ROOT SPACE DECOMPOSITION OF SEMISIMPLE LIE ALGEBRAS AND ABSTRACT ROOT SYSTEMS

FEDERICO BINDA   ZIYANG GAO

Abstract. The lecture divides in two parts. In the first part, our main subject is the root space decomposition of semisimple Lie algebras, a useful method to describe the representations of a Lie algebra. We will mainly focus on the case of $\mathbb{C}$. We will first characterize all irreducible representations of $sl(2, F)$ in terms of highest weight, then study the general root space decomposition. The notion of root system will be introduced here. The second part talks about the basic concept of a root system. We follow the axiomatic approach (as in Serre [3], Humphreys [2]). We introduce bases, the Weyl group and we explain its action on the set of bases (or, equivalently, on the Weyl chambers). Finally we introduce the classification theorem, using the Cartan matrix, Coxeter graphs and Dynkin diagrams.

1. Representation of $sl(2, F)$

In this section, we always assume that F is algebraically closed of character 0. Denote by $L = sl(2, F)$ with standard basis

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then $[h, x] = 2x, [h, y] = -2y, [x, y] = h$.

Now for any L-module V(i.e. a representation $sl(2, F) \to gl(V)$), h acts diagonally on V since it is semisimple (see [2, §7]). So if we let $V_\lambda = \{v \in V | h.v = \lambda v\}$, then $V = \oplus V_\lambda$. When $V_\lambda \neq 0$, we call λ a *weight* of h and V_λ a *weight space*. We call n =max$\{\lambda | V_\lambda \neq 0\}$ (if it makes sense, i.e. all λ's are real) the *highest weight*. Note the highest weight really exists since $\dim V < \infty$.

1.0.1. **Lemma.** *If $v \in V_\lambda$, then $x.v \in V_{\lambda+2}$ and $y.v \in V_{\lambda-2}$.*

1.0.2. **Theorem.** *For any non-negative integer n, there exists a unique(up to isomorphism) irreducible module of $sl(2, F)$ with the highest weight n, whose dimension is $n + 1$ and weights are $n, n - 2, ..., -n$. Thus all its weight spaces have dimension 1.*

Proof. (Sketch) Suppose $\alpha : sl(2, F) \to gl(V)$ is an irreducible module with highest weight n. Take $0 \neq v_0 \in V_n$, define inductively $v_{k+1} = \frac{1}{k+1} y.v_k$, and denote by $v_{-1} = 0$. Now we claim that

$$\begin{aligned} h.v_k &= (n - 2k)v_k, \\ x.v_k &= (n - k + 1)v_{k-1}, \\ y.v_k &= (k + 1)v_{k+1}. \end{aligned}$$

Then there exists k_0 s.t. $v_{k_0} \neq 0$ but $v_{k_0+1} = 0$. Now, by the second equation above, $k_0 = n$. So span$\{v_0, ..., v_n\}$ is an invariant space of V, hence V =span$\{v_0, ..., v_n\}$. Under this basis, the matrixes of $\alpha(h), \alpha(x), \alpha(y)$ are

$$\begin{pmatrix} n & & & & 0 \\ & n-2 & & & \\ & & n-4 & & \\ & & & \ddots & \\ 0 & & & & -n \end{pmatrix}, \begin{pmatrix} 0 & n & & & 0 \\ & 0 & n-1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & n & 0 \end{pmatrix}$$

respectively. This proves the uniqueness of the irreducible module with highest weight n.

Vice versa, use the three matrix above to define a module of $sl(2, F)$. It satisfies the conditions required. ☐

1.0.3. **Corollary.** *For a L-module V, in any decomposition of V into direct sum of irreducible submodules, the number of summands is precisely* $\dim V_0 + \dim V_1$.

2. Root Space Decomposition

In this section, L denotes a semisimple Lie algebra.

2.1. Maximal toral subalgebras and roots.

2.1.1. **Definition.** A *toral subalgebra* of L is a nonzero subalgebra of L consisting of semisimple elements. A *maximal toral subalgebra* H of L is a toral subalgebra not properly included in any other.

2.1.2. **Lemma.** *A toral subalgebra of L is always abelian.*

Proof. Let T be toral. We have to show that $\mathrm{ad}_T x = 0$ for all $x \in T$. Now since F is algebraically closed, $\mathrm{ad}_T x$ is diagonalizable. Hence it suffices to show that $\mathrm{ad}_T x$ has no nonzero eigenvalues.

Suppose, on the contrary, that $[x, y] = ay (a \neq 0)$ for some nonzero $y \in T$. Then $\mathrm{ad}_T y(x) = -ay$ and this is an eigenvector of $\mathrm{ad}_T y$ of eigenvalue 0. On the other hand, we can write x a a linear combination of eigenvectors of $\mathrm{ad}_T y$; after applying $\mathrm{ad}_T y$ to x, all that is left is a combination of eigenvetors which belong to nonzero eigenvalues, if any. Contradiction! ☐

First of all, since L is semisimple, there exists a toral subalgebra (see [2]). By this lemma, we can see that $\mathrm{ad}_L(H)$ is a commuting family of semisimple endomorphisms of L, thus there exists a basis of L s.t. all $\mathrm{ad}_L(H)$'s are diagonal matrices under this basis. Hence if we denote by $L_\alpha = \{x \in L | [h, x] = \alpha(h)x, \forall h \in H\}$, where $\alpha \in H^*$, we have $L = \oplus L_\alpha$. Now let $\Phi = \{0 \neq \alpha \in H^* | L_\alpha \neq 0\}$ be the set of all roots of L(relative to H), then we have a root space decomposition

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

2.1.3. **Remark.** Before proceeding, I would like to make a note about the relationship between Maximal Connected Abelian Lie subgroups of a Lie group and Maximal Toral Subalgebras of its Lie algebra. This makes sense due to Ado's Theorem. It is clear that they are 1-to-1 corresponded.

