STABILITY CONDITIONS ON KUZNETSOV COMPONENTS

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ABSTRACT. We introduce a general method to induce Bridgeland stability conditions on semiorthogonal decompositions. In particular, we prove the existence of Bridgeland stability conditions on the Kuznetsov component of the derived category of many Fano threefolds (including all but one deformation type of Picard rank one), and of cubic fourfolds. As an application, in the appendix, written jointly with Xiaolei Zhao, we give a variant of the proof of the Torelli theorem for cubic fourfolds by Huybrechts and Rennemo.

CONTENTS

1. INTRODUCTION

Main results. Let X be a smooth Fano variety and let $\mathrm{D}^{\mathrm{b}}(X)$ denote its bounded derived category of coherent sheaves. Let $E_1, \dots, E_m \in \mathrm{D}^{\mathrm{b}}(X)$ be an exceptional collection in $\mathrm{D}^{\mathrm{b}}(X)$. We call its right

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2010 *Mathematics Subject Classification.* 14C34, 14D20, 14F05, 18E30.

Key words and phrases. Bridgeland stability, cubic fourfolds, Fano threefolds, Torelli theorem, semiorthogonal decomposition.

A. B. is supported by the ERC grant WallXBirGeom, no. 337039. M. L. is partially supported by the grant MTM2015-65361-P MINECO/FEDER, UE and the grant number 230986 of the Research Council of Norway. E. M. is partially supported by the NSF grant DMS-1523496 and he is holding a Poincaré Chair from the Institut Henri Poincaré and the Clay Mathematics Institute. P. S. is partially supported by the grants FIRB 2012 "Moduli Spaces and Their Applications" and PRIN 2015 "Geometria delle varietà proiettive".

orthogonal complement

$$\begin{aligned}\mathcal{K}u(X) &= \langle E_1, \dots, E_m \rangle^{\perp} \\ &= \left\{ C \in \mathrm{D}^{\mathrm{b}}(X) \,:\, \mathrm{Hom}(E_i, C[p]) = 0,\ \forall i = 1, \dots, m,\ \forall p \in \mathbb{Z} \right\}\end{aligned}$$

a *Kuznetsov component* of X. In a series of papers, Kuznetsov has shown that much of the geometry of Fano varieties, and their moduli spaces, can be captured efficiently by $\mathcal{K}u(X)$, for appropriate exceptional collections.

On the other hand, stability conditions on triangulated categories as introduced by Bridgeland in [Bri07] and wall-crossing have turned out to be an extremely powerful tool for the study of moduli spaces of stable sheaves. We connect these two developments with the following two results:

Theorem 1.1. *Let X be a Fano threefold of Picard rank 1. Assume that X is not the intersection $X_{2,3}$ of a quadric and a cubic in $\mathbb{P}^5$. Then the Kuznetsov semiorthogonal component $\mathcal{K}u(X)$ has a Bridgeland stability condition.*

The most interesting cases of Theorem 1.1 are Fano threefolds of index two, and those of index one and even genus. We refer to Section 6 for an overview of the classification of Fano threefolds of Picard rank one, and the exceptional collections appearing implicitly in Theorem 1.1. Our results also apply to a number of Fano threefolds of higher Picard rank.

Theorem 1.2. *Let X be a cubic fourfold. Then $\mathcal{K}u(X)$ has a Bridgeland stability condition.*

Here $\mathcal{K}u(X)$ is defined by the semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle,$$

where H is a hyperplane section. Here $\mathcal{K}u(X)$ is a K3 category (i.e., the double shift $[2]$ is a Serre functor); conjecturally [Kuz10, Conjecture 1.1] is the derived category of a K3 surface if and only if X is rational. Our results also give the first stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ when X does not contain a plane.

Background and motivation. The study of Kuznetsov components of derived categories of Fano varieties started with [BO95], and has seen a lot of recent interest, see e.g., [Kuz04, Kuz05, IKP14] for threefolds, and [Kuz10, AT14, Add16] for the cubic fourfold, as well as [Kuz09a, Kuz14, Kuz15] for surveys. The interest in their study comes from a variety of directions. They are part of Kuznetsov's powerful framework of Homological Projective Duality [Kuz07]. They often seem to encode the most interesting and geometric information about $\mathrm{D}^{\mathrm{b}}(X)$ and moduli spaces of sheaves on X; e.g., several recent constructions of hyperkähler varieties associated to moduli spaces of sheaves on the cubic fourfold are induced by the projection to the K3 category $\mathcal{K}u(X)$ [KM09] (where moduli spaces naturally come with a holomorphic symplectic structure, due to the fact that $\mathcal{K}u(X)$ is a K3 category). In the case of Fano threefolds, there are a number of unexpected equivalences (some conjectural) between Kuznetsov components of pairs of Fano threefolds of index one and two, see [Kuz09a] for the theory, and [KPS16] for an application to Hilbert schemes. In the case of cubic fourfolds, as mentioned above, they conjecturally determine rationality of X. Finally, they are naturally related to Torelli type questions: on the one hand, they still encode much of the cohomological information of X; on the other hand, one can hope to recover X directly from $\mathcal{K}u(X)$ (in some cases when equipped with some additional data); see [BMMS12] for such a result for cubic threefolds, and [HR16] for many hypersurfaces, including cubic fourfolds.

Perhaps the most natural way to recover geometry from $\mathcal{K}u(X)$ is to study moduli spaces of stable objects—which explains the interest in the question on the existence of stability conditions on $\mathcal{K}u(X)$. Indeed, this question was first raised for cubic threefolds in [Kuz04], and for cubic fourfolds by Addington and Thomas [AT14] and Huybrechts [Huy15], and in the generality of Theorems 1.1 and 1.2 by Kuznetsov in his lecture series [Kuz16].

Prior work. When X is a Fano threefold of Picard rank one, stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ have been constructed in [Li15]. However, in general these do not descend to stability conditions on the semiorthogonal component $\mathcal{K}u(X)$, and due to their importance for moduli spaces, a direct construction of stability conditions on $\mathcal{K}u(X)$ is of independent interest.

For Fano threefolds of index two, our Theorem 1.1 is referring to the decomposition $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(H)\rangle$. Their deformation type is determined by $d = H^3 \in \{1, 2, 3, 4, 5\}$. The result is straightforward from prior descriptions of $\mathcal{K}u(X)$ for $d \geq 4$, due to [BMMS12] for cubic threefolds ($d = 3$) and new for $d \in \{1, 2\}$. The most interesting cases of index one are those of even genus $g_X = \frac{1}{2}H^3+1$, for which Mukai [Muk92] constructed an exceptional rank two vector bundle $\mathcal{E}_2$ of slope $-\frac{1}{2}$; in these cases our Theorem refers to the semiorthogonal decomposition $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{K}u(X), \mathcal{E}_2, \mathcal{O}_X\rangle$. The result is straightforward from previous descriptions of $\mathcal{K}u(X)$ for $g_X \in \{10, 12\}$, due to [BMMS12] for $g_X = 8$, and new for $g_X = 6$.

For cubic fourfolds containing a plane, stability conditions on $\mathcal{K}u(X)$ were constructed in [MS12], and *Gepner point* stability conditions[1] were obtained in [Tod16]. In this case, $\mathcal{K}u(X)$ is equivalent to the derived category over a K3 surface with a Brauer twist.

Open question. Our methods are currently unable to handle the case of Fano threefolds of index one and genus two, i.e., the complete intersection $X_{2,3}$ of degree $(2, 3)$ in $\mathbb{P}^5$. We have learned from Chunyi Li that he has been able to prove a stronger Bogomolov-Gieseker inequality for slope-stable sheaves (and *tilt*-stable complexes) on $X_{2,3}$, which allows him to treat this case; see Remark 6.12.

Applications. Kuznetsov conjectured an equivalence between $\mathcal{K}u(Y_d)$ and $\mathcal{K}u(X_{4d+2})$ for appropriate pairs Y_d and X_{4d+2}, where Y_d is a Fano threefold of Picard rank one, index two and degree d, and X_{4d+2} is of index one and genus $2d + 2$ (degree $4d + 2$). Our results may be helpful in reproving known cases, and proving new cases of these equivalences, by identifying moduli spaces of stable objects in both categories. We illustrate this for $d = 4$, see Example 6.6.

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[1] These are stability conditions invariant, up to rescaling, under the functor (1) appearing in Theorem A.1.

For cubic fourfolds, we show in the appendix, written jointly with Xiaolei Zhao, that the existence of stability conditions on $\mathcal{K}u(X)$ is already enough to reprove a categorical Torelli theorem for very general cubic fourfolds (which is a special case of [HR16, Corollary 2.10]):

Theorem A.1. *Let X and Y be smooth cubic fourfolds. Assume that $H^*_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{Z})$ has no (-2)-classes. Then $X \cong Y$ if and only if there is an equivalence $\Phi\colon \mathcal{K}u(X) \to \mathcal{K}u(Y)$ whose induced map $H^*_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{Z}) \to H^*_{\mathrm{alg}}(\mathcal{K}u(Y),\mathbb{Z})$ commutes with the action of* (1).

Here (1) denotes the autoequivalence of $\mathcal{K}u(X)$ induced by $\_ \otimes \mathcal{O}_X(1)$ on $\mathrm{D}^{\mathrm{b}}(X)$; the numerical Grothendieck group of $\mathcal{K}u(X)$ is denoted by $H^*_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{Z})$.

The idea is quite simple: we show that the projection of ideal sheaves of lines on X to $\mathcal{K}u(X)$ are stable for *all* stability conditions on $\mathcal{K}u(X)$; therefore, the Fano variety of lines can be recovered from $\mathcal{K}u(X)$. An additional argument based on the compatibility with (1) shows that the polarization coming from the Plücker embedding is preserved. By a classical argument, this is enough to recover X.

We also show that, with the same arguments as in [HR16], Theorem A.1 is enough to reprove the classical Torelli theorem for cubic fourfolds.

Approach. We establish general methods for inducing t-structures and stability conditions on $\mathcal{K}u(X)$ from $\mathrm{D}^{\mathrm{b}}(X)$. The former, see Corollary 4.4, is a mild generalization of a well-known construction that first appeared in [VdB00]. The latter, Proposition 5.1, gives in addition the existence of Harder-Narasimhan filtrations and the support property on the subcategory $\mathcal{K}u(X)$ (and thus a stability condition) given an appropriate *weak stability condition* on $\mathrm{D}^{\mathrm{b}}(X)$.

The crucial assumption for both methods is that the relevant *heart* $\mathcal{A}$ in $\mathrm{D}^{\mathrm{b}}(X)$ contains the exceptional objects $E_1,\dots,E_m$, while its shift $\mathcal{A}[1]$ contains their Serre duals $S(E_1),\dots,S(E_m)$. For Fano threefolds, we show that in all our cases this can be achieved in two steps: first we start with ordinary slope-stability for coherent sheaves, and tilt to obtain a new heart $\mathrm{Coh}^{\beta}(X)$; we then use a weak stability condition on $\mathrm{Coh}^{\beta}(X)$, called *tilt-stability* in [BMT14], and tilt again to arrive at a situation where our general method applies; see Section 6.

In order to prove Theorem 1.2 with the same method, we would have to tilt a third time; this would require a conjectural Bogomolov-Gieseker type inequality [PT15] that is not currently known for any fourfold. Instead, we use the rational fibration in conics $X \dashrightarrow \mathbb{P}^3$ and Kuznetsov's theory of derived categories of quadric fibrations [Kuz08] to reinterpret $\mathcal{K}u(X)$ as a semiorthogonal component in the derived category $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3,\mathcal{B}_0)$ of modules over the associated sheaf of Clifford algebras on $\mathbb{P}^3$, see Section 7. After establishing a sharp Bogomolov-Gieseker type inequality for slope-stable modules over the Clifford algebra in Section 8, we are in position apply the same method as before in Section 9.

Further directions. It would be powerful to extend the fundamental results (due to Mukai, Huybrechts, O'Grady, Yoshioka, Toda) on moduli spaces of sheaves and stable objects on K3 surfaces to stable objects in $\mathcal{K}u(X)$ for the cubic fourfold.

In particular, the work in progress [BLM+17] uses deformation arguments to study non-emptiness of a moduli space $M_\sigma(\mathbf{v})$ of σ-semistable objects for every class $\mathbf{v}$ in the numerical Grothendieck group of $\mathcal{K}u(X)$ with Mukai self-pairing $\mathbf{v}^2 \geq -2$; in other words, the goal is to extend the deformation arguments of [GH96, O'G97, Yos01] from families of K3 surfaces to families of Kuznetsov components of cubic fourfolds, as well as prove the basic existence result for moduli spaces of stable complexes in [Tod08]. This would, for example, allow one to naturally reprove and complete a result by Addington and Thomas [AT14]: the Kuznetsov component $\mathcal{K}u(X)$ is equivalent to the derived category of a K3 surface if and only if the numerical Grothendieck group $H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ (equipped with a natural Mukai pairing) contains a hyperbolic plane U. In fact, the class $\mathbf{w} \in U$ of square zero would provide a K3 surface as the moduli space of stable objects $M_\sigma(\mathbf{w})$ for a generic stability conditions σ. This would make Kuznetsov's conjecture [Kuz10, Conjecture 1.1] equivalent to a folklore conjecture characterizing rationality of X in terms of its Hodge structure. It would also provide the full strength of the results of [BM14b] on the birational geometry of moduli spaces in $\mathcal{K}u(X)$.

It would also be interesting to extend a number of results previously obtained for (general) cubic hypersurfaces containing a plane to arbitrary cubic fourfolds; for example the embedding of X itself into an 8-dimensional hyperkähler variety constructed as a moduli space of stable objects in $\mathcal{K}u(X)$ [Ouc14], or the description of the Fano variety of lines as a moduli space of stable objects [MS12] (without the assumption on (-2)-classes appearing in the proof of Theorem A.1). Finally, the 8-dimensional hyperkähler variety associated to cubic fourfolds via the Hilbert scheme of twisted cubics [LLSVS13] can be described as a moduli space of tilt-stable and Bridgeland stable complexes in $\mathrm{D}^{\mathrm{b}}(X)$ [LLMS16], for X very general. It would be interesting to describe it also as a moduli space of stable objects in $\mathcal{K}u(X)$, for any X.

Acknowledgements. We are very grateful to Xiaolei Zhao for agreeing to make the results of the appendix part of this paper. We are also indebted to Marcello Bernardara for very valuable help in a long discussion on derived categories of quadric fibrations and modules over the Clifford algebra, and to Howard Nuer and Alex Perry for many suggestions and conversations. Moreover, the paper benefited from many useful discussions with Asher Auel, Agnieszka Bodzenta, François Charles, Igor Dolgachev, Frédéric Han, Daniel Huybrechts, Alexander Kuznetsov, Chunyi Li, Richard Thomas, and Yukinobu Toda.

The authors would like to acknowledge the following institutions, where parts of this paper have been written: Institut Henri Poincaré, Northeastern University, Université Paris Diderot, and Université Paul Sabatier.

We work over the complex numbers $\mathbb{C}$.

2. REVIEW ON TILT AND BRIDGELAND STABILITY

We begin with a quick review about weak and Bridgeland stability conditions. Let $\mathcal{D}$ be a triangulated category and let $K(\mathcal{D})$ denote the Grothendieck group of $\mathcal{D}$. Fix a finite rank lattice Λ and a surjective group homomorphism $v\colon K(\mathcal{D}) \twoheadrightarrow \Lambda$.

Weak stability conditions. A weak stability condition has two ingredients: a heart of a bounded t-structure and a weak stability function.

Definition 2.1 ([Bri07, Lemma 3.2])**.** A *heart of a bounded t-structure* is a full subcategory $\mathcal{A} \subset \mathcal{D}$ such that

(a) for $E, F \in \mathcal{A}$ and $n < 0$ we have $\mathrm{Hom}(E, F[n]) = 0$, and

(b) for every $E \in \mathcal{D}$ there exists a sequence of morphisms

$$0 = E_0 \xrightarrow{\phi_1} E_1 \to \ldots \xrightarrow{\phi_m} E_m = E$$

such that the cone of ϕ_i is of the form $A_i[k_i]$ for some sequence $k_1 > k_2 > \cdots > k_m$ of integers and objects $A_i \in \mathcal{A}$.

We write $H^{-k_i}_{\mathcal{A}}(E) = A_i$ for the cohomology objects of E with respect to the bounded t-structure.

Definition 2.2. Let $\mathcal{A}$ be an abelian category. We say that a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ is a *weak stability function* on $\mathcal{A}$ if, for $E \in \mathcal{A}$, we have $\Im Z(E) \geq 0$, with $\Im Z(E) = 0 \Rightarrow \Re Z(E) \leq 0$.

If moreover, for $E \neq 0$, $\Im Z(E) = 0 \Rightarrow \Re Z(E) < 0$, we say that Z is a *stability function* on $\mathcal{A}$.

Definition 2.3. A *weak stability condition* on $\mathcal{D}$ is a pair $\sigma = (\mathcal{A}, Z)$ consisting of the heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}$ and a group homomorphism $Z\colon \Lambda \to \mathbb{C}$ such that (a)–(c) below are satisfied:

(a) The composition $K(\mathcal{A}) = K(\mathcal{D}) \xrightarrow{v} \Lambda \xrightarrow{Z} \mathbb{C}$ is a weak stability function on $\mathcal{A}$. By abuse of notation, we will write $Z(E)$ instead of $Z \circ v([E])$ for any $E \in \mathcal{D}$.

The function Z allows one to define a *slope* for any $E \in \mathcal{A}$ by setting

$$\mu_\sigma(E) := \begin{cases} -\frac{\Re Z(E)}{\Im Z(E)} & \text{if } \Im Z(E) > 0 \\ +\infty & \text{otherwise} \end{cases}$$

and a notion of stability: An object $0 \neq E \in \mathcal{A}$ is σ*-semistable* if for every proper subobject F, we have $\mu_\sigma(F) \leq \mu_\sigma(E)$. We will often use the notation μ_Z as well.

(b) (HN-filtrations) We require any object A of $\mathcal{A}$ to have a Harder-Narasimhan filtration in σ-semistable ones.

(c) (Support property) There exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that $Q|_{\ker Z}$ is negative definite, and $Q(E) \geq 0$, for all σ-semistable objects $E \in \mathcal{A}$.

As usual, given a non-zero object $E \in \mathcal{A}$, we will denote by $\mu^+_\sigma(E)$ (resp. $\mu^-_\sigma(E)$) the biggest (resp. smallest) slope of a Harder-Narasimhan factor.

Remark 2.4. If in fact Z is a stability function on $\mathcal{A}$, then the pair σ defines a *Bridgeland stability condition*, see [Bri07, Proposition 5.3].

Remark 2.5. If Z has discrete image in $\mathbb{C}$, and if $\mathcal{A}$ is noetherian, then the existence of Harder-Narasimhan filtrations is automatic by [Bri07, Lemma 2.4] (see [BM11, Proposition B.2]).

Remark 2.6. If Λ has rank two, and if $Z\colon \Lambda \to \mathbb{C}$ is injective, then the support property is trivially satisfied for any non-negative quadratic form Q on $\Lambda \otimes \mathbb{R} \cong \mathbb{R}^2$.

Remark 2.7. For the purpose of Theorems 1.1 and 1.2, we will choose Λ to be the numerical K-group $K_{\mathrm{num}}(\mathcal{D})$ of $\mathcal{D}$: it is defined as the quotient of $K(\mathcal{D})$ by the kernel of the Euler characteristic pairing $\chi(E,F) = \sum_i (-1)^i \dim \operatorname{Ext}^i(E,F)$.

Example 2.8. To fix notation, we first recall slope-stability as a weak stability condition. Let X be an n-dimensional smooth projective variety and let H be an hyperplane section. For $j = 0, \dots, n$, consider the lattices $\Lambda_H^j \cong \mathbb{Z}^{j+1}$ generated by vectors of the form

$$\left(H^n \operatorname{ch}_0(E), H^{n-1} \operatorname{ch}_1(E), \dots, H^{n-j} \operatorname{ch}_j(E)\right) \in \mathbb{Q}^{j+1}$$

together with the natural map $v_H^j\colon K(X) \to \Lambda_H^j$.

Then the pair $(\mathcal{A} = \operatorname{Coh}(X), Z_H)$ with

$$Z_H(E) = \mathfrak{i}\, H^n \operatorname{ch}_0(E) - H^{n-1} \operatorname{ch}_1(E)$$

defines slope-stability as a weak stability condition with respect to Λ_H^1; here, by Remark 2.6, we can choose $Q = 0$. We write μ_H for the associated slope function.

Slope-semistable sheaves satisfy a further inequality, which will allow us in Proposition 2.11 to improve our positivity condition, by changing the bounded t-structure. More precisely, by the Bogomolov-Gieseker inequality, for any slope-semistable sheaf E, we have $\Delta_H(E) \geq 0$, where

$$\Delta_H(E) = \left(H^{n-1} \operatorname{ch}_1(E)\right)^2 - 2H^n \operatorname{ch}_0(E) H^{n-2} \operatorname{ch}_2(E). \tag{1}$$

Tilting. Assume that we are given a weak stability condition $\sigma = (\mathcal{A}, Z)$, and let $\mu \in \mathbb{R}$. We can form the following subcategories of $\mathcal{A}$ (where $\langle \dots \rangle$ denotes the extension closure):

$$\begin{aligned}
\mathcal{T}_\sigma^\mu &= \{E : \text{ All HN factors } F \text{ of } E \text{ have slope } \mu_\sigma(F) > \mu\} \\
&= \langle E\colon E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) > \mu \rangle, \\
\mathcal{F}_\sigma^\mu &= \{E : \text{ All HN factors } F \text{ of } E \text{ have slope } \mu_\sigma(F) \leq \mu\} \\
&= \langle E\colon E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) \leq \mu \rangle.
\end{aligned}$$

It follows from the existence of Harder-Narasimhan filtrations that $(\mathcal{T}_\sigma^\mu, \mathcal{F}_\sigma^\mu)$ forms a torsion pair in $\mathcal{A}$ in the sense of [HRS96]. In particular, we can obtain a new heart of a bounded t-structure by tilting:

Proposition and Definition 2.9 ([HRS96])**.** *Given a weak stability condition $\sigma = (Z, \mathcal{A})$ and a choice of slope $\mu \in \mathbb{R}$, there exists a heart of a bounded t-structure defined by*

$$\mathcal{A}_\sigma^\mu = \langle \mathcal{T}_\sigma^\mu, \mathcal{F}_\sigma^\mu[1] \rangle.$$

We will call $\mathcal{A}_\sigma^\mu$ the heart obtained by tilting $\mathcal{A}$ with respect to the stability condition σ at the slope μ.

Now return to the setting of slope stability as in Example 2.8, and choose a parameter $\beta \in \mathbb{R}$. Then we can apply Proposition 2.9 and obtain:

Definition 2.10. We write $\mathrm{Coh}_H^\beta(X) \subseteq \mathrm{D}^{\mathrm{b}}(X)$ for the heart of a bounded t-structure obtained by tilting $\mathrm{Coh}(X)$ with respect to slope-stability at the slope $\mu = \beta$.

In particular, $\mathrm{Coh}_H^\beta(X)$ contains slope-semistable sheaves F of slope $\mu(F) > \beta$, and shifts $F[1]$ of slope-semistable sheaves F of slope $\mu(F) \leq \beta$. In our setting, the polarization will often be unique, in which case we drop the subscript H from the notation.

For a coherent sheaf E on X, we define the vector

$$\mathrm{ch}^\beta(E) = e^{-\beta H}\,\mathrm{ch}(E) = \left(\mathrm{ch}_0^\beta(E), \mathrm{ch}_1^\beta(E), \dots, \mathrm{ch}_n^\beta(E)\right) \in H^*(Y, \mathbb{R}).$$

Proposition 2.11 ([BMT14, BMS16])**.** *Given $\alpha > 0, \beta \in \mathbb{R}$, the pair $\sigma_{\alpha,\beta} = (\mathrm{Coh}^\beta(X), Z_{\alpha,\beta})$ with $\mathrm{Coh}^\beta(X)$ as constructed above, and*

$$Z_{\alpha,\beta}(E) := \mathfrak{i}\, H^{n-1}\,\mathrm{ch}_1^\beta(E) + \frac{1}{2}\alpha^2 H^n\,\mathrm{ch}_0^\beta(E) - H^{n-2}\,\mathrm{ch}_2^\beta(E)$$

defines a weak stability condition on $\mathrm{D}^{\mathrm{b}}(X)$ with respect to Λ_H^2. The quadratic form Q can be given by the discriminant Δ_H as defined in (1)*.*

These stability conditions vary continuously as $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ varies.

In particular, this means that the family of stability conditions $\sigma_{\alpha,\beta}$ satisfies wall-crossing: for every fixed class $v \in \Lambda_H^2$, there is a locally finite wall-and-chamber structure on $\mathbb{R}_{>0} \times \mathbb{R}$ controlling stability of objects of class v.

Geometry of walls. It is very helpful to visualize the structure of this family of stability conditions, and the associated walls, via the cone associated to the quadratic form Δ_H. Consider $\mathbb{R}^3 = \Lambda_H^2 \otimes \mathbb{R}$ with coordinates $\left(H^n\,\mathrm{ch}_0, H^{n-1}\,\mathrm{ch}_1, H^{n-2}\,\mathrm{ch}_2\right)$, and with the quadratic form of signature (2, 1) induced by Δ_H. The map

$$(\alpha, \beta) \mapsto \mathrm{Ker}\, Z_{\alpha,\beta} \subset \mathbb{R}^3$$

assigns to each point in the upper half plane $\mathbb{R}_{>0} \times \mathbb{R}$ a line contained in the negative cone of Δ_H; this induces a homeomorphism between the upper half plane and the projectivization of the negative cone of Δ_H. The kernels $\mathrm{Ker}\, Z_{\alpha,\beta}$ with a fixed $\beta = \mu$ lie all in the same plane passing through $(0, 0, 0)$ and $(0, 0, 1)$. The quadric $\Delta_H(\_) = 0$ contains, of course, the vectors $v_H(L)$ for any line bundle L proportional to L, as well as $(0, 0, 1)$.

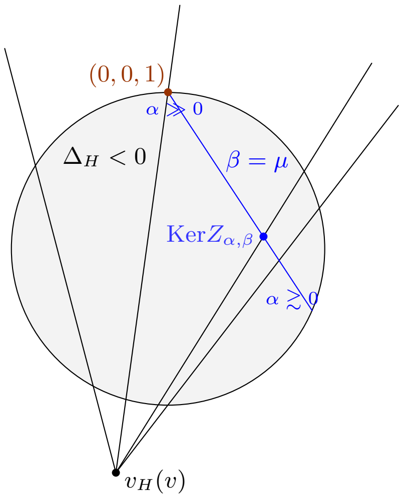

FIGURE 1. Walls in the cross-section of the negative cone

Now fix a Chern character v. Then the walls of tilt-stability correspond to hyperplanes $\mathcal{W}$ in $\mathbb{R}^3$ containing $v_H(v)$: a stability conditions $\sigma_{\alpha,\beta}$ is contained in the wall if and only if $\mathrm{Ker}\, Z_{\alpha,\beta}$ is contained in $\mathcal{W}$. Moreover, Proposition 2.12 below will translate into the statement that for $\mathrm{Ker}\, Z_{\alpha,\beta}$ near $(0,0,1)$, slope-stable vector bundles of a fixed class are $\sigma_{\alpha,\beta}$-stable. In Figure 1, we draw a cross-section of the negative cone.

Basic properties of tilt-stability. We recall here further properties of tilt-stability that we will use later. The first is a well-known variant of [Bri08, Lemma 14.2].

Proposition 2.12. *Let E be a slope-stable vector bundle. Then E is $\sigma_{\alpha,\beta}$-stable for all α sufficiently big, and for all $\beta \in \mathbb{R}$.*

The next property is a consequence of Bogomolov-Gieseker inequality for tilt-stability.

Proposition 2.13 ([BMT14, Proposition 7.4.1] or [BMS16, Corollary 3.11])**.** *Let E be a slope-stable vector bundle with $\Delta_H(E) = 0$. Then E is $\sigma_{\alpha,\beta}$-stable for all $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$.*

Conversely, let E be a $\sigma_{\alpha,\beta}$-stable object with $\Delta_H(E) = 0$ and $\beta > \mu_H(E)$. Then $E = F[1]$ is the shift of a slope-semistable vector bundle F.

We also need the following variant of Proposition 2.11, that appears implicitly, but not explicitly, in [BMT14] for $\mu = 0$. It is also a consequence of the general results in [PT15], which are, however, depending on a conjectural Bogomolov-Gieseker type inequality involving ch_3; we will give a proof without such an assumption.

Choose a weak stability condition $\sigma_{\alpha,\beta}$ as in Proposition 2.11, and $\mu \in \mathbb{R}$. Following Proposition/Definition 2.9, we obtain a tilted heart, which we will denote by

$$\mathrm{Coh}^{\mu}_{\alpha,\beta}(X) := \mathcal{A}^{\mu}_{\sigma_{\alpha,\beta}}.$$

Let $u \in \mathbb{C}$ be the unit vector in the upper half plane with $\mu = -\frac{\Re u}{\Im u}$. Then it is straightforward to see that

$$Z^{\mu}_{\alpha,\beta} := \frac{1}{u} Z_{\alpha,\beta}$$

is a weak stability function on $\mathrm{Coh}^{\mu}_{\alpha,\beta}(X)$.

Proposition 2.14. *The pair* $(\mathrm{Coh}^{\mu}_{\alpha,\beta}(X), Z^{\mu}_{\alpha,\beta})$ *is a weak stability condition on* $\mathrm{D}^{\mathrm{b}}(X)$.

In the context of stability conditions, this statement would follow automatically from Proposition 2.11 via the $\widetilde{\mathrm{GL}^{+}_{2}}(\mathbb{R})$-action on the space of stability conditions; however, due to the special treatment of objects $E \in \mathcal{A}$ with $Z(E) = 0$, no such action exists a priori on the set of weak stability conditions.

Proposition 2.14 will be a consequence of the following general result.

Lemma 2.15. *Let* $\sigma = (\mathcal{A}, Z)$ *be a weak stability condition. Let* $\mathcal{A}_0 \subset \mathcal{A}$ *be the abelian subcategory whose objects have* $Z = 0$. *Assume the following:*

(a) $\mathcal{A}_0$ *is noetherian.*

(b) *For* $A \in \mathcal{A}$, *there exists a maximal subobject* $\widetilde{A} \hookrightarrow A$, $\widetilde{A} \in \mathcal{A}_0$, *such that* $\mathrm{Hom}(\mathcal{A}_0, A/\widetilde{A}) = 0$.

(c) *For* $A \in \mathcal{A}$ *with* $\mu^{+}_{\sigma}(A) < +\infty$, *there exists* $A \hookrightarrow \widehat{A} \in \mathcal{A}$ *such that* $\mathrm{Hom}(\mathcal{A}_0, \widehat{A}[1]) = 0$ *and* $\widehat{A}/A \in \mathcal{A}_0$.

Then, given $\mu \in \mathbb{R}$, *we have*

$$\mathcal{A}_0 = \{E \in \mathcal{A}^{\mu}_{\sigma} \,:\, Z(E) = 0\}$$

and, for all $B \in \mathcal{A}^{\mu}_{\sigma}$, *there exists a maximal subobject* $\widetilde{B} \hookrightarrow B$, $\widetilde{B} \in \mathcal{A}_0$, *such that* $\mathrm{Hom}(\mathcal{A}_0, B/\widetilde{B}) = 0$.

Proof. First of all, note that $\mathcal{A}_0$ is closed under subobjects, quotients, and extensions in $\mathcal{A}$.

To prove the first statement, note that $\mathcal{A}_0 \subset \mathcal{T}^{\mu}_{\sigma} \subset \mathcal{A}^{\mu}_{\sigma}$ and so $\mathcal{A}_0 \subset \{E \in \mathcal{A}^{\mu}_{\sigma} \,:\, Z(E) = 0\}$. For the reverse inclusion note that $\frac{1}{u}Z$ is a weak stability function on $\mathcal{A}^{\mu}_{\sigma}$, where $u \in \mathbb{C}$ is the unit vector in the upper half plane with $\mu = -\frac{\Re u}{\Im u}$. Thus, for $E \in \mathcal{A}^{\mu}_{\sigma}$ with $\frac{1}{u}Z(E) = 0$, we must have $\frac{1}{u}Z(H^{-1}_{\mathcal{A}}(E)) = \frac