

THE LOOP AND SUSPENSION FUNCTORS & FIBRATION SEQUENCES

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In the first part, we will consider a fixed model category \mathcal{C} and $f, g : A \rightrightarrows B$ two maps in \mathcal{C} where A is cofibrant and B is fibrant.

Notation 1. Left and right homotopies are denoted as follows:

$$\begin{array}{ccc}
 A \sqcup A & \xrightarrow{f+g} & B \\
 \downarrow \nabla & \searrow \partial_0 + \partial_1 & \uparrow h \\
 A & \xleftarrow{\sigma} & A \times I
 \end{array}
 \qquad
 \begin{array}{ccc}
 B^I & \xleftarrow{s} & B \\
 \uparrow k & \searrow (d_0, d_1) & \downarrow \Delta \\
 A & \xrightarrow{(f, g)} & B \times B
 \end{array}$$

Definition 2. (1) Let $h : A \times I \rightarrow B$ and $h' : A \times I' \rightarrow B$ be two left homotopies from f to g . By a left homotopy from h to h' we mean a diagram

$$\begin{array}{ccc}
 A \times I \sqcup_{A \sqcup A} A \times I' & \xrightarrow{h+h'} & B \\
 \downarrow \sigma + \sigma' & \searrow j_0 + j_1 & \uparrow H \\
 A & \xleftarrow{\tau} & A \times J
 \end{array}$$

where $j_0 + j_1$ is a cofibration and τ is a weak equivalence. Here $A \times I \sqcup_{A \sqcup A} A \times I'$ is the push-out of the maps $\partial_0 + \partial_1 : A \sqcup A \rightarrow A \times I$ and $\partial'_0 + \partial'_1 : A \sqcup A \rightarrow A \times I'$.

(2) Dually, let $k : A \rightarrow B^I$ and $k' : A \rightarrow B^{I'}$ be two rightarrow homotopies from f to g . By a right homotopy from k to k' we mean a diagram

$$\begin{array}{ccc}
 B^J & \xleftarrow{t} & B \\
 \uparrow K & \searrow (\tilde{j}_0, \tilde{j}_1) & \downarrow (s, s') \\
 A & \xrightarrow{(k, k')} & B^I \times_{B \times B} B^{I'}
 \end{array}$$

where $\tilde{j}_0 + \tilde{j}_1$ is a fibration and t is a weak equivalence. Here $B^I \times_{B \times B} B^{I'}$ is the pullback of the maps $(d_0, d_1) : B^I \rightarrow B \times B$ and $(d'_0, d'_1) : B^{I'} \rightarrow B \times B$.

Definition 3. Let $h : A \times I \rightarrow B$ be a left homotopy from f to g and let $k : A \rightarrow B^I$ be a right homotopy from f to g . By a correspondence between h and k we mean a map $H : A \times I \rightarrow B^I$ s.t. $H\partial_0 = k, H\partial_1 = sg, d_0H = h$ and $d_1H = g\sigma$. It's good to bear in mind the following diagrams:

$$\begin{array}{ccc}
 f & \xrightarrow{h} & g \\
 & & \uparrow k \\
 & & f \\
 & & \downarrow k \\
 & & B^I
 \end{array}
 \qquad
 \begin{array}{ccc}
 g & \xrightarrow{g\sigma} & g \\
 \uparrow k & & \uparrow sg \\
 f & \xrightarrow{h} & g \\
 & & \downarrow H \\
 & & B^I
 \end{array}$$

with the horizontal lines referring to left homotopies and the vertical lines referring to right homotopies.

Lemma 4. *Given B^I and $h : A \times I \rightarrow B$, there is a right homotopy k corresponding to h . Dually, given $A \times I$ and a right homotopy $k : A \rightarrow B^I$, there is a left homotopy $h : A \times I \rightarrow B$ corresponding to k .*

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{sg} & B^I \\ \partial_1 \downarrow & & \downarrow (d_0, d_1) \\ A \times I & \xrightarrow{(h, g\sigma)} & B \times B \end{array}$$

Since the left arrow is a trivial cofibration, there exists an $H : A \times I \rightarrow B^I$ s.t. every triangle commutes. Now $k = H\partial_0$ satisfies the condition. \square

Lemma 5. *Suppose $h : A \times I \rightarrow B$ and $h' : A \times I' \rightarrow B$ are two left homotopies from f to g and that $k : A \rightarrow B^I$ is a right homotopy from f to g . Suppose that h and k correspond. Then h' and k correspond iff h' is left homotopic to h .*

Proof. Let $H : A \times I \rightarrow B^I$ be a correspondence between h and k , and $H' : A \times I' \rightarrow B^I$ be a correspondence between h' and k . Let $A \times J$, $j_0 + j_1$ and τ be as before. Then the dotted arrow K exists in the diagram

$$\begin{array}{ccc} A \times I \sqcup_{A \sqcup A} A \times I' & \xrightarrow{H+H'} & B^I \\ j_0+j_1 \downarrow & \nearrow K & \downarrow d_1 \\ A \times J & \xrightarrow{g\tau} & B \end{array}$$

and $d_0K : A \times J \rightarrow B$ is a left homotopy from h to h' .

Conversely, suppose given $H : A \times I \rightarrow B^I$ a correspondence between h and k , and a left homotopy $K : A \times J \rightarrow B$ from h to h' . Then $j_0 : A \times I \rightarrow A \times J$ is a cofibration since it's the composition of $j_0 + j_1$ and $A \times I \rightarrow A \times I \sqcup_{A \sqcup A} A \times I'$ which is the pushout of $\partial_0 + \partial_1$. Also j_0 is trivial since $\tau j_0 = \sigma$. Hence the dotted arrow φ exists in the diagram

$$\begin{array}{ccc} A \times I & \xrightarrow{H} & B^I \\ j_0 \downarrow & \nearrow \varphi & \downarrow (d_0, d_1) \\ A \times J & \xrightarrow{(K, g\tau)} & B \times B \end{array}$$

and $\varphi j_1 : A \times I' \rightarrow B^I$ is a correspondence between h' and k . \square

Corollary 6. *\sim_l is an equivalence relation on the class of left homotopies from f to g and the equivalence classes form a set $\pi_1^l(A, B; f, g)$. Dually right homotopy classes of right homotopies form a set $\pi_1^r(A, B; f, g)$. Correspondence yields a bijection $\pi_1^l(A, B; f, g) \simeq \pi_1^r(A, B; f, g)$.*

Proof. Lemma 5 yields the equivalence relation assertion. Lemma 4 shows that to give an equivalent class of h is equivalent to give a $k : A \rightarrow B^I$ with fixed B^I , and thus the equivalence classes form a set. The last assertion is clear from Lemma 5 and its dual. \square

By Corollary 6, we may simply write the set as $\pi_1(A, B; f, g)$ and refer to an element of this set as a homotopy class of homotopies from f to g .

Definition 7. (1) Let $f_1, f_2, f_3 \in \text{Hom}(A, B)$, let $h : A \times I \rightarrow B$ be a left homotopy from f_1 to f_2 and let $h' : A \times I' \rightarrow B$ be a left homotopy from f_2 to f_3 . By the composition of h and h' , we mean the homotopy $h'' : A \times I'' \rightarrow B$ where $A \times I''$ is defined by pushout

$$\begin{array}{ccc}
 A & \xrightarrow{\partial'_0} & A \times I' \\
 \partial_1 \downarrow & & \downarrow \text{in}_2 \\
 A \times I & \xrightarrow{\text{in}_1} & A \times I'' \\
 & \searrow \sigma' & \downarrow \sigma'' \\
 & & A
 \end{array}$$

and $h''\text{in}_1 = h, h''\text{in}_2 = h'$. Note that $A \times I''$ is also a cylinder object with $\partial''_0 = \text{in}_1\partial_0, \partial''_1 = \text{in}_2\partial'_1, \sigma''\text{in}_1 = \sigma$ and $\sigma''\text{in}_2 = \sigma'$. This composition is denoted by $h \cdot h'$, and it gives a left homotopy from f_1 to f_3 .

(2) If $f, g \in \text{Hom}(A, B)$ and $h : A \times I \rightarrow B$ is a left homotopy from f to g , then by the inverse of h , we mean the left homotopy $h' : A \times I' \rightarrow B$ from g to f , where $A \times I'$ is the cylinder object for A given by $A \times I' = A \times I, \partial'_0 = \partial_1, \partial'_1 = \partial_0, \sigma' = \sigma$ and where $h' = h$. This inverse is denoted by h^{-1} .

Hence, we have the following pictures:

$$\begin{array}{ccccc}
 f_1 & \xrightarrow{h} & f_2 & \xrightarrow{h'} & f_3 \\
 & & g & \xrightarrow{h^{-1}} & f
 \end{array}$$

Proposition 8. *Composition of left homotopies induces maps $\pi_1^l(A, B; f_1, f_2) \times \pi_1^l(A, B; f_2, f_3) \rightarrow \pi_1^l(A, B; f_1, f_3)$ and similarly for right homotopies. Composition of left and right homotopies is compatible with the correspondence bijection of Corollary 6. Finally the category with objects $\text{Hom}(A, B)$, with a morphism from f to g defined to be an element of $\pi_1(A, B; f, g)$ and with composition of morphisms defined to be induced by composition of homotopies, is a groupoid, with the inverse of an element of $\pi_1^l(A, B; f, g)$ represented by h being represented by h^{-1} .*

Proof. Let h (resp. k) be a left (resp. right) homotopy from f_1 to f_2 , let h' (resp. k') be a left (resp. right) homotopy from f_2 to f_3 , and let H (resp. H') be a correspondence between h and k (resp. h' and k'). Then we have the following correspondence between

$h \cdot h'$ and $k \cdot k'$:

$$\begin{array}{c}
 \begin{array}{ccc}
 & \overline{f_3\sigma} & \overline{f_3\sigma'} \\
 k' \left| \begin{array}{c} k'\sigma \\ \hline H' \end{array} \right| s'f_3 \\
 \hline \overline{f_2\sigma} & & \overline{h'} \\
 k \left| \begin{array}{c} H \\ \hline sh' \end{array} \right| sf_3 \\
 \hline h & & h'
 \end{array}
 \end{array}$$

Taking Lemma 5 into account, this proves the first two assertions of the proposition.

Composition is associative because $(h \cdot h') \cdot h''$ and $h \cdot (h' \cdot h'')$ are both represented by the picture

$$\cdot \overline{h} \cdot \overline{h'} \cdot \overline{h''} \cdot$$

If $h : A \times I \rightarrow B$ is a left homotopy from f to g and $H : A \times I \rightarrow B^I$ is a correspondence of h with some right homotopy k , then the diagrams

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \overline{g\sigma} & \overline{g\sigma} \\
 k \left| \begin{array}{c} k\sigma \\ \hline H \end{array} \right| sg \\
 \hline \overline{f\sigma} & & \overline{h}
 \end{array} & & \begin{array}{ccc}
 & \overline{g\sigma} & \overline{g\sigma} \\
 k \left| \begin{array}{c} H \\ \hline sg\sigma \end{array} \right| sg \\
 \hline \overline{h} & & \overline{g\sigma}
 \end{array}
 \end{array}$$

and Lemma 5 give $f\sigma \cdot h \sim h$, $h \cdot g\sigma \sim h$, proving the existence of identities and hence that $\text{Hom}(A, B)$ is a category. Finally let $H' : A \times I' \rightarrow B^I$ be $H : A \times I \rightarrow B^I$, where $A \times I'$ is $A \times I$ with $\partial'_0 = \partial_1$, $\partial'_1 = \partial_0$, and $\sigma' = \sigma$, and let $H'' : A \times I' \rightarrow B^I$ be a correspondence of $h^{-1} : A \times I' \rightarrow B$ with some $k'' : A \rightarrow B^I$, and let $\tilde{H} : A \times I \rightarrow B^I$ be H'' . Then the diagrams

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \overline{g\sigma} & \overline{g\sigma} \\
 sg \left| \begin{array}{c} H' \\ \hline k \end{array} \right| H \\
 \hline g \xrightarrow{h^{-1}} f \xrightarrow{h} g
 \end{array} & & \begin{array}{ccc}
 & \overline{f\sigma} & \overline{f\sigma} \\
 sf \left| \begin{array}{c} \tilde{H} \\ \hline k'' \end{array} \right| H'' \\
 \hline f \xrightarrow{h} \quad \quad \quad \xrightarrow{h^{-1}} f
 \end{array}
 \end{array}$$

show that $h^{-1} \cdot h \sim g\sigma$ and $h \cdot h^{-1} \sim f\sigma$ proving the last assertion of the proposition. \square

It is clear that if $i : A' \rightarrow A$ is a map of cofibrant objects, then there is a functor $i^* : \text{Hom}(A, B) \rightarrow \text{Hom}(A', B)$ which sends f into fi and a right homotopy $k : A \rightarrow B^I$ into $ki : A' \rightarrow B^I$. Similarly if $j : B \rightarrow B'$ is a map of fibrant objects, there is a functor $j_* : \text{Hom}(A, B) \rightarrow \text{Hom}(A, B')$.

Lemma 9. *The diagram*

$$\begin{array}{ccc}
 \pi_1(A, B; f, g) & \xrightarrow{i^*} & \pi_1(A', B; fi, gi) \\
 \downarrow j_* & & \downarrow j_* \\
 \pi_1(A, B'; jf, jg) & \xrightarrow{i^*} & \pi_1(A', B'; jfi, jgi)
 \end{array}$$

commutes.

Proof. Let $\alpha \in \pi_1(A, B; f, g)$ and represent α by $h : A \times I \rightarrow B, k : A \rightarrow B^I$, and let H be a correspondence between h and k . By Lemma 4, we may assume that $\sigma : A \times I \rightarrow A$ is a trivial fibration and $s : B \rightarrow B^I$ is a trivial cofibration. So we may choose dotted arrows in the diagrams:

$$\begin{array}{ccc} A' \sqcup A' & \xrightarrow{\partial_0 i + \partial_1 i} & A \times I \\ \partial'_0 + \partial'_1 \downarrow & \nearrow \varphi & \downarrow \sigma \\ A' \times I & \xrightarrow{i\sigma'} & A \end{array} \quad \begin{array}{ccc} B & \xrightarrow{s'j} & (B')^I \\ s \downarrow & \nearrow \psi & \downarrow (d'_0, d'_1) \\ B^I & \xrightarrow{(jd_0, jd_1)} & B' \times B' \end{array}$$

Then ψH is a correspondence between jh and ψk . Hence ψk represents $j_*\alpha$ and so ψki represents $i^*j_*\alpha$. Similarly, $H\varphi$ is a correspondence between ki and $h\varphi$. Hence $h\varphi$ represents $i^*\alpha$ and so $jh\varphi$ represents $j_*i^*\alpha$. Finally $\psi H\varphi$ is a correspondence between ψki and $jh\varphi$ which shows that $i^*j_*\alpha = j_*i^*\alpha$. \square

Notation 10. A pointed category is a category \mathcal{C} with a zero object $*$. If X and Y are arbitrary objects of \mathcal{C} , we denote by $0 \in \text{Hom}_{\mathcal{C}}(X, Y)$ the composition $X \rightarrow * \rightarrow Y$. In a pointed model category, if $A \in \mathcal{C}_c$ and $B \in \mathcal{C}_f$, we will abbreviate $\pi_1(A, B; 0, 0)$ to $\pi_1(A, B)$. It's a group by Proposition 8.

Theorem 11. *Let \mathcal{C} be a pointed model category. Then there is a functor $(H \circ \mathcal{C})^\circ \times H \circ \mathcal{C} \rightarrow \mathcal{G}rp$, sending $(A, B) \mapsto [A, B]_1$, where $[A, B]_1$ is determined up to canonical isomorphism by $[A, B]_1 = \pi_1(A, B)$ if A is cofibrant and B is fibrant. Furthermore, there are two functors $\Sigma, \Omega : H \circ \mathcal{C} \rightarrow H \circ \mathcal{C}$ (they are called the suspension functor and the loop functor respectively) and canonical isomorphisms*

$$[\Sigma A, B] \simeq [A, B]_1 \simeq [A, \Omega B]$$

of functors $(H \circ \mathcal{C})^\circ \times (H \circ \mathcal{C}) \rightarrow \mathcal{S}ets$ where $[X, Y] = \text{Hom}_{H \circ \mathcal{C}}(X, Y)$.

Proof. Let A be cofibrant. Choose a cylinder object $A \times I$ and let $A \times I \xrightarrow{\pi} \Sigma A$ be the cofibre of $\partial_0 + \partial_1 : A \sqcup A \rightarrow A \times I$, then ΣA is cofibrant. We shall define a bijection

$$\rho : \pi(\Sigma A, B) \xrightarrow{\sim} \pi_1(A, B)$$

which is a natural transformation of functors to (sets) as B runs over \mathcal{C}_f (Here, $\pi(X, Y) := \text{Hom}_{\mathcal{C}}(X, Y) / \sim$). Let $\varphi : \Sigma A \rightarrow B$ be a map and let $\rho(\varphi)$ be the element of $\pi_1(A, B)$ represented by $\varphi\pi : A \times I \rightarrow B$. If $\varphi, \varphi' \in \text{Hom}(\Sigma A, B)$ and $\varphi \sim \varphi'$, then there is a right homotopy $h : \Sigma A \rightarrow B^I$ from φ to φ' . Let $H : A \times I \rightarrow B^I$ be a correspondence of $\varphi'\pi$ with some right homotopy k from 0 to 0 and consider the diagram

$$\begin{array}{ccc} & \xrightarrow{0\sigma} & \\ k \left[\begin{array}{c} H \\ \varphi'\pi \end{array} \right] & & s0 \\ & \xrightarrow{\varphi\pi} & \end{array}$$

This shows that $\varphi\pi$ corresponds to $s0 \cdot k$ and $\varphi'\pi$ corresponds to k , as $s0 \cdot k$ and k represents the same element of $\pi_1(A, B)$ so do $\varphi\pi$ and $\varphi'\pi$ and hence $\rho(\varphi) = \rho(\varphi')$. This shows that ρ is well-defined. ρ is surjective by the definition of ΣA . Finally, if $\rho(\varphi) = \rho(\varphi')$, then there is a left homotopy $H : A \times J \rightarrow B$ from $\varphi\pi$ to $\varphi'\pi$. Let $H' : A \times J \rightarrow B$ be given by $H'j_0 = H'j_1 = \varphi\pi$ and let K be the dotted arrow in the diagram

$$\begin{array}{ccc} A \times I & \xrightarrow{s\varphi\pi} & B^I \\ j_0 \downarrow & \nearrow K & \downarrow (d_0, d_1) \\ A \times J & \xrightarrow{(H, H')} & B \times B \end{array}$$

(j_0 was shown to be a trivial cofibration in the proof of Lemma 5.) Then $Kj_1 : A \times I \rightarrow B^I$ is a right homotopy from $\varphi\pi$ to $\varphi'\pi$ s.t. $Kj_1(\partial_0 + \partial_1) = 0$ and so induces a right homotopy $\Sigma A \rightarrow B^I$ from φ to φ' . This shows ρ is injective.

Dually if we choose a path object B^I and let ΩB be the fibre of $(d_0, d_1) : B^I \rightarrow B \times B$, then ΩB is fibrant and there is a bijection

$$\pi(A, \Omega B) \xrightarrow{\sim} \pi_1(A, B)$$

which is a natural transformation of functors as A runs over \mathcal{C}_C .

For general A, B , use the cofibrant replacement functor and the fibrant replacement functor, then we can extend the functor we've obtained (from $(H \circ \mathcal{C}_c)^\circ \times H \circ \mathcal{C}_f$ to $\mathcal{G}rp$) to a functor $(A, B) \mapsto [A, B]_1$ from $(H \circ \mathcal{C})^\circ \times H \circ \mathcal{C}$ to $\mathcal{G}rp$ unique up to canonical isomorphism (note that it doesn't have to be unique), and the bifunctor $[\cdot, \cdot]_1$ is representable in the first and second variables. \square

Remarks 12. Actually, here we kind of abuse the notations of writing Σ for both the functors on $H \circ \mathcal{C}$ and writing ΣA for the cofibre of $A \sqcup A \rightarrow A \times I$ when $A \in \mathcal{C}_c$. Actually, the former one is a left derived functor. So if we should encounter a situation where this abuse of notations would lead to confusion, we shall denote the former one by $L\Sigma$. Similarly, $R\Omega$ will be used for the loop functor on $H \circ \mathcal{C}$ if necessary.

Now we proceed to develop an extra structure on $H \circ \mathcal{C}$, namely the long exact sequences for fibrations and cofibrations. From now on, \mathcal{C} denotes a fixed pointed model category.

Notation 13. If $\alpha : X \rightarrow Y$ is a monomorphism in a category and $\beta : Z \rightarrow Y$ is a map, then by $\alpha^{-1}\beta$ we mean the unique map $\gamma : Z \rightarrow X$ with $\alpha\gamma = \beta$, if such a map exists.

Let $p : E \rightarrow B$ be a fibration where B is fibrant and let $i : F \rightarrow E$ be the inclusion of the fibre of p into E , then F and E are both fibrant. Let $B \xrightarrow{s^B} B^I \xrightarrow{(d_0^B, d_1^B)} B \times B$ be a factorization of Δ_B into a weak equivalence followed by a fibration. We shall construct an object E^I which is nicely related to B^I .

Let $E \times_B B^I$ (resp. $B^I \times_B E$) denote the fibre product of $p : E \rightarrow B$ and $d_0^B : B^I \rightarrow B$ (resp. $d_1^B : B^I \rightarrow B$), and let the fibre product sign $\times_B B^I$ to the left (resp. $d_1^B : B^I \rightarrow B$ to the right) of B^I denote fibre products with d_0^B (resp. d_1^B) in what follows.

Let $E \xrightarrow{s^E} E^I \xrightarrow{(d_0^E, p^I, d_1^E)} E \times_B B^I \times_B E$ be a factorization of $(1_E, s^B p, 1_E)$ into a weak equivalence followed by a fibration. The notation E^I, s^E , etc. is justified because s^E is a weak equivalence and (d_0^E, d_1^E) is a fibration since it is the composition of (d_0^E, p^I, d_1^E) and $(pr_1, pr_3) : E \times_B B^I \times_B E \rightarrow E \times E$, which is the base extension of (d_0^B, d_1^B) by $p \times p$. A

similar argument shows that $(d_0^E, p^I) : E^I \rightarrow E \times_B B^I$ ($E \times_B B^I \times_B E \rightarrow E \times_B B^I$ is the base extension of $p : E \rightarrow B$ by $d_1^B pr_2$) and (p^I, d_1^E) are fibrations.

The map $pr_1 : E \times_B B^I \rightarrow E$ is the base extension of d_0^B by p and hence is a trivial fibration. Hence the fibration $(d_0^E, p^I) : E^I \rightarrow E \times_B B^I$ is trivial since $1_E = pr_1(d_0^E, p^I)s^E$.

Lemma 14. *The diagram*

$$(14.1) \quad \begin{array}{ccc} F \times_E E^I \times_E F & \xrightarrow{pr_2} & E^I \\ \pi \downarrow & & \downarrow (d_0^E, p^I) \\ F \times \Omega B & \xrightarrow{i \times j} & E \times_B B^I \end{array}$$

is cartesian where $\pi = (pr_1, j^{-1}p^I pr_2)$ and where $j : \Omega B \hookrightarrow B^I$ is the fibre of (d_0^B, d_1^B) .

Proof. (1) Claim: $pd_0^E = d_0^B p^I$, $pd_1^E = d_1^B p^I$.

The first equation can be shown by the commutative diagram:

$$\begin{array}{ccc} E^I & & \\ \downarrow (d_0^E, p^I) & \searrow d_0^E & \\ E \times_B B^I & \xrightarrow{p_1} & E \\ \downarrow p_2 & \lrcorner & \downarrow p \\ B^I & \xrightarrow{d_0^B} & B \end{array}$$

and the second part can be proved similarly.

(2) Show that $j^{-1}p^I pr_2$ is well-defined. In fact, by the commutative diagram:

$$\begin{array}{ccc} F \times_E E^I & \xrightarrow{pr_1} & F \\ pr_2 \downarrow & \lrcorner & \downarrow i \\ E^I & \xrightarrow{d_0^E} & E \\ \downarrow p^I & & \downarrow p \\ B^I & \xrightarrow{d_0^B} & B \end{array}$$

we have that $d_0^B p^I pr_2 = p i pr_1 = 0$. Similarly, $d_1^B p^I pr_2 = 0$. So the dotted arrow exists:

$$\begin{array}{ccc} & \Omega B & \longrightarrow & * \\ & \downarrow j & \lrcorner & \downarrow \\ E^I & \xrightarrow{p^I pr_2} & B^I & \xrightarrow{(d_0^B, d_1^B)} & B \times B \end{array}$$

(3) By the diagram above, we have $(d_0^E, p^I)pr_2 = (d_0^E pr_2, p^I pr_2) = (i pr_1, j j^{-1} p^I pr_2) = (i \times j)\pi$.

(4) We have the following diagram which concludes:

$$\begin{array}{ccc}
X & & \\
\downarrow \lambda & \searrow \varphi & \\
F \times_E E^I \times_E F & \xrightarrow{pr_2} & E^I \\
(\psi_1, \psi_2) \downarrow \pi & & \downarrow (d_0^E, p^I) \\
F \times \Omega B & \xrightarrow{i \times j} & E \times_B B^I
\end{array}$$

where $\lambda = (\psi_1, \varphi, i^{-1}d_1^E \varphi)$. Here $i^{-1}d_1^E \varphi$ can be defined since $pd_1^E \varphi = d_1^B p^I \varphi = d_1^B j \psi_2 = 0$ (The last equation can be deduced from the following diagram).

$$\begin{array}{ccc}
\Omega B & \longrightarrow & * \\
\downarrow j & & \downarrow \\
X \xrightarrow{\psi_2} B^I & \xrightarrow{(d_0^B, d_1^B)} & B \times B
\end{array}$$

□

By this lemma, we can see that π is a trivial fibration, and thus we can obtain in $H \circ \mathcal{C}$ a map

$$m : F \times \Omega B \rightarrow F$$

given by the coposition $F \times \Omega B \xrightarrow{\pi^{-1}} F \times_E E^I \times_E F \xrightarrow{pr_3} F$.

In fact, m may be defined in another way.

Proposition 15. *Let A be cofibrant and let the map $m_* : [A, F] \times [A, \Omega B] \rightarrow [A, F]$ be denoted by $(\alpha, \lambda) \mapsto \alpha \cdot \lambda$. If $\alpha \in [A, F]$ is represented by $u : A \rightarrow F$, if $\lambda \in [A, \Omega B] = [A, B]_1$ is represented by $h : A \times I \rightarrow B$ with $h(\partial_0 + \partial_1) = 0$, and if h' is a dotted arrow in the diagram*

$$\begin{array}{ccc}
A & \xrightarrow{iu} & E \\
\downarrow \partial_0 & \nearrow h' & \downarrow p \\
A \times I & \xrightarrow{h} & B
\end{array}$$

then $\alpha \cdot \lambda$ is represented by $i^{-1}h'\partial_1 : A \rightarrow F$.

Proof. Let $H : A \times I \rightarrow B^I$ be a correspondence of h with $k : A \rightarrow B^I$. Let K be a lifting in

$$\begin{array}{ccc}
A & \xrightarrow{s^E h' \partial_1} & E^I \\
\downarrow \partial_1 & \nearrow K & \downarrow (d_0^E, p^I) \\
A \times I & \xrightarrow{(h', H)} & E \times_B B^I
\end{array}$$

Picture:

$$\begin{array}{ccccc}
0 & \xrightarrow{0\sigma} & 0 & & d_1^E K \partial_0 & \xrightarrow{d_1^E K} & h' \partial_1 \\
k \downarrow & & H & \left| \begin{array}{c} s^{B0} \\ \leftarrow p^I \\ K \partial_0 \end{array} \right. & & K & \left| \begin{array}{c} \\ \\ s^E h' \partial_1 \end{array} \right. \\
0 & \xrightarrow{h} & 0 & & iu & \xrightarrow{h'} & h' \partial_1
\end{array}$$

Now $K\partial_0 : A \rightarrow E^I$ induces a map $K\partial_0 : A \rightarrow F \times_E E^I \times_E E$ s.t. $\pi K\partial_0 = (u, j^{-1}k)$ (see 14.1) and hence by the definition of m we have that $\alpha \cdot \lambda$ is represented by $i^{-1}d_1^E K\partial_0 : A \rightarrow F$. But $i^{-1}d_1^E K : A \times I \rightarrow F$ is a homotopy from $i^{-1}d_1^E K\partial_0$ to $i^{-1}h'\partial_1$ and this proves the proposition. \square

Proposition 16. *The map m is independent of the choice of $p^I : E^I \rightarrow B^I$ and is a right action of the group object ΩB on F in $H \circ \mathcal{C}$.*

Proof. m is independent of p^I by Proposition 15 since the diagram there is independent of p^I . On the other hand, let $\alpha, \lambda, u, h, h'$ be as in Proposition 15, let $\lambda_1 \in [A, B]_1$ be represented by $h_1 : A \times I \rightarrow B$ and let h'_1 be a dotted arrow in the first diagram:

$$\begin{array}{ccc} A & \xrightarrow{h'\partial_1} & E \\ \partial_0 \downarrow & \nearrow h'_1 & \downarrow p \\ A \times I & \xrightarrow{h} & B \end{array} \quad \begin{array}{ccc} A & \xrightarrow{iu} & E \\ \partial'_0 \downarrow & \nearrow h' \cdot h'_1 & \downarrow p \\ A \times I' & \xrightarrow{h \cdot h_1} & B \end{array}$$

s.t. $i^{-1}h'_1\partial_1$ represents $(\alpha \cdot \lambda) \cdot \lambda_1$ by Proposition 15. As the composite homotopy $h \cdot h_1$ represents $\lambda \cdot \lambda_1$, the second diagram and Proposition 15 show that $i^{-1}(h' \cdot h'_1)\partial'_1$ represents $\alpha \cdot (\lambda \cdot \lambda_1)$. But $(h' \cdot h'_1)\partial'_1 = h'_1\partial_1$, hence $(\alpha \cdot \lambda) \cdot \lambda_1 = \alpha \cdot (\lambda \cdot \lambda_1)$ and m is an action as claimed. \square

Definition 17. By a fibration sequence in $H \circ \mathcal{C}$, we mean a diagram in $H \circ \mathcal{C}$ of the form

$$X \times \Omega Z \rightarrow X \rightarrow Y \rightarrow Z$$

which for some fibration $p : E \rightarrow B$ in \mathcal{C}_f is isomorphic to the diagram

$$(17.1) \quad F \times \Omega B \xrightarrow{m} F \xrightarrow{i} E \xrightarrow{p} B$$

constructed before.

Proposition 18. *If 17.1 is a fibration sequence, so is*

$$(18.1) \quad \Omega B \times \Omega E \xrightarrow{n} \Omega B \xrightarrow{\partial} F \xrightarrow{i} E$$

where ∂ is the composition $\Omega B \xrightarrow{(0, id)} F \times \Omega B \xrightarrow{m} F$ and where $n_* : [A, \Omega B] \times [A, \Omega E] \rightarrow [A, \Omega B]$ is given by $(\lambda, \mu) \mapsto ((\Omega p)_* \mu)^{-1} \cdot \lambda$.

Proof. We may assume that (17.1) is the sequence constructed above from a fibration p . Let $p^I : E^I \rightarrow B^I$ be as in the definition of m . Then $pr_1 : E \times_B B^I \times_B * \rightarrow E$ is the base extension of (d_0^B, d_1^B) by $(p, 0) : E \rightarrow B \times B$ and hence is a fibration; so we get a fibration sequence

$$(18.2) \quad \Omega B \times \Omega E \xrightarrow{n} \Omega B \xrightarrow{(0, j, 0)} E \times_B B^I \times_B * \xrightarrow{pr_1} E.$$

We calculate n by Proposition 15. Let $\lambda \in [A, \Omega B]$ be represented by $u : A \rightarrow \Omega B$, let $\mu \in [A, \Omega E]$ be represented by $h : A \times I \rightarrow E$ and let $(h, H, 0)$ be a lifting in

$$\begin{array}{ccc} A & \xrightarrow{(0, ju, 0)} & E \times_B B^I \times_B * \\ \partial_0 \downarrow & \nearrow (h, H, 0) & \downarrow pr_1 \\ A \times I & \xrightarrow{h} & E \end{array} \quad \begin{array}{ccc} & \xrightarrow{0\sigma} & \\ ju \downarrow & \square & \downarrow H\partial_1 \\ & \xrightarrow{ph} & \end{array}$$

where $H : A \times I \rightarrow B^I$ is pictured at the right. By Proposition 15, $j^{-1}H\partial_1$ represents $n_*(\lambda, \mu)$ in $[A, \Omega B]$. Letting $H' : A \times I \rightarrow B^I$ be a correspondence of $H\partial_1$ with $h' : A \times I \rightarrow B$, we obtain the correspondence

$$\begin{array}{ccc} & \xrightarrow{0\sigma} & \xrightarrow{0\sigma} \\ ju \left[\begin{array}{c|c} H & H' \\ \hline ph & h' \end{array} \right] & & s^{B0} \end{array}$$

of ju with $ph \cdot h'$, which shows that $\lambda = (\Omega p)_* \mu \cdot n_*(\lambda, \mu)$ or $n_*(\lambda, \mu) = [(\Omega p)_* \mu]^{-1} \cdot \lambda$. Thus the map n in (18.2) is the same as that in (18.1).

The map $F \xrightarrow{(i,0,0)} E \times_B B^I \times_B *$ is a weak equivalence since it may be factored $F \xrightarrow{(s^E i, 1_F)} E^I \times_E F \simeq E^I \times_B * \rightarrow E \times_B B^I \times_B *$ where the last map is a trivial fibration (base extension of $E^I \xrightarrow{(d_0^E, p^I)} E \times_B B^I$) and where the first map is a section of the trivial fibration $E^I \times_E F \xrightarrow{pr_2} F$ (base extension of d_1^E .) We shall show that the diagram in $H \circ \mathcal{C}$

$$(18.3) \quad \begin{array}{ccc} & \Omega B & \\ \partial \swarrow & & \searrow (0,j,0) \\ F & \xrightarrow{(i,0,0)} & E \times_B B^I \times_B * \end{array}$$

commutes. Let $\lambda \in [A, \Omega B]$ be represented by $k : A \rightarrow B^I$ and let $H : A \times I \rightarrow B^I$ be a correspondence of k with h . Then $\partial_* \alpha = 0 \cdot \alpha$ is represented by $i^{-1}h'\partial_1 : A \rightarrow F$ where h' is the dotted arrow in

$$\begin{array}{ccc} A & \xrightarrow{0} & E \\ \partial_0 \downarrow & \nearrow h' & \downarrow p \\ A \times I & \xrightarrow{h} & B \end{array}$$

So $(i, 0, 0)_* \partial_* \lambda$ is represented by $A \xrightarrow{(h'\partial_1, 0, 0)} E \times_B B^I \times_B *$, $(0, j, 0)_* \lambda$ is represented by $A \xrightarrow{(0, k, 0)} E \times_B B^I \times_B *$, and $(h', H, 0) : A \times I \rightarrow E \times_B B^I \times_B *$ is a left homotopy between these maps, showing that the triangle (18.3) commutes in $H \circ \mathcal{C}$. As $pr_1 \circ (i, 0, 0) = i$, we see that $1_{\Omega B}$, $(i, 0, 0)$, and 1_E give an isomorphism of (18.3) with the fibration sequence (18.2), and so by definition (18.1) is a fibration sequence. \square

Proposition 19. *Let 17.1 be a fibration sequence in $H \circ \mathcal{C}$, let $\partial : \Omega B \rightarrow F$ be defined as in Proposition 18 and let A be any object of $H \circ \mathcal{C}$. Then the sequence*

$$\begin{aligned} \dots &\rightarrow [A, \Omega^{q+1} B] \xrightarrow{(\Omega^q \partial)_*} [A, \Omega^q F] \xrightarrow{(\Omega^q i)_*} [A, \Omega^q E] \xrightarrow{(\Omega^q p)_*} \dots \\ &\rightarrow [A, \Omega E] \xrightarrow{(\Omega p)_*} [A, \Omega B] \xrightarrow{\partial_*} [A, F] \xrightarrow{i_*} [A, E] \xrightarrow{p_*} [A, B] \end{aligned}$$

is exact in the following sense:

- (1) $(p_*)^{-1}(0) = \text{Im}(i_*)$
- (2) $i_* \partial_* = 0$ and $i_* \alpha_1 = i_* \alpha_2 \iff \alpha_2 = \alpha_1 \cdot \lambda$ for some $\lambda \in [A, \Omega B]$
- (3) $\partial_*(\Omega i)_* = 0$ and $\partial_* \lambda_1 = \partial_* \lambda_2 \iff \lambda_2 = (\Omega p)_* \mu \cdot \lambda_1$ for some $\mu \in [A, \Omega E]$

- (4) The sequence of group homomorphisms from $[A, \Omega E]$ to the left is exact in the usual sense.

Proof. We may assume (17.1) is the sequence constructed from the fibration p .

- (1) Clearly $pi = 0$. If $p_*\alpha = 0$ represent α by $u : A \rightarrow E$, let $h : A \times I \rightarrow B$ be s.t. $h\partial_0 = pu, h\partial_1 = 0$. Since ∂_0 is a trivial cofibration, we can find a $k : A \times I \rightarrow E$ s.t. $\partial_0 k = u, pk = h$. Then if β is represented by $i^{-1}k\partial_1$, we have $i_*\beta = \alpha$.
- (2) With the notation of Proposition 15, we have that h' is a homotopy from iu which represents $i_*\alpha$ to $h'\partial_1$ which represents $i_*(\alpha \cdot \lambda)$. Hence $i_*(\alpha \cdot \lambda) = i_*\alpha$ and in particular $i_*\partial_*\lambda = i_*(0 \cdot \lambda) = i_*0 = 0$, so $i_*\partial_* = 0$. Conversely, given α_1, α_2 with $i_*\alpha_1 = i_*\alpha_2$, represent α_i by u_i and let $h : A \times I \rightarrow E$ be s.t. $h\partial_0 = iu_1, h\partial_1 = iu_2$ whence if λ is the class of ph , $\alpha_1 \cdot \lambda = \alpha_2$ by Proposition 15.
- (3) This follows from (2) and Proposition 18.
- (4) This can be shown by repeated use of Proposition 18.

□

Proposition 20. *The class of fibration sequences in $H \circ \mathcal{C}$ has the following properties:*

- (1) Any map $f : X \rightarrow Y$ may be embedded in a fibration sequence $F \times \Omega Y \rightarrow F \rightarrow X \xrightarrow{f} Y$.
- (2) Given a diagram of solid arrows

$$\begin{array}{ccccccc}
 F \times \Omega B & \xrightarrow{m} & F & \xrightarrow{i} & E & \xrightarrow{p} & B \\
 \downarrow \gamma \times \Omega \alpha & & \downarrow \gamma & & \downarrow \beta & & \downarrow \alpha \\
 F' \times \Omega B' & \xrightarrow{m'} & F' & \xrightarrow{i'} & E' & \xrightarrow{p'} & B'
 \end{array}$$

where the rows are fibration sequences, the dotted arrow γ exists.

- (3) In any diagram above where the rows are fibration sequences, if α and β are isomorphisms, so is γ .
- (4) Proposition 18.

Proof. (1) Any map in $H \circ \mathcal{C}$ is isomorphic to a fibration of objects in \mathcal{C}_{cf} .

(3) If A is any object in $H \circ \mathcal{C}$, then Proposition 19 gives a diagram

$$\begin{array}{ccccccccc}
 [A, \Omega E] & \longrightarrow & [A, \Omega B] & \longrightarrow & [A, F] & \longrightarrow & [A, E] & \longrightarrow & [A, B] \\
 \downarrow \wr & & \downarrow \wr & & \downarrow \gamma_* & & \downarrow \wr & & \downarrow \wr \\
 [A, \Omega E'] & \longrightarrow & [A, \Omega B'] & \longrightarrow & [A, F'] & \longrightarrow & [A, E'] & \longrightarrow & [A, B']
 \end{array}$$

where the rows are exact in the sense of Proposition 19. However, this is enough to conclude by the usual 5-lemma argument that $\gamma_* : [A, F] \rightarrow [A, F']$ is a bijection for all A and hence that γ is an isomorphism.

(2) We may suppose by replacing the diagram by an isomorphic diagram if necessary that the rows are constructed in the standard way from fibrations p and p' in \mathcal{C}_f . Let $\tilde{B} \xrightarrow{u} B$ be a trivial fibration with \tilde{B} cofibrant and let $\tilde{E} \xrightarrow{v} E \times_B \tilde{B}$ be a trivial fibration with \tilde{E} cofibrant, then $pr_1 : E \times_B \tilde{B} \rightarrow E$ is a trivial fibration

and $pr_2 : E \times_B \tilde{B} \rightarrow \tilde{B}$ is a fibration. So we obtain a diagram

$$\begin{array}{ccccc} \tilde{F} & \xrightarrow{\tilde{i}} & \tilde{E} & \xrightarrow{pr_2 v} & \tilde{B} \\ \downarrow \epsilon & & \downarrow pr_1 v & & \downarrow u \\ F & \xrightarrow{i} & E & \xrightarrow{p} & B \end{array}$$

in \mathcal{C} , where $pr_1 v$ and u are weak equivalences. It follows easily from the calculation given in Proposition 15 that

$$\begin{array}{ccc} \tilde{F} \times \Omega \tilde{B} & \xrightarrow{\tilde{m}} & \tilde{F} \\ \downarrow & & \downarrow \\ F \times \Omega B & \xrightarrow{m} & F \end{array}$$

commutes. Hence by (3) the \sim sequence is isomorphic to first row of the original diagram and so we may suppose that the rows of the original diagram are not only constructed in the standard way from fibrations p and p' but that E and B are in \mathcal{C}_{cf} . Then α and β are represented by maps u and v in \mathcal{C} with $p'v \sim up$. As E is cofibrant, we may modify v s.t. $p'v = up$. Then we may take $\gamma : F \rightarrow F'$ in the original diagram to be the map in \mathcal{C} induced by v . By Proposition 15, both squares commute. □