2.1.4. **Proposition.** *(1)* $\forall \alpha, \beta \in H^*, [L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.
(2) If $\alpha + \beta \neq 0$*, then* $L_\alpha \perp L_\beta$ *with respect to the Killing from* κ.
(3) $L_0 = C_L(H) = H$.

Proof. (1) Use Jacobi-identity.
(2) Find $h \in H$ s.t. $(\alpha+\beta)(h) \neq 0$. Then if $x \in L_\alpha, y \in L_\beta$, associativity of the killing-form allows us to write $\kappa([h,x],y) = -\kappa([x,h],y) = -\kappa(x,[h,y])$, so $(\alpha+\beta)(h)\kappa(x,y) = 0$, which forces $\kappa(x,y) = 0$.
(3) Use the note above. □

2.1.5. **Corollary.** *The restriction of* κ *to* H *is nondegenerate.*

Proof. Note that $H = L_0$, κ nondegenerate and L_0 is orthogonal to all L_α for $\alpha \in \Phi$. □

2.2. **Connection with Representation.** In this subsection, we denote by t_α the dual element of $\alpha \in H^*$ in H. More explicitely, since $\kappa|_H$ is nondegenerate, every $\varphi \in H^*$ corresponds to a unique element $t_\varphi \in H$ satisfying $\varphi(h) = \kappa(t_\varphi, h)$ for all $h \in H$.

2.2.1. **Proposition.** *The following statements hold:*
(1) Φ *spans* H^*.
(2) *If* $\alpha \in \Phi$*, then* $-\alpha \in \Phi$.
(3) $\kappa(t_\alpha, t_\alpha) \neq 0, \forall \alpha \in \Phi$*. So* $h_\alpha := \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$ *is well-defined.*
(4) *If* $\alpha \in \Phi$ *and* x_α *is any nonzero element of* L_α*, then there exists* $y_\alpha \in L_{-\alpha}$ *s.t.* $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ *span a three dimensional simple subalgebra* $\mathfrak{L}_\alpha$ *of* L *isomorphic to* $sl(2,F)$ *via* $x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proof. (1) If not, then by duality there exists a nonzero $h \in H$ s.t. $\alpha(h) = 0$ for all $\alpha \in \Phi$. But this means that $[h, L_\alpha] = 0$ for all $\alpha \in \Phi$. Since $[h,H] = 0$, this in turn forces $[h,L] = 0$, or $h \in Z(L) = 0$. Contradiction.
(2) Just note that $\kappa(L_\alpha, L_\beta) = 0$ if $\alpha + \beta \neq 0$.
(3) First note that for $x \in L_\alpha$ and $y \in L_{-\alpha}$, we have $[x,y] = \kappa(x,y)t_\alpha$. Now for a nonzero $x \in L_\alpha$, we can find a $y \in L_{-\alpha}$ s.t. $\kappa(x,y) \neq 0$. Modifying one by a scalar, we may assume that $\kappa(x,y) = 1$, so $[x,y] = t_\alpha$. Suppose $\alpha(t_\alpha) = 0$, then $[t_\alpha, x] = [t_\alpha, y] = 0$. It follows that the subspace S of L spanned by x, y, t_α is a 3-dimensional solvable algebra, $S \cong \mathrm{ad}_L S \subset gl(L)$. In particular, $\mathrm{ad}_L s$ is nilpotent for all $s \in [S,S]$, so $\mathrm{ad}_L t_\alpha$ is both semisimple and nilpotent, i.e., $\mathrm{ad}_L t_\alpha = 0$. This says that $t_\alpha \in Z(L) = 0$, contradiction.
(4) Find a $y_\alpha \in L_{-\alpha}$ satisfying the property. The rest is computation.

□

2.3. **Summary of Properties of Φ.** In this section, we denote by $\langle \beta, \alpha \rangle = \frac{2\kappa(\beta,\alpha)}{\kappa(\alpha,\alpha)}$ if $\alpha, \beta \in \Phi$.

2.3.1. **Theorem.** Φ *satisfies the following four properties:*
i) $span(\Phi) = H^*$, $0 \notin \Phi$ *and* Φ *is finite;*
ii) For any $\alpha \in \Phi$ *and* $c \in \mathbb{R}$*, we have* $c\alpha \in \Phi \iff c = \pm 1$;
iii) If $\alpha, \beta \in \Phi$*, then* $\beta - \langle \beta, \alpha \rangle \alpha \in \Phi$;
iv) If $\alpha, \beta \in \Phi$*, then* $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Proof. i) Proven. ii) ($\Leftarrow$) Already seen.

($\Rightarrow$) Consider $\mathfrak{L}_\alpha$ as in (4) of the lemma above. It's not hard to see that $V = F \cdot h_\alpha \oplus \bigoplus_{c \in \mathbb{R}\setminus\{0\}} L_{c\alpha}$ is the invariant space of $\mathfrak{L}_\alpha$ under the adjoint representation. Consider the sub-representation of $\mathfrak{L}_\alpha$ on V and the action of h_α on V. If $c\alpha \in \Phi$, then for $x \in L_{c\alpha}$, we have $[h_\alpha, x] = 2cx$, making $2c$ a weight corresponding to the weight space $L_{c\alpha}$. Note that all nonzero weights of V are of this form, and the weight space of 0 is just $F \cdot h_\alpha$.

Note that $\mathfrak{L}_\alpha$ is an invariant subspace of V. Since the representation of $sl(2, F)$ is completely reducible, there exists an invariant subspace V' of V s.t. $V = \mathfrak{L}_\alpha \oplus V'$. Since 0 is not a weight of V', V' has no even weights. Hence the only even weights of V are ± 2 and 0. So $V = \mathfrak{L}_\alpha$. Note that this furthermore implies that $\dim L_\alpha = 1$.

iii)& iv) It's OK when $\beta = \pm\alpha$. For the other cases, β and α are linearly independent. We can easily see that $\bigoplus_{k\in\mathbb{Z}} L_{\beta+k\alpha}$ is invariant under the adjoint representation of $\mathfrak{L}_\alpha$ on L. Let's consider the sub-representation of $\mathfrak{L}_\alpha$ on this subspace. Sice for every $x \in L_{\beta+k\alpha}$ we have $[h_\alpha, x] = (\langle\beta,\alpha\rangle + 2k)x$, so $\beta + k\alpha \in \Phi \iff \langle\beta,\alpha\rangle + 2k$ is a weight. In the case $k = 0$, $\beta \in \Phi$, so $\langle\beta,\alpha\rangle$ is a weight, which implies that $\langle\beta,\alpha\rangle \in \mathbb{Z}$. Note that this also indicates $-\langle\beta,\alpha\rangle$ is a weight, which corresponds to the case $k = -\langle\beta,\alpha\rangle$. Hence, $\beta - \langle\beta,\alpha\rangle\alpha \in \Phi$. $\square$

2.3.2. **Corollary** (of the process of proof)**.** *$\forall\alpha \in \Phi$, $\dim L_\alpha = 1$.*

3. Abstract root systems

3.0.3. **Definition.** A subset Φ of an euclidean space $\mathcal{E}$ (i.e. a finite dimensional vector space over $\mathbb{R}$ endowed with a positive definite symmetric bilinear form (α,β)) is called a *root system* in $\mathcal{E}$ if the following axioms are satisfied:

(R1) Φ is finite and spans $\mathcal{E}$, $0 \notin \Phi$.
(R2) if $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.
(R3) if $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.
(R4) if $\alpha, \beta \in \Phi$, $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} =: \langle\beta,\alpha\rangle \in \mathbb{Z}$.

Axiom (R4) in the definition above restricts the possible angles occurring between pairs of roots rather strictly. Recall that the cosine of the angle ϑ between α and $\beta \in \mathcal{E}$ is given by the formula $\cos\vartheta = \frac{(\alpha,\beta)}{\|\alpha\|\|\beta\|}$. Therefore, using the definition, $\langle\beta,\alpha\rangle = 2\frac{\|\beta\|}{\|\alpha\|}\cos\vartheta$ and $4\cos^2\vartheta = \langle\beta,\alpha\rangle\langle\alpha,\beta\rangle \in \mathbb{Z}_{\geq 0}$. Since $0 \leq \cos^2\vartheta \leq 1$ and since $\langle\beta,\alpha\rangle$ has the same sign of $\langle\alpha,\beta\rangle$, the only possible values for $\langle\alpha,\beta\rangle$ are $0, \pm1, \pm2, \pm3$. As a consequence, we have the following simple result:

3.0.4. **Lemma.** *Let α, β be non-proportional roots in Φ. If $(\alpha,\beta) > 0$ (i.e. if the angle between the vectors α and β is strictly acute), then $\alpha - \beta$ is a root. If $(\alpha,\beta) < 0$, then $\alpha + \beta$ is a root.*

3.0.5. **Definition.** The *Weyl group* $\mathcal{W}$ of a root system Φ consists of the subgroup of $GL(\mathcal{E})$ generated by all the reflections σ_α for $\alpha \in \Phi$.

For a given α, the reflection σ_α fixes the hyperplane $P_\alpha = \{x \in \mathcal{E} \mid (\beta,\alpha) = 0\}$ normal to α and maps $\alpha \mapsto -\alpha$. It's easy to see that an explicit formula for σ_α is given by $\sigma_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha$. By axiom (R3), $\mathcal{W}$ permutes the (finite) set Φ. This allow us to identify the Weyl group with a subgroup of the symmetric group on Φ. In particular, $\mathcal{W}$ is finite.

3.1. **Bases and Weyl chambers.** From property (R1) above, it is clear that any root system also contains a basis for $\mathcal{E}$. We can refine this concept with a natural notion of a base for the root system

3.1.1. **Definition.** A subset Δ of Φ is called a *base* for Φ if

(B1) Δ is a basis of E.
(B2) Each root $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ with integral coefficients k_α that are either all nonpositive or all nonnegative.

The roots in a base are called *simple*. Clearly the expression for β in (B2) is unique. This allow us to define the *height* of a root β (relative to a fixed basis Δ) to be $\mathrm{ht}(\beta) = \sum_{\alpha\in\Delta} k_\alpha$. We say that a root is *positive* (resp. *negative*) if all k_α are ≥ 0 (resp. ≤ 0). In order to distinguish the two cases, we may write $\beta \succ 0$ or $\beta \prec 0$ respectively. The collection of positive and negative roots, relative to Δ, is denoted by Φ^+ and Φ^- (notice that $-\Phi^+ = \Phi^-$).

It is not clear from the definition that a base actually exists for a root system Φ. Indeed, all possible bases are related in a precise way and there is a concrete method for constructing them. For each $\gamma \in \mathcal{E}$, define $\Phi^+(\gamma)$ as the set of roots lying on the positive side of the hyperplane orthogonal to γ, i.e. $\Phi^+(\gamma) = \{\alpha \in \Phi \,|\, (\gamma, \alpha) > 0\}$. Similarly we define $\Phi^-(\gamma) = \{\alpha \in \Phi \,|\, (\gamma, \alpha) < 0\}$. We call $\gamma \in \mathcal{E}$ *regular* if $\gamma \in \mathcal{E} \setminus \bigcup_{\alpha\in\Phi} P_\alpha$ and *singular* otherwise. When γ is regular we clearly have $\Phi = \Phi^+(\gamma) \cup \Phi^-(\gamma)$. Finally, we call $\alpha \in \Phi^+(\gamma)$ *decomposable* if $\alpha = \beta_1 + \beta_2$ for some $\beta_i \in \Phi^+(\gamma)$ and *indecomposable* otherwise. Now we can give the following statement:

3.1.2. **Theorem.** *Let $\gamma \in \mathcal{E}$ be regular. Then the set $\Delta(\gamma)$ of all indecomposable roots in $\Phi^+(\gamma)$ is a base for Φ and every base is obtained in this manner.*

The hyperplanes P_α ($\alpha \in \Phi$) define a partition of $\mathcal{E}$ into finitely many regions. The connected components of $\mathcal{E} \setminus \bigcup_{\alpha\in\Phi} P_\alpha$ are called the *Weyl chambers* of $\mathcal{E}$. For a given base Δ of $\mathcal{E}$, the unique Weyl chamber containing all vectors $\gamma \in \mathcal{E}$ which satisfies the inequalities $(\gamma, \alpha) > 0$ ($\alpha \in \Delta$) is called the *fundamental Weyl chamber* relative to Δ. Each regular $\gamma \in \mathcal{E}$ belongs to precisely one Weyl chamber, denote $\mathfrak{C}(\gamma)$. If $\mathfrak{C}(\gamma) = \mathfrak{C}(\gamma')$, then γ and γ' lie on the same side of each hyperplane P_α ($\alpha \in \Phi$), which is equivalent to require that $\Phi^+(\gamma) = \Phi^+(\gamma')$ or $\Delta(\gamma) = \Delta(\gamma')$. Hence we have just proved the following:

3.1.3. **Lemma.** *The set of Weyl chambers is in natural 1-1 correspondence with the set of bases of Φ.*

3.2. **The Weyl group.** The Weyl group sends one Weyl chamber onto another: if $\sigma \in \mathcal{W}$ and $\gamma \in \mathcal{E}$ is regular, then $\sigma(\mathfrak{C}(\gamma)) = \mathfrak{C}(\sigma(\gamma))$. It is easy to show that $\mathcal{W}$ permutes the bases: since $\sigma \in \mathcal{W}$ is both invertible and orthogonal (i.e. preserves the inner product on $\mathcal{E}$), σ sends Δ to $\sigma(\Delta)$, which is again a base. This two actions are in fact compatible with the above correspondence between the Weyl chambers and bases. We will prove that $\mathcal{W}$ permutes the bases of Φ (or, equivalently, the Weyl chambers) in a simply transitive way. We will first prove the statement for $\mathcal{W}'$, the subgroup of $\mathcal{W}$ generated by the reflections σ_α for α in a base Δ.

3.2.1. **Theorem.** *Given Δ and Δ' bases for a root system Φ, we have $\Delta' = \sigma(\Delta)$ for some $\sigma \in \mathcal{W}'$.*

Proof. First, recall that a base Δ for a given root system Φ is uniquely determined by its fundamental Weyl chamber: we may represent it by selecting a vector γ which is regular. Furthermore, it is thus sufficient to prove that $\mathcal{W}'$ acts transitively on Weyl chambers, i.e. for any base Δ, there exists $\sigma \in \mathcal{W}'$ with $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$. This actually says that we can "move" the Weyl chamber represented by γ onto the fundamental Weyl chamber of the base Δ.

Define $\delta = \frac{1}{2}\sum_{\alpha \succ 0} \alpha$. Then, choosing $\sigma \in \mathcal{W}'$ such that $(\sigma(\gamma), \delta)$ is as large as possible, we have $(\sigma(\gamma), \delta) \geq (\sigma_\alpha \sigma(\gamma), \delta)$ for all $\alpha \in \Delta$ (since of course $\sigma_\alpha \sigma \in \mathcal{W}'$ for $\alpha \in \Delta$). We now need the following lemma:

3.2.2. **Lemma.** *Let α be a simple root. Then σ_α permutes the positive roots other than α.*

Using this result, we have that $\sigma_\alpha(\delta) = \delta - \alpha$. Now, by the linearity of the symmetric form $(\ ,\ )$ (together with the fact that $(\sigma(\gamma), \sigma(\alpha)) = (\gamma, \alpha)$), we get:

$$(\sigma(\gamma), \delta) \geq (\sigma_\alpha \sigma(\gamma), \delta) = (\sigma(\gamma), \sigma_\alpha(\delta)) = (\sigma(\gamma), \delta - \alpha) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha),$$

forcing $(\sigma(\gamma), \alpha) \geq 0$. However, since γ is regular, we cannot have $(\sigma(\gamma), \alpha) = 0$ for any α, because then γ would be orthogonal to $\sigma^{-1}\alpha$. So all inequalities are strict and therefore $\sigma(\gamma)$ lies in the fundamental Weyl chamber $\mathfrak{C}(\Delta)$ as desired. $\square$

To see that the Weyl group itself acts transitively on bases, it remains only to show that $\mathcal{W}$ is indeed generated by a set of simple rotations, i.e. $\mathcal{W} = \mathcal{W}'$. To prove this statement we need two lemmas.

3.2.3. **Lemma.** *For all $\alpha \in \Phi$, there exists $\sigma \in \mathcal{W}'$ such that $\sigma(\alpha) \in \Delta$.*

Proof. Since $\mathcal{W}'$ acts transitively on bases (by the previous theorem), it suffices to prove that each root α belongs to some base Δ'. Since the only multiple of α that appear in Φ are $\pm\alpha$, the hyperplanes P_β ($\beta \neq \pm\alpha$) are all distinct from P_α, i.e. the hyperplane fixed by $\sigma_{\pm\alpha}$. So there exists $\gamma \in P_\alpha$, $\gamma \notin P_\beta$ for $\beta \neq \pm\alpha$ (such a γ must exist: argue by contradiction). Taking γ' close enough to γ such that $(\gamma', \alpha) = \varepsilon > 0$ while $|(\gamma', \beta)| > \varepsilon$ for all $\beta \neq \pm\alpha$, we have that $\alpha \in \Phi^+(\gamma')$ cannot be decomposable, otherwise we would have β_1, β_2 such that $(\gamma', \alpha) = (\gamma', \beta_1) + (\gamma', \beta_2)$, leading to a contradiction. Then α must belong to the base $\Delta(\gamma')$. $\square$

3.2.4. **Lemma.** *The set of "simple reflections", i.e. σ_α for $\alpha \in \Delta$, generates $\mathcal{W}$.*

Proof. To prove $\mathcal{W} = \mathcal{W}'$, it is enough to show that each reflection σ_α (for $\alpha \in \Phi$ any root) is in $\mathcal{W}'$. Using the previous lemma, we have that there exists $\sigma \in \mathcal{W}'$ such that $\beta := \sigma(\alpha) \in \Delta$. Then $\sigma_\beta = \sigma_{\sigma(\alpha)} = \sigma\sigma_\alpha\sigma^{-1}$, so that $\sigma_\alpha = \sigma^{-1}\sigma_\beta\sigma \in \mathcal{W}'$. $\square$

This completes the proof that $\mathcal{W}$ acts transitively on all possible bases of Φ.

When $\sigma \in \mathcal{W}$ is written as $\sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ with $\alpha_i \in \Delta$ and t minimal, we call the expression *reduced* of *length* $\ell(\sigma) = t$ (relative to a fixed base Δ). By definition, $\ell(1) = 0$. If we define the number $n(\sigma)$ to be the number of positive roots α for which $\sigma(\alpha)$ is negative, we can characterize the length in another way:

3.2.5. **Lemma.** *For all $\sigma \in \mathcal{W}$, $\ell(\sigma) = n(\sigma)$.*

3.3. Irreducible root systems.

3.3.1. **Definition.** A root system Φ is called *irreducible* if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other.

Suppose Δ is a base of Φ. We claim that Φ is irreducible if and only if Δ cannot be partitioned in the same way (i.e. into two proper subsets such that each root in one set is orthogonal to each root in the other). For the "if" part, suppose that $\Phi = \Phi_1 \cup \Phi_2$ with $(\Phi_1, \Phi_2) = 0$. Unless Δ is completely contained in Φ_1 or in Φ_2, this induces a similar partition of Δ. But $\Delta \subset \Phi_1$ implies $(\Delta, \Phi_2) = 0$ and so $(\mathcal{E}, \Phi_2) = 0$ (since Δ spans $\mathcal{E}$), i.e. $\Phi_2 = 0$. For the "only if" part, let Φ be irreducible but $\Delta = \Delta_1 \cup \Delta_2$, with $(\Delta_1, \Delta_2) = 0$. Each root is conjugate to a simple root (by lemma 3.2.3), so we can write $\Phi = \Phi_1 \cup \Phi_2$, where Φ_i is the set of roots having a conjugate in Δ_i. It is easy to see that if $(\alpha, \beta) = 0$ (i.e. $\alpha \in P_\beta$ or, equivalently, $\beta \in P_\alpha$), then the two rotations σ_α and σ_β commute, i.e. $\sigma_\alpha \sigma_\beta = \sigma_\beta \sigma_\alpha$. Since $\mathcal{W}$ is generated by σ_α for $\alpha \in \Delta$, we can use the explicit formula for computing simple reflections to show that each root in Φ_i is obtained from one in Δ_i by adding or subtracting (multiples of) elements of Δ_i. In other words, Φ_i lies in the subspace $\mathcal{E}_i$ of $\mathcal{E}$ spanned by Δ_i and so, by the linearity of the inner product, we see that $(\Phi_1, \Phi_2) = 0$. This forces $\Phi_1 = \emptyset$ or $\Phi_2 = \emptyset$, so that $\Delta_1 = \emptyset$ or $\Delta_2 = \emptyset$.

4. Classification

In this section, Φ denotes a root system of rank $\ell = \dim_{\mathbb{R}} \mathcal{E}$, $\mathcal{W}$ its Weyl group and Δ a fixed base of Φ.

4.1. **The Cartan matrix.** Fix an ordering $(\alpha_1, \dots, \alpha_\ell)$ of the simple roots. Then we may define the matrix $(\langle \alpha_i, \alpha_j \rangle) = C_{ij}$. This is call the *Cartan matrix* of Φ. In general, the Cartan matrix is not symmetric; however, it has several immediately observable features. For example, the elements of the diagonal are all 2, and all the off-diagonal entries are integers of absolute value ≤ 3. When $\ell \leq 2$ we can describe all possible root systems Φ by simply drawing a picture. If $\ell = 1$ we have only one possibility, labelled A_1.

For $\ell = 2$ the situation is more complicated. However, as we mentioned in section 1, there is just a limited number of possibilities for possible angles occurring between pairs of roots (and this fact limits severely the possible root systems in rank 2). Indeed, when $\alpha \neq \pm\beta$ and $\|\beta\| \geq \|\alpha\|$, we have:

Table 1.

$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$	$\langle \boldsymbol{\beta}, \boldsymbol{\alpha} \rangle$	$\boldsymbol{\vartheta}$	$\|\boldsymbol{\beta}\|^2 / \|\boldsymbol{\alpha}\|^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

If α and β are both simple roots and $\langle\alpha,\beta\rangle$ and $\langle\beta,\alpha\rangle$ are strictly positive (i.e., the angle between α and β is acute), then (from the table) one of them, say $\langle\beta,\alpha\rangle$, is equal to 1. But then we have

$$\sigma_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha = \beta - \alpha$$

so that $\pm(\beta-\alpha)$ are roots. One of them, say $\alpha-\beta$, must be positive. But then $\alpha = (\alpha-\beta)+\beta$, contradicting the simplicity of α. We conclude that $\langle\beta,\alpha\rangle$ and $\langle\alpha,\beta\rangle$ are both negative. From this it follows that there are actually exactly four (non isomorphic) possibilities in rank 2.

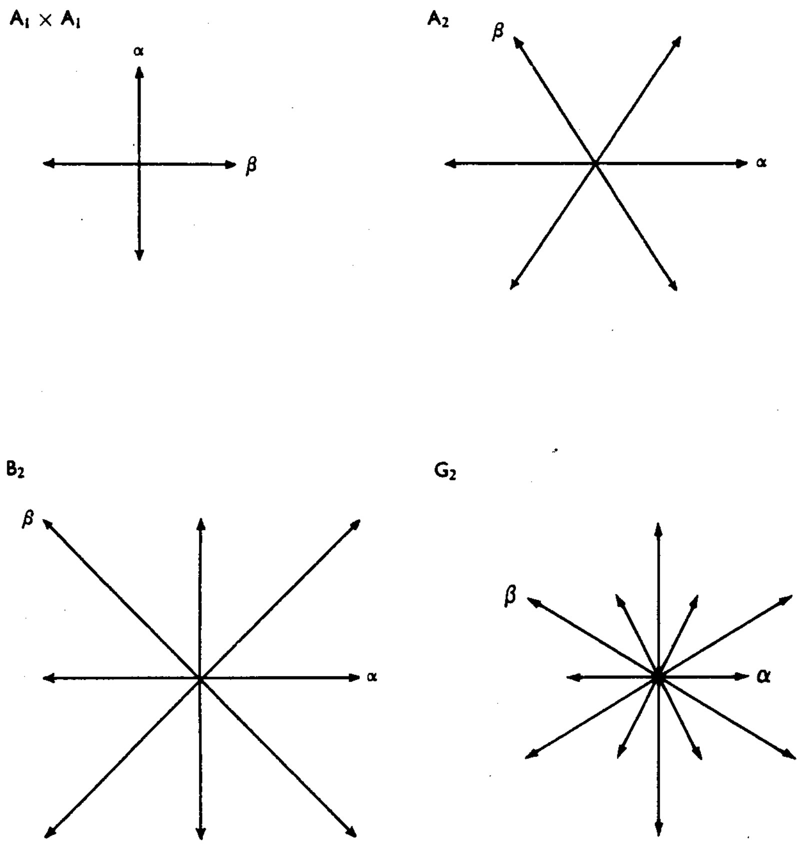


FIGURE 1. Root systems in rank $\ell = 2$

It worth noticing that A_2, B_2 and G_2 are irreducible, while $A_1\times A_1$ is not. For the systems of rank 2, the possible Cartan matrices are the following:

$$A_1\times A_1 \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; A_2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}; B_2 \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}; G_2 \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

The matrix clearly depends on the chosen ordering, but this turns out not be a serious problem. The important point is that the Cartan matrix is independent on the choice of the base Δ, thanks to the fact that $\mathcal{W}$ acts transitively on the collection of bases: if Δ' is another base of Φ and if $\sigma \in \mathcal{W}$ is such that $\sigma(\Delta) = \Delta'$, we can use again the explicit formula for a reflection (and the linearity of the inner product) to show that the integers $\langle \alpha_i, \alpha_j \rangle$ and $\langle \sigma(\alpha_i), \sigma(\alpha_j) \rangle$ are equal. We can actually say more: it turns out that the Cartan matrix characterize Φ completely.

4.1.1. **Proposition.** *Let $\Phi' \subset \mathcal{E}'$ be a root system for an Euclidean space $\mathcal{E}'$ with base $\Delta' = (\alpha_1^{'}, \ldots, \alpha_\ell^{'})$. If $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_i^{'}, \alpha_j^{'} \rangle$ for $1 \leq i, j \leq \ell$, then the bijection $\alpha_i \mapsto \alpha_i^{'}$ extends in a unique way to an isomorphism $\varphi \colon \mathcal{E} \to \mathcal{E}'$ mapping Φ to Φ' and satisfying $\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$. Therefore the Cartan matrix of Φ determines Φ up to isomorphism.*

Proof. Since Δ is a basis of $\mathcal{E}$ as real vector space, (and, similarly, Δ' is a basis of $\mathcal{E}'$ as real vector space) there is a unique isomorphism of vector space that satisfies $\alpha_i \mapsto \alpha_i^{'}$. Call it φ. Now we use the assumption $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_i^{'}, \alpha_j^{'} \rangle$ for $1 \leq i, j \leq \ell$ and the formula for simple reflections. If $\alpha, \beta \in \Delta$, we have:

$$\sigma_{\varphi(\alpha)}(\varphi(\beta)) = \sigma_{\alpha'}(\beta') = \beta' - \langle \beta^{'}, \alpha^{'} \rangle \alpha^{'} = \varphi(\beta) - \langle \beta, \alpha \rangle \varphi(\alpha) = \varphi(\beta - \langle \beta, \alpha \rangle \alpha) = \varphi(\sigma_\alpha(\beta)).$$

In other words, we have the following commutative diagram (for all $\alpha \in \Delta$)

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\varphi"] \arrow[d, "\sigma_\alpha"'] & \mathcal{E}' \arrow[d, "\sigma_{\alpha'}"] \\ \mathcal{E} \arrow[r, "\varphi"] & \mathcal{E}'. \end{tikzcd}$$

As the Weyl groups $\mathcal{W}$ and $\mathcal{W}'$ are generated by simple reflections (i.e. by reflections σ_α for $\alpha \in \Delta$), this commutativity gives rise to an isomorphism between the two groups, given by the map $\sigma \mapsto \varphi \circ \sigma \circ \varphi^{-1}$. Then, as for all $\beta \in \Phi$ there exists $\sigma \in \mathcal{W}$ such that $\alpha := \sigma(\beta) \in \Delta$, we have $\varphi(\beta) = (\varphi \circ \sigma \circ \varphi^{-1})(\varphi(\alpha)) \in \Phi'$. Then φ maps all $\beta \in \Phi$ to Φ'. Finally, using again the formula for the reflections, we see that φ preserves all Cartan integers. $\square$

Hence, the proposition shows that is possible (in principle) to recover Φ from the knowledge of the Cartan matrix.

4.2. **Coxeter graphs and Dynkin diagrams.** As mentioned previously, root systems provide a (relatively) simple way of classifying semisimple Lie algebras: a non-trivial result (that we are not going to prove, see [2, Chap. IV]) states that two semisimple Lie algebras having the same root system — that is, the set of roots of L relative to a maximal toral subalgebra H of L (which turns out to be a root system in our sense up to an extension of the base field) — are isomorphic. On the other hand, the root systems may themselves be classified by means of particular diagrams, called *Dinkyn diagrams*. Each such diagram belongs to one of finitely many families of graphs.

The first step in defining such diagrams are *Coxeter graphs*. If α and β are positive distinct roots, we know that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$. We define the *Coxeter graph* of Φ to be a graph having ℓ vertices. Each vertex corresponds to a root $\alpha_i \in \Delta$. Two vertex, corresponding to α and β respectively ($\alpha \neq \beta$), are connected by $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ edges (so that no edge exists between the vertices α_i and α_j if $C_{ij} = 0$).

Irreducible root systems play a fundamental role in the classification process that we will describe below. In particular we will use the following result about the lengths:

4.2.1. **Lemma.** *Let Φ be an irreducible root system. Then at most two root lengths occur in Φ and all roots of a given length are conjugate under the action of $\mathcal{W}$.*

In case Φ is irreducible with 2 distinct root lengths, we will speak about *short* and *long* roots (referring to the corresponding lengths). If all roots are of equal length, we conventionally call all of them long.

If all roots have equal length, i.e. $\|\alpha_i\| = \|\alpha_j\|$ for all i, j, of course we have $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle = -1$ (recall that $\langle \alpha, \beta \rangle < 0$ and $\langle \beta, \alpha \rangle < 0$ if α and β are simple roots). In this case, the Coxeter graph determines the Cartan integers. However, this is not always the case. For example, if $\ell = 2$, we have the following possibilities: If more than one root

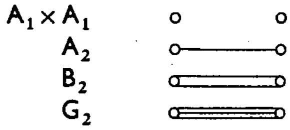


FIGURE 2. Coxeter graphs in rank $\ell = 2$

length occurs, the graph fails to tell us which of a pair of vertices should correspond to a short (simple) root and which to a long (when these vertices are connected by two or three edges). This is the situation of B_2 or G_2.

When this occurs, we put an arrowhead on the lines joining the vertices pointing towards the shorter root. The resulting diagram is called the *Dynkin diagram* of the root system. For example we have:

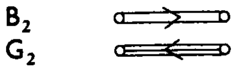
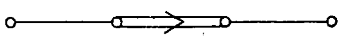


FIGURE 3. Dynkin diagrams B_2 and G_2

To see that the Cartan matrix can be recovered from the Dynkin diagram, consider another example:



It's easy to see that the associated Cartan matrix is:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

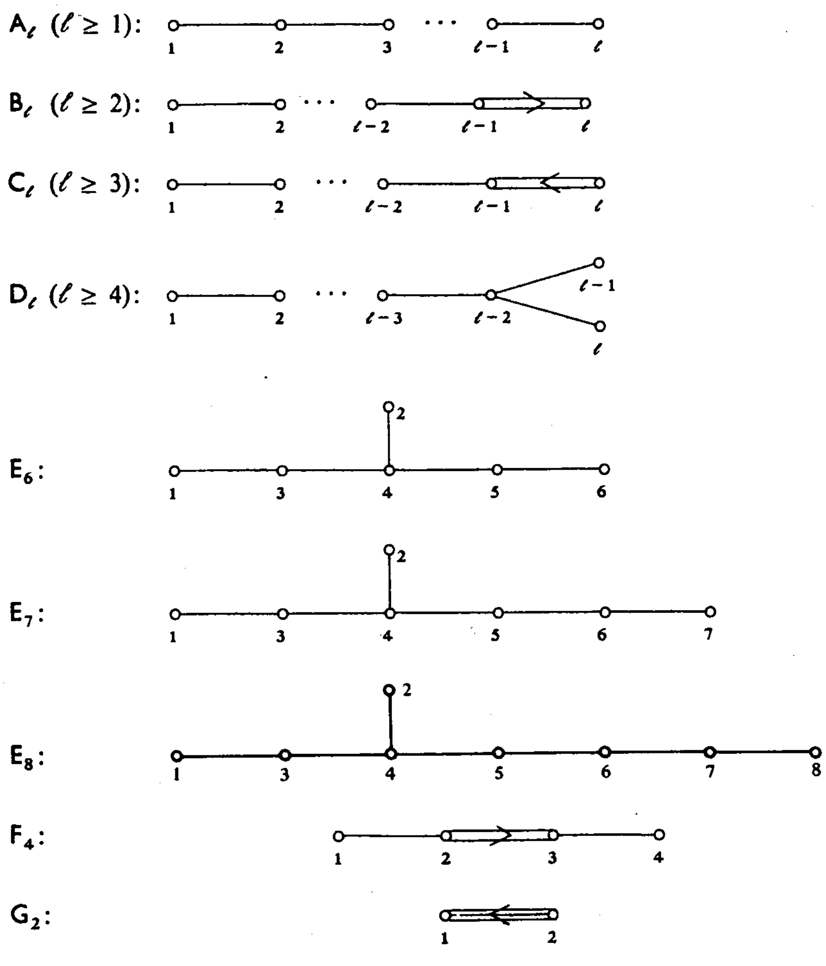
Recall that a root system is called irreducible if it cannot be written as the union of two proper orthogonal subsets. We proved that this condition is equivalent to the same condition written for a base Δ. It is clear, by definition, that a root system is irreducible if and only if its Coxeter graph is connected. In general (since a root system is always a finite set), there will be a (finite) number of connected components of the Coxeter graph, corresponding to a partition of $\Delta = \Delta_1 \cup \ldots \cup \Delta_t$ and to a partition of $\mathcal{E}$ into

$\mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_t$, where $\mathcal{E}_i$ is the space span of Δ_i (the sum is clearly direct, since all subspaces are orthogonal). It is then easy to show that each root lies in one of the $\mathcal{E}_i$, i.e. we have a partition of $\Phi = \Phi_1 \cup \ldots \cup \Phi_t$. Thus we have proved the following statement:

4.2.2. **Proposition.** *Each root system Φ decomposes (uniquely, up to isomorphism) as the union of irreducible root systems Φ_i, each of them is a root system of a subspace $\mathcal{E}_i$ of $\mathcal{E}$ such that $\mathcal{E} = \mathcal{E}_1 \oplus \ldots \mathcal{E}_t$.*

The consequence of the previous proposition is that the classification of irreducible root system is equivalent to the classification of connected Dynkin diagrams. In particular, we have:

4.2.3. **Theorem.** *If Φ is an irreducible root system of rank ℓ, its Dynkin diagram is one of the following (ℓ vertices in each case):*



4.3. **Notes.** The discussion in this paper follows [2], chap. II& III, §7, 8, 9, 10 and 11. All the proofs that are not given here can be found, for example, there. The classical reference for the subject is the book of Serre [3]. Milne's course notes [1] follow, more or less, the same approach of [2], with more emphasis on the connection with algebraic groups.

References

[1] J. S. Milne (2011), *Algebraic Groups, Lie Groups, and their Arithmetic Subgroups*, available online at www.jmilne.org/math/.
[2] J. Humphreys (1972), *Introduction to Lie Algebras and Representation Theory*, GTM, Springer-Verlag, NY.
[3] J. P. Serre (1966), *Algébres de Lie semi-simples complexes*, W.A. Benjamin, NY.