i+1-p}$. Thus we get a decreasing filtration of complexes

$$K = F^0K \supset F^1K \supset \ldots \supset F^iK \supset \ldots$$

which is an $\mathbf{x}$-good filtration of K, i.e. for all p the filtration $(F^iK_p)_i$ of K_p is $\mathbf{x}$-good.

Let $gr(A)$ be the graded ring associated with the $\mathbf{x}$-adic filtration of A, that is

$$gr(A) = G_{\mathbf{x}}(A) = \bigoplus_p \mathbf{x}^i/\mathbf{x}^{i+1}.$$

So $gr_0(A) = A/\mathbf{x}$ and $gr_1(A) = \mathbf{x}/\mathbf{x}^2$. Let $\xi_1, \ldots, \xi_r$ be the classes of $x_1, \ldots, x_r$ in the quotient $\mathbf{x}/\mathbf{x}^2$ and denote by $\Xi = (\xi_1, \ldots, \xi_r)$ the ideal of $\mathbf{x}/\mathbf{x}^2$ generated by $\xi_1, \ldots, \xi_r$. Let now $gr(M) = G_{\mathbf{x}}(M)$ be the graded module associated with the $\mathbf{x}$-adic filtration of M. Hence $gr(M)$ is a graded $gr(A)$-module.

It worth noticing that the Koszul complex $K(\Xi, gr(M))$ is a graded complex, isomorphic to $gr(K) = \bigoplus_i F^iK/F^{i+1}K$ (this can be seen by direct computation, using the explicit description of the modules of the K_p as free modules generated by the elements $e_{i_1} \otimes \ldots \otimes e_{i_p}$, $i_1 < i_2 < \ldots < i_p$, where $e_i = e_{x_i}$).

The homology modules $H_p(\Xi, gr(M))$ are all finitely generated modules over $gr(A)/\Xi \cong A/\mathbf{x}$ (since A is Noetherian). Since $\operatorname{Ann} M$ annihilates the modules $H_p(\mathbf{x}, M)$, using the description of $K(\Xi, gr(M))$ in terms of the complex associated to the $\mathbf{x}$-adic filtration $gr(K)$) we see that $\operatorname{Ann} M$ actually annihilates $H_p(\Xi, gr(M))$. Hence we can give them a structure of $A/(\mathbf{x} + \operatorname{Ann} M)$-module. Since $M/\mathbf{x}M$ has finite length, the unique prime ideal that contains both $\mathbf{x}$ and $\operatorname{Ann} M$ is $\mathfrak{m}$. Hence $A/(\mathbf{x}+\operatorname{Ann} M)$ is Artinian (Noetherian of $\dim 0$) and then, by a well-known result, $H_p(\Xi, gr(M))$ have finite length (for all p).

As a consequence, since $H_p(\Xi, gr(M))$ is the direct sum of the $H_p(F^iK/F^{i+1}K)$, we have that there exists an $m \geq 0$ such that $H_p(F^iK/F^{i+1}K) = 0$ for all $i > m$ and all p. Without loss of generality, we can assume $m \geq r$.

Notice that we can actually say more:

Claim. *for all p and for $i > m$ we have $H_p(F^iK) = 0$.*

This can be shown in two steps: first, arguing by induction, it is possible to see that $H_p(F^iK/F^{i+j}K) = 0$ for $p \in \mathbb{Z}$, $i > m$ and $j \geq 0$. Then the claim follows from a corollary to Artin-Rees lemma (for details about this step of the proof, see [9, p. 58, §3.5-3.6]).

The claim can be reformulated as follows: for all $p \in \mathbb{Z}$, the map, induced in homology by $K \to K/F^iK$, $H_p(\mathbf{x}, M) = H_p(K) \to H_p(K/F^iK)$ is an isomorphism for $i > m$.

The next step is to show that the Euler-Poincaré characteristic $\chi(\mathbf{x}, M)$ is equal, for $i > m$ to the characteristic $\chi(K/F^iK)$. Indeed, by the claim, we have that

$$\chi(\mathbf{x}, M) = \sum_{p=0}^{r} (-1)^p \text{length}(H_p(\mathbf{x}, M)) = \sum_{p=0}^{r} (-1)^p \text{length}(H_p(K/F^iK)).$$

The complex K/F^iK has finite length. Since the length is additive on short exact sequences, a simple computation shows that the Euler-Poincaré characteristic of a complex of finite length is equal to the characteristic of the corresponding homology complex.

Hence we have:

$$\chi(\mathbf{x}, M) = \sum_{p=0}^{r} (-1)^p \text{length}(H_p(K/F^iK)) = \sum_{p=0}^{r} (-1)^p \text{length}(K_p/F^iK_p) = \chi(K/F^iK).$$

But $K_p/F^iK_p = K_p/\mathbf{x}^{i-p}K_p \cong K_p \otimes_A A/\mathbf{x}^{i-p}$. Since $K_p = K_p(\mathbf{x}, M)$ is a direct sum of $\binom{r}{p}$ modules $e_{i_1} \otimes_A \cdots \otimes_A e_{i_p} \otimes_A M$ where $i_1 < i_2 < \ldots < i_p$, $K_p \otimes_A A/\mathbf{x}^{i-p}$ is isomorphic to $\binom{r}{p}$ copies of $M \otimes_A A/\mathbf{x}^{i-p} \cong M/\mathbf{x}^{i-p}M$. Hence, if $i > m$, we have

$$\chi(\mathbf{x}, M) = \sum_{p=0}^{r} (-1)^p \binom{r}{p} \text{length}(M/\mathbf{x}^{i-p}M).$$

If i is large enough we can rewrite this equality as

$$\chi(\mathbf{x}, M) = \sum_{p=0}^{r} (-1)^p \binom{r}{p} P_{\mathbf{x}}(M)(i-p)$$

with the Hilbert-Samuel polynomials $P_{\mathbf{x}}(M)$. By direct computation, the right side is equal to $e_r(\mathbf{x}, M)$. □

As a consequence (by the properties of $e_r(\mathfrak{q}, M)$ proved in section 1)

3.3. **Corollary.** *We have* $\chi(\mathbf{x}, M) > 0$ *if* $\dim_A(M) = r$ *and* $\chi(\mathbf{x}, M) = 0$ *if* $\dim_A(M) < r$.

4. Reduction to the diagonal

In this paragraph we present the so-called *reduction to the diagonal*: it is a simple but clever argument that reduced the question of intersection of arbitrary affine varieties to a question of intersection of an algebraic set with a linear variety. Let k be a field. First of all, recall that a subset $V \subset \mathbb{A}^n_k(k)$ is called an affine algebraic variety if the ideal $\mathcal{I}(V) = \mathfrak{p}$ is prime. It is well known that an affine algebraic set V is irreducible if and only if $\mathcal{I}(V)$ is prime (hence varieties are irreducible algebraic sets). So let $V \subset \mathbb{A}^n_k(k)$ be an affine variety and let $\mathfrak{p} = \mathcal{I}(V)$. We define the dimension $\dim V$ to be the dimension of the Noetherian ring $A = k[X_1, \ldots, X_n]/\mathfrak{p}$ (which is indeed a domain).

Let U, V be two affine varieties. The set $U \cap V$ needs not to be irreducible, but is clearly an algebraic set (actually $U \cap V = \mathcal{V}(\mathfrak{p} + \mathfrak{q})$ where $\mathfrak{p}$ and $\mathfrak{q}$ are the ideals $\mathcal{I}(V)$ and $\mathcal{I}(U)$ resp.): consider an irreducible component W of $V \cap U$. Then we have the inequality

$$\dim W \geq \dim V + \dim U - n.$$

Following Serre, this result can be restated in a purely algebraic language as follows:

4.1. **Proposition.** *Let* $\mathfrak{p}$ *and* $\mathfrak{q}$ *be two prime ideals of the polynomial ring* $A = k[X_1, \ldots, X_n]$. *Let* $\mathfrak{P}$ *be a minimal element of* $\mathcal{V}(\mathfrak{p} + \mathfrak{q})$. *Then we have*

$$\text{ht}(\mathfrak{P}) \leq \text{ht}(\mathfrak{p}) + \text{ht}(\mathfrak{q}).$$

As we said above, the idea behind the proof is to look at $A \otimes_k A$ and at the two prime ideals corresponding to the product $V \times U$ and to the diagonal Δ respectively. We need two lemmas:

4.2. **Lemma.** *Let A', A'' be domains which are finitely generated k-algebras. For every minimal prime ideal $\mathfrak{p}$ of $A' \otimes_k A''$, we have*

$$\dim(A' \otimes_k A''/\mathfrak{p}) = \dim(A' \otimes_k A'') = \dim A' + \dim A''.$$

Proof. By Noether normalization lemma, there exist two polynomial k-algebras $B' = k[X_1, \dots, X_n]$ and $B'' = k[Y_1, \dots, Y_m]$ such that A' is an integral extension of B' and A'' is an integral extension of B''. By the Going Up theorem, this implies $\dim A' = \dim B'$ and $\dim A'' = \dim B''$. Notice that the tensor product

$$B' \otimes_k B'' = k[X_1, \dots, X_n] \otimes_k k[Y_1, \dots, Y_m] \cong k[X_1, \dots, X_n, Y_1, \dots, Y_m]$$

is again a polynomial ring. Therefore, since we know the dimension of a polynomial algebra, we have $\dim(B' \otimes_k B'') = \dim B' + \dim B''$.

Now, since A' is integral over B', then A' is finitely generated over B'. Similarly, A'' is finitely generated over B''. But then the ring $A' \otimes_k A''$ is finitely generated over the product $B' \otimes_k B''$ by the (tensor) products of the generators of A' over B' and of A'' over B'', hence integral. Again by the Going Up theorem, we have $\dim(A' \otimes_k A'') = \dim(B' \otimes_k B'')$. So we have:

$$\dim(A' \otimes_k A'') = \dim(B' \otimes_k B'') = \dim B' + \dim B'' = \dim A' + \dim A''.$$

Let $\mathfrak{p}$ be a minimal prime of $A' \otimes_k A''$. Let K', K'', L', L'' be the field of fractions of A', A'', B', B''. We have the diagram of injections

$$\begin{array}{ccccc}
0 & \longrightarrow & L' \otimes_k L'' & \longrightarrow & K' \otimes_k K'' \\
 & & \uparrow & & \uparrow \\
0 & \longrightarrow & B' \otimes_k B'' & \longrightarrow & A' \otimes_k A'' \\
 & & \uparrow & & \uparrow \\
 & & 0 & & 0
\end{array}$$

As K' is an L' vector space, it is free over L'. Similarly K'' is free over L''. Since direct sums commute with tensor products, we have that $K' \otimes_k K''$ is free over $L' \otimes_k L''$; in particular, it is a torsion-free module over the polynomial algebra $B' \otimes_k B''$. Notice that the intersection of the prime ideal $\mathfrak{p}$ with $B' \otimes_k B''$ is 0. In fact, let $x \in \mathfrak{p} \cap B' \otimes_k B''$: notice that we can think at x as an element of $A' \otimes_k A''$. Since $\mathfrak{p}$ is a minimal prime, then it is associated to (0). Hence $x \in \mathfrak{p}$ implies that x is a zero-divisor in $A' \otimes_k A''$. Therefore there exists $y \in A' \otimes_k A''$ such that $xy = 0$. But then y is a $B' \otimes_k B''$-torsion element of $K' \otimes_k K''$, so $x = 0$.

As a consequence, we have that $A' \otimes_k A''/\mathfrak{p}$ is integral over $B' \otimes_k B''$ and so, finally, we get

$$\dim(B' \otimes_k B'') = \dim(A' \otimes_k A''/\mathfrak{p})$$

and this completes the proof. ☐

4.3. **Lemma.** *Let A be a k-algebra, let $C = A \otimes_k A$ and let $\varphi \colon C \to A$ be the homomorphism defined by $\varphi(a \otimes b) = ab$. Then:*

i) The kernel $\mathfrak{d}$ of φ is the ideal of C generated by the elements $1 \otimes a - a \otimes 1$, for $a \in A$.

ii) If $\mathfrak{p}$ and $\mathfrak{q}$ are two ideals of A, the image via φ of the ideal $\mathfrak{p} \otimes A + A \otimes \mathfrak{q}$ is equal to $\mathfrak{p} + \mathfrak{q}$.

Proof. It's a simple computation. For details, see [9, p. 48]. □

Now we can return to the proof of the proposition.

Proof. Consider the exact sequence

$$0 \to \mathfrak{p} \otimes_k A + A \otimes_k \mathfrak{q} \to A \otimes_k A \to A/\mathfrak{p} \otimes_k A/\mathfrak{q} \to 0.$$

Let $I = \varphi^{-1}(\mathfrak{P})$ where $\varphi \colon A \otimes_k A \to A$ is the multiplication map $\varphi(a \otimes b) = ab$. Notice that $I \in \mathcal{V}(\mathfrak{d})$ (i.e. $I \supset \mathfrak{d} = \operatorname{Ker} \varphi$) and $I \in \mathcal{V}(\mathfrak{q})$ where $\mathfrak{q} := \mathfrak{p} \otimes A + A \otimes \mathfrak{q}$. Thus $I \in \mathcal{V}(\mathfrak{q}) \cap \mathcal{V}(\mathfrak{d}) = \mathcal{V}(\mathfrak{q} + \mathfrak{d})$. Notice that I is actually a minimal prime ideal of $\mathcal{V}(\mathfrak{q} + \mathfrak{d})$, since any J prime that contains $\mathfrak{q}$ must contain also I (by definition). Let now $\mathfrak{Q}$ be the image of I in $A/\mathfrak{p} \otimes_k A/\mathfrak{q}$: it is prime and actually a minimal prime ideal of $\mathcal{V}(\mathfrak{d}')$ where $\mathfrak{d}'$ is the image of $\mathfrak{d}$ in $A/\mathfrak{p} \otimes_k A/\mathfrak{q}$. So we have the following situation:

$$0 \to \mathfrak{q} \to \mathfrak{d} \subset I \to \mathfrak{d}' \subset \mathfrak{Q} \to 0$$

and $(\mathfrak{d} \subset I) \xrightarrow{\varphi} (0 \subset \mathfrak{P})$. By lemma 4.3, we have that the kernel $\mathfrak{d}$ is generated by the n elements $X_i \otimes 1 + 1 \otimes X_i$. Hence we can deduce that $\operatorname{ht}(\mathfrak{Q}) \leq n$. If $\mathfrak{Q}_0$ is a minimal prime ideal of $A/\mathfrak{p} \otimes_k A/\mathfrak{q}$ contained in $\mathfrak{Q}$, then again we have $\operatorname{ht}(\mathfrak{Q}/\mathfrak{Q}_0) \leq n$. So, according to lemma 4.2, we have

$$\dim((A/\mathfrak{p} \otimes_k A/\mathfrak{q})/\mathfrak{Q}_0) = \dim(A/\mathfrak{p}) + \dim(A/\mathfrak{q}).$$

Moreover, since also $A/\mathfrak{p} \otimes_k A/\mathfrak{q}$ is finitely generated as k-algebra, we can use the formula

$$\operatorname{ht}(\mathfrak{Q}/\mathfrak{Q}_0) = \dim((A/\mathfrak{p} \otimes_k A/\mathfrak{q})/\mathfrak{Q}_0) - \dim((A/\mathfrak{p} \otimes_k A/\mathfrak{q})/\mathfrak{Q})$$

and so, using the fact that $\dim((A/\mathfrak{p} \otimes_k A/\mathfrak{q})/\mathfrak{Q}) = \dim A/\mathfrak{P}$, we get

$$n \geq \operatorname{ht}(\mathfrak{Q}/\mathfrak{Q}_0) = \dim A/\mathfrak{p} + \dim A/\mathfrak{q} - \dim A/\mathfrak{P}$$

so that

$$n - \dim A/\mathfrak{P} \leq n - \dim A/\mathfrak{p} + n - \dim A/\mathfrak{q}$$

i.e. $\operatorname{ht}(\mathfrak{P}) \leq \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q})$. □

As we mentioned above, the proof is an algebraic analogue of the set-theoretic formula $V \cap W = (V \times W) \cap \Delta$. Notice also that the first lemma shows that every irreducible component of the product of two algebraic (affine) varieties has dimension equal to the sum of the dimensions of the two varieties.

5. Multiplicity of a module and intersection multiplicity of two modules

Let A be a Noetherian ring. Assume that all A-modules mentioned in this section are finitely generated.

5.1. **Definition.** An element of the free abelian group $Z(A)$ generated by the elements of $\operatorname{Spec}(A)$ is called a *cycle* of A. We say that a cycle Z is *positive* if it is of the form

$$Z = \sum_{\mathfrak{p}} n(\mathfrak{p})\mathfrak{p} \quad \text{with } n(\mathfrak{p}) \geq 0 \text{ for every } \mathfrak{p} \in \operatorname{Spec}(A)$$

For each i, we denote by $Z_i(A)$ the free Abelian group with basis consisting of all prime ideals $\mathfrak{p}$ such that the dimension of $A/\mathfrak{p}$ is i. The group $Z_i(A)$ is called the group of cycles of A of dimension i. If we assume that A is local of dimension n (that implies, in particular, that the dimension of $A/\mathfrak{p}$ is bounded by n for all $\mathfrak{p} \in \operatorname{Spec}(A)$), we have that the group $Z(A)$, also called the group of cycles of A, is the direct sum of its subgroups $Z_i(A)$, $0 \leq i \leq n$.

In order to understand the definition, we can consider the following example: let $A = k[X, Y]$ be the polynomial ring in two variables over a field k. It is well-known that the dimension of $k[X, Y]$ is, as one might expect, 2. As a consequence, the groups of cycles $Z_k(A)$ are zero for all k except for $k = 0, 1$ and 2. The group $Z_0(A)$ consists of the free Abelian group generated by the set of prime ideals such that $A/\mathfrak{p}$ has dimension 0. Since $A/\mathfrak{p}$ is a domain, this implies that is actually a field. Hence $Z_0(A)$ is the free Abelian group on the set of maximal ideals of A. If k is algebraically closed, by the Nullstellensatz, this set is in one-to-one correspondence with the set of points (a, b) in $k \times k$ (actually it is the set of closed points of $\operatorname{Spec}(A)$). $Z_1(A)$ has a basis consisting of primes $\mathfrak{p}$ with codimension 1, which correspond to irreducible curves. Finally, $Z_2(A)$ is free of rank 1, generated by the (0) ideal.

Let M be an A-module: we wish to define the group of cycles of M. A first, naive, idea is to consider the set of associated primes of M and then take the free module generated by them. Anyway, it turns out to be more useful to make a different choice. Notice (see [9, V.A.1]) that the category of A-modules M such that $\dim_A(M) \leq i$ is abelian. Given an exact sequence of A-modules

$$0 \to M' \to M \to M'' \to 0$$

with M' and M'' of dimension less or equal than i, then, since $\operatorname{Supp}(M) = \operatorname{Supp}(M') \cup \operatorname{Supp}(M'')$, also M has dimension less or equal than i. We provisionally denote by $K_i(A)$ the category of A-modules M with $\dim_A(M) \leq i$. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $\dim(A/\mathfrak{p}) = i$ and let $M \in K_i(A)$. Then the module $M_\mathfrak{p}$ over $A_\mathfrak{p}$ is of finite length (maybe zero): in fact, if $\mathfrak{p} \notin \operatorname{Supp}(M)$, then $M_\mathfrak{p} = 0$. If $\mathfrak{p} \in \operatorname{Supp}(M)$ we have[4] that $\dim(A/\mathfrak{p}) = i = \dim_A(M)$ and so $\operatorname{Supp}(M_\mathfrak{p}) = \{\mathfrak{p}A_\mathfrak{p}\}$. Denote the length of $M_\mathfrak{p}$ as $A_\mathfrak{p}$-module by $\operatorname{length}(M_\mathfrak{p})$. This length satisfies the following property: if

$$0 \subset M_0 \subset \ldots \subset M_i \subset \ldots \subset M_s = M$$

is a finite filtration of M whose quotients M_i/M_{i-i} are of the form $A/\mathfrak{q}$, where $\mathfrak{q} \in \operatorname{Spec}(A)$, then there are exactly $\operatorname{length}(M_\mathfrak{p})$ quotients of the form $A/\mathfrak{p}$.

Thus, we can finally define the function $z_i \colon K_i(A) \to Z_i(A)$ by

$$z_i(M) = \sum_{\dim A/\mathfrak{p} = i} \operatorname{length}(M_\mathfrak{p})\mathfrak{p}.$$

We define $z_i(M)$ to be the cycle of dimension i associated to M. The function z_i is zero on $K_{i-1}(A)$. As we have already observed in the example above, if A is a domain, then $Z_n(A) \cong \mathbb{Z}$ for $n = \dim(A)$. Similarly, $z_n(M)$ is (by definition) the *rank* of the A-module M.

[4]since $\dim_A(M) = \sup_{\mathfrak{p} \in \operatorname{Supp}(M)}(\operatorname{coht} \mathfrak{p})$

Assume A is local and let $\mathfrak{m}$ be its maximal ideal. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal. Recall that by the so-called Dimension theorem, for every non-zero A-module M, the Hilbert-Samuel polynomial $P_{\mathfrak{q}}(M)$ is of degree equal to $d = \dim_A M$. Moreover, as we have seen in section 1, its leading coefficient is $e(\mathfrak{q}, M)/d!$. More generally, we defined $e_i(\mathfrak{q}, M)$ for all positive integer i and M module such that $d = \dim_A M \leq i$ to be 0 if $d < i$ and $e(\mathfrak{q}, M)$ if $d = i$. Hence $e_i(\mathfrak{q}, M)$ is an additive function on $K_i(A)$ which is zero on $K_{i-1}(A)$. It can be proved[5] that any such function on $K_i(A)$ factors through $z_i(M)$ defined above. Thus we have the following additive formula:

$$e_i(\mathfrak{q}, M) = \sum_{\dim A/\mathfrak{p}=i} \text{length}(M_{\mathfrak{p}}) e_i(\mathfrak{q}, A/\mathfrak{p})$$

Let $\mathbf{x}$ be an ideal of definition of A, generated by $x_1, \ldots, x_n$ where $n = \dim A$. According to theorem 3.2, the i-th homology module of the Koszul complex $K(\mathbf{x}, M)$ has finite length for every A-module M and every $i \geq 0$ and we have

$$\chi(\mathbf{x}, M) = e_n(\mathbf{x}, M) = \sum_{i=0}^{n} (-1)^i \mathcal{L}_i(\mathbf{x}, M)$$

where $\mathcal{L}_i(\mathbf{x}, M) := \text{length}(H_i(\mathbf{x}, M))$.

In section 4, we introduced the viewpoint of the reduction to the diagonal: in particular, given the k-algebra $A = k[X_1, \ldots, X_n]$, we introduced $A \otimes_k A$ as the coordinate ring of the product $\mathbb{A}^n_k(k) \times \mathbb{A}^n_k(k)$. Similarly, $A/\mathfrak{p} \otimes_k A/\mathfrak{q}$ can be seen as the coordinate ring of $U \times V$ and $(A \otimes_k A)/\mathfrak{d}$ as the coordinate ring of the diagonal Δ. In particular, A can be identified with $(A \otimes_k A)/\mathfrak{d}$ and this gives a $A \otimes_k A$-module structure to A. The isomorphism of $U \cap V$ with the intersection $(U \times V) \cap \Delta$ is the expressed in algebraic form as the base change[6]

$$A/\mathfrak{p} \otimes_A A/\mathfrak{q} \cong (A/\mathfrak{p} \otimes_k A/\mathfrak{q}) \otimes_{A \otimes_k A} A.$$

This idea can be generalized as follows: let A be a k-algebra (k field, not necessarily algebraically closed) and let M, N be two A-modules. Let $B = A \otimes_k A$ and let $\mathfrak{d}$ be the ideal generated by $a \otimes 1 - 1 \otimes a$, $a \in A$ (is the kernel of the multiplication map φ of section 4). Then $A \cong B/\mathfrak{d}$ has a B module structure and we have the formula (see [3, Chap. IX, 2.8] and [9, p. 101])

$$\text{Tor}_n^B(M \otimes_k N, A) \cong \text{Tor}_n^A(M, N).$$

This is the new "reduction to the diagonal" argument. Using this isomorphism we get a new way of computing the $\text{Tor}_n^A(M, N)$ using the Koszul complex. In fact, from the definition of the Tor functor we have that given a projective resolution

$$\to P_n \to \ldots \to P_0 \to A$$

of A as B-module, there is a natural isomorphism between the homology modules of the complex $(M \otimes_k N) \otimes_B P$ and the modules $\text{Tor}_n^B(M \otimes_k N, A)$. Using the reduction to the diagonal argument, we get the isomorphism (natural in M and N) between $\text{Tor}_n^A(M, N)$ and $H_n((M \otimes_k N) \otimes_B P)$. In particular, if A is the polynomial ring $k[X_1, \ldots, X_n]$, a

[5]This is more or less the universal property of the Grothendieck group of A. See [1, p.88]

[6]Recall that the support of a tensor product of modules $M \otimes_A N$ is simply the intersection of the supports of the two modules M and N.

projective resolution of A as $B = A \otimes_k A$ module is given (again, see [9] for details) by the Koszul complex $K^B((X_i \otimes 1 - 1 \otimes X_i), B)$.

Now we need one last tool:

5.2. **Proposition.** *Let A be a Noetherian ring such that, for every maximal ideal $\mathfrak{m}$, $A_\mathfrak{m}$ is a domain. Then A is a direct product of a finite number of domains.*

Proof. Since A is Noetherian, we can consider the decomposition of the nilradical of A as a finite intersection of primes: $\sqrt{(0)} = \mathrm{Nil}(A) = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n$: they are exactly the minimal prime ideals of A. Let $\mathfrak{m}$ be a maximal ideal of A. Then $A_\mathfrak{m}$ is a domain and so it has exactly one minimal prime ideal (actually (0)). As a consequence, $\mathfrak{m}$ must contain exactly one of the $\mathfrak{p}$'s. Hence there is no maximal ideal $\mathfrak{m}$ such that $\mathfrak{m} \supset \mathfrak{p}_i + \mathfrak{p}_j$ for $i \neq j$ and so $\mathfrak{p}_i + \mathfrak{p}_j = A$, i.e. they are pairwise coprime. By CRT we get

$$A/\bigcap_{i=1}^{n} \mathfrak{p}_i = A/\,\mathrm{Nil}(A) = \prod_{i=1}^{n} A/\mathfrak{p}_i.$$

We claim that $\mathrm{Nil}(A) = 0$, i.e. A is reduced. In fact, let $x \in \mathrm{Nil}(A)$ and let $\mathfrak{m}$ a maximal ideal of A containing the annihilator $\mathrm{Ann}(x)$. Then $x \in A_\mathfrak{m}$ is not zero but x is still nilpotent there, contradicting the assumption that $A_\mathfrak{m}$ is a domain. □

5.3. **Definition.** A Noetherian ring A is called a *regular ring* if it has finite global homological dimension.

Equivalently, A is regular if the localization at every prime ideal is a regular local ring. Since a regular local ring is a domain, from the previous proposition we get that every regular ring A is a direct product of a finite number of regular domains. We need also the following

5.4. **Definition.** A domain A is called of *equal characteristic* if, for every prime $\mathfrak{p}$, $A/\mathfrak{p}$ and A have the same characteristic. If A is a regular ring, $A \cong \prod_{i=1}^n B_i$, we say that A is of equal characteristic if the components B_i are domains of equal characteristic, which is to say if, for every prime $\mathfrak{p}$, the ring $A_\mathfrak{p}$ is of equal characteristic.

We are finally ready to state the following result:

5.5. **Theorem.** *If A is a regular ring of equal characteristic of dimension n, M and N are two finitely generated A-modules and $\mathfrak{q}$ a minimal prime ideal of* $\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$*, then:*

i) The Euler-Poincaré characteristic $\chi_\mathfrak{q}(M, N) := \sum_{i=1}^n (-1)^i \mathrm{length}(\mathrm{Tor}_i^A(M, N)_\mathfrak{q})$ *is well-defined and* ≥ 0.

ii) $\dim_{A_\mathfrak{q}} M_\mathfrak{q} + \dim_{A_\mathfrak{q}} N_\mathfrak{q} \leq \mathrm{ht}_A(\mathfrak{q})$

iii) $\dim_{A_\mathfrak{q}} M_\mathfrak{q} + \dim_{A_\mathfrak{q}} N_\mathfrak{q} < \mathrm{ht}_A(\mathfrak{q})$ *if and only if* $\chi_\mathfrak{q}(M, N) = 0$.

Where $\mathrm{Tor}_i^A(M, N)_\mathfrak{q} = \mathrm{Tor}_i^{A_\mathfrak{q}}(M_\mathfrak{q}, N_\mathfrak{q})$ *is the localization at* $\mathfrak{q}$.

The theorem is a generalization of the results that we collected along the way (in particular theorem 3.2). We don't give a proof, since it requires results and definitions about completed tensor product and completed Tor_i. Details can be found in [9, pp. 102-106].

6. The Tor formula

In this section we will describe the connection with algebraic geometry. Let $X = \mathbb{A}^n_k(k)$ be the n-dimensional affine space over a field k. We can assume, for simplicity, that the field k is algebraically closed. Let U, V be two irreducible varieties and let W be an irreducible component of $U \cap V$. Suppose that the local ring A of $\mathbb{A}^n_k(k)$ at W

$$A = \{f/g \mid f, g \in k[X_1, \dots, X_n], g(W) \neq 0\} = k[X_1, \dots, X_n]_{\mathfrak{p}_W}$$

(where $\mathfrak{p}_W$ is the prime ideal corresponding to W) is a regular local ring. Then (see section 4) we have:

$$\dim U + \dim V \leq n + \dim W.$$

When the equality holds in this formula, the intersection is called *proper at* W (and, following Serre, we say that U and V intersect properly at W).

Let now $\mathfrak{p}_U$ and $\mathfrak{p}_V$ be the prime ideals of the regular local ring A corresponding to U and V respectively. If we apply theorem 5.5 to the modules $A/\mathfrak{p}_U$ and $A/\mathfrak{p}_V$ (finitely generated over A) we get that the Euler-Poincaré characteristic:

$$\chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V) = \sum_{i=0}^{n} (-1)^i \mathrm{length}(\mathrm{Tor}_i^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)) \tag{6.1}$$

is well defined; it is an integer ≥ 0.

Indeed the hypothesis of the theorem are satisfied: A is regular and, for $M = A/\mathfrak{p}_U$, $N = A/\mathfrak{p}_V$, we have $\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N) = \{\mathfrak{m}\}$ where $\mathfrak{m}$ is the maximal ideal of A, since $M \otimes_A N \cong A/(\mathfrak{p}_U + \mathfrak{p}_V)$ has finite length. Hence $\mathfrak{m}$ is the (unique) minimal prime ideal of $\mathrm{Supp}(M \otimes_A N)$: using the Koszul complex and the previous results, we get that the modules $\mathrm{Tor}_i^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)$ have finite length, i.e. the Euler-Poincaré characteristic is well-defined. Finally, $\chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V) \geq 0$ using part i) of the theorem.

6.1. **Definition.** The non negative integer $\chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)$ in formula 6.1 is Serre's definition of *intersection multiplicity* of U and V at W.

For[7] $X = \mathbb{A}^n_k(k)$, the function $I(X, U \cdot V, W) := \chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)$ satisfies the formal properties of an *intersection multiplicity* (see below).

Let X be a non singular affine variety of dimension n and let A be its coordinate ring. If $a \in \mathbb{N}$ and if M is an A-module of dimension less or equal than a, we defined, at the beginning of section 5, the cycle $z_a(M) = \sum_{\dim A/\mathfrak{q}=a} \mathrm{length}(M_\mathfrak{q})\mathfrak{q}$. It's a positive cycle of dimension a (i.e. $z_a(M) \in Z_i(A)$) which is zero if (and only if) $\dim M < a$. Using the function I defined above, we want to define a *product of cycles* in $Z(A)$.

6.2. **Proposition.** *Let $a, b, c \in \mathbb{N}$ such that $a + b = n + c$. Let M, N be two A-modules such that* $\dim M \leq a$, $\dim N \leq b$ *and* $\dim M \otimes_A N \leq c$. *Then the cycles*

$$z_a(M) = \sum_{\dim A/\mathfrak{q}=a} \mathrm{length}(M_\mathfrak{q})\mathfrak{q}, \qquad z_b(N) = \sum_{\dim A/\mathfrak{p}=b} \mathrm{length}(N_\mathfrak{p})\mathfrak{p}$$

[7] We can restate the above results in the (slightly) more general situation where X is an algebraic variety and U, V, W are three irreducible subvarieties of X, W being an irreducible component of $U \cap V$.

are defined and the intersection cycle (defined by linearity using the function $I(X, U \cdot V, W) := \chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)$*)* $z_a(M) \cdot z_b(N)$ *is defined and coincides with the cycle*

$$z_c(\mathrm{Tor}^A(M,N)) := \sum(-1)^i z_c(\mathrm{Tor}_i^A(M,N)).$$

Proof. The result follows from a direct computation: for details see [9, p. 113]. □

6.3. **Remark.** In the case $M = A/\mathfrak{p}_U$, $N = A/\mathfrak{p}_V$ with $\dim M = a$, $\dim N = b$ the condition $a + b = c + n$ is exactly the case of U and V which intersect properly at an irreducible subvariety W of dimension c. Notice also that $M \otimes_A N \cong A/(\mathfrak{p}_U + \mathfrak{p}_V)$.

The product of cycles defined using the function I has the so called *fundamental properties of intersection theory*; namely it's commutative, associative, and satisfies two more properties (the product formula and the reduction to the diagonal). This should convince us that Serre's definition of intersection multiplicity makes sense. Actually, it's not hard to show that the following result holds:

6.4. **Theorem.** *With the above notations, we have:*

(1) *If* U *and* V *do not intersect properly at* W*, we have* $\chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V) = 0$
(2) *If* U *and* V *intersect properly at* W*,* $\chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V) > 0$ *and coincides with the intersection multiplicity in the sense of Samuel.*

We will not give the general definition of Samuel's intersection multiplicity: the key point here is that Samuel's definition is strictly geometric, while Serre's definition is purely algebraic.

Here there is an idea of the proof. First consider the case where U is a complete intersection in W, i.e. the ideal $\mathfrak{p}_U$ in the local ring A of X at W is generated by h elements $x_1, \ldots, x_h$ with $h = \dim X - \dim U = \dim V - \dim W$. We have (see section 2)

$$\mathrm{Tor}_i^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V) = H_i(\mathbf{x}, A/\mathfrak{p}_V).$$

Hence

$$\chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V) = \sum_i (-1)^i \mathrm{length}(H_i(\mathbf{x}, A/\mathfrak{p}_V))$$

and, by theorem 3.2, this gives $\chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V) = e_n(\mathbf{x}, A/\mathfrak{p}_V)$ (where $n = \dim X$ and $\mathbf{x}$ denotes the ideal of $A/\mathfrak{p}_V$ generated by the images of the x_i's). This number coincides (actually we can take it as a definition) with Samuel's definition of multiplicity of U and V on W in the complete intersection case. The general case can be reduced to the previous one using the reduction to the diagonal, which holds for both I and Samuel's multiplicity. Actually the diagonal Δ is non singular and so is locally a complete intersection.

7. Examples and applications

In this section we prove that Serre's definition of intersection multiplicity is the "right one" at least in the case of affine plane curves. Consider the 2-dimensional affine plane $\mathbf{A}_k^2(k)$ over an algebraically closed field k. Let $\mathcal{O}_P \cong k[X,Y]_{(X-a,Y-b)}$ be the local ring at the point $P = (a,b) \in \mathbf{A}_k^2(k)$. Let U and V be two irreducible curves and let P be a point in the intersection. Let $\mathfrak{p}_U$ and $\mathfrak{p}_V$ be the prime ideals in $\mathcal{O}_P$ corresponding to U

and V respectively. From classical algebraic geometry, the intersection multiplicity of U and V at P, $\mu_P(U,V)$ is defined as follows:

$$\mu_P(U,V) = \dim_k(\mathcal{O}_P/(\mathfrak{p}_U + \mathfrak{p}_V)) = \dim_k(\mathcal{O}_P/\mathfrak{p}_U \otimes_{\mathcal{O}_P} \mathcal{O}_P/\mathfrak{p}_V).$$

Then we have the following

7.1. **Proposition.** *Let $A = \mathcal{O}_P$. Then we have $\mu_P(U,V) = \chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)$.*

Proof. Recall (see section 2) that, the Koszul complex of a commutative ring A for $x \in A$ is such that $H_i(x, M) \cong \mathrm{Tor}_i^A(A/x, M)$ for all i and M.

We use the definition of $\chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)$; clearly $\dim A = 2$, hence

$$\chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V) =$$
$$= \mathrm{length}(\mathrm{Tor}_0^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)) - \mathrm{length}(\mathrm{Tor}_1^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)) + \mathrm{length}(\mathrm{Tor}_2^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V))$$
$$= \mathrm{length}(A/\mathfrak{p}_U \otimes A/\mathfrak{p}_V) - \mathrm{length}(\mathrm{Tor}_1^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)) + \mathrm{length}(\mathrm{Tor}_2^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V))$$

Let $f \in A$ be the generator of the principal[8] ideal $\mathfrak{p}_V$ and let $M = A/\mathfrak{p}_U$. f is not a zero divisor of M, since the two curves U and V are both irreducible and distinct (in particular, they do no share irreducible components). So, using the Koszul complex, we get:

$$(A/\mathfrak{p}_U \otimes A/\mathfrak{p}_V) \cong H_0(f, M) \cong M/(f)M$$
$$(\mathrm{Tor}_1^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)) \cong H_1(f, M) \cong \mathrm{Ker}(m \to fm) = 0$$
$$(\mathrm{Tor}_2^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)) \cong H_2(f, M) = 0$$

Moreover

$$M/(f)M = (A/\mathfrak{p}_U)/((f)A/\mathfrak{p}_U) \cong A/(\mathfrak{p}_V + \mathfrak{p}_U) \cong A/\mathfrak{p}_V \otimes_A A/\mathfrak{p}_U$$

hence we have done, since clearly $\dim_k(A/(\mathfrak{p}_V + \mathfrak{p}_U)) = \mathrm{length}_A(A/\mathfrak{p}_V \otimes_A A/\mathfrak{p}_U)$. □

7.2. **Example.** To see why the higher Tor are actually useful (so that Serre's "Tor-formula" is really the right one for computing the intersection numbers), we conclude with the following example (taken from [5] and [4, p. 428]).

Consider the affine space $\mathbb{A}^4_k(k)$ (over a fixed field k that may be chosen algebrically closed). Let V be the plane $V = \mathcal{V}(x_1 - x_3, x_2 - x_4)$ and let $U = \mathcal{V}(x_1x_3, x_1x_4, x_2x_3, x_2x_4)$ be the union of two planes meeting at a point. Notice that $(x_1, x_2) \cap (x_3, x_4) = (x_1, x_2) \cdot (x_3, x_4) = (x_1x_3, x_1x_4, x_2x_3, x_2x_4)$, i.e. $U = U_1 \cup U_2$ where $U_1 = \mathcal{V}(x_1, x_2)$ and $U_2 = \mathcal{V}(x_3, x_4)$ are the two irreducible components. We want to (naively) compute the intersection multiplicity at the point $P = (0, 0, 0, 0)$ for the two varieties using the definition given for curves: in other words we have to compute the dimension as k-vector space of the ring

$$\frac{k[x_1, x_2, x_3, x_4]_{(x_1,x_2,x_3,x_4)}}{(x_1 - x_3, x_2 - x_4) + (x_1x_3, x_1x_4, x_2x_3, x_2x_4)}.$$

Actually (since localization commutes with the quotient), this dimension coincides with

$$\dim_k\Big(\frac{k[x_1, x_2, x_3, x_4]}{(x_1 - x_3, x_2 - x_4) + (x_1x_3, x_1x_4, x_2x_3, x_2x_4)}\Big)_{(\overline{x_1},\overline{x_2},\overline{x_3},\overline{x_4})} := \dim_k(D)$$

[8] It's the localization of the principal ideal defining the curve V, hence it's still principal.

However $D \simeq (k[\overline{x_1}, \overline{x_2}]/(\overline{x_1}^2, \overline{x_1 x_2}, \overline{x_2}^2))_{\overline{x_1}, \overline{x_2}}$ which has dimension 3. This number is what we would call the "intersection multiplicity" at P of the two varieties after this naive computation. Since the variety U is not irreducible, it is quite natural to check what happens if we consider the two irreducible components. So let us compute the intersection of U_1 with V at P and of U_2 with V at P. For example,

$$\frac{k[x_1, x_2, x_3, x_4]}{(x_1 - x_3, x_2 - x_4) + (x_1, x_2)} \simeq k$$

and clearly this does not change after localizing at $\overline{x_1}$, so the intersection multiplicity of $V \cap U_1$ at P is 1. Similarly for $V \cap U_2$. So, since V meets each component of U in one point P, we have, by linearity, that the intersection number is simply the sum $1 + 1 = 2 = I(\mathbb{A}^4, U \cdot V, P)$. Clearly 2, that is the correct number, is different from 3, that is the result of our first computation.

References

[1] M. F. Atiyah and I. Macdonald (1969), *Introduction to commutative algebra*, Addison-Wesley, Reading, Mass.
[2] D. Eisenbud (2004), *Commutative algebra with a view toward algebraic geometry*, Springer, NY.
[3] H. Cartan and S. Eilenberg (1956), *Homological Algebra*, Princeton Math. Ser. 19, Princeton.
[4] R. Hartshorne (1997), *Algebraic Geometry*, GTM, Springer-Verlag, NY.
[5] W. Fulton (1997), *Intersection Theory*, Springer-Verlag, NY.
[6] H. Matsumura (1986), *Commutative ring theory*, Cambridge Univ. Press, Cambridge.
[7] A. Petrov (2008), *Two Ideas from Intersection Theory*, Available at http://math.arizona.edu/ noni/.
[8] P. Roberts (1998), *Multiplicities and Chern Classes in Local Algebra*, Cambridge Univ. Press, Cambridge.
[9] J. P. Serre (2000), *Local Algebra*, Springer Monographs in Mathematics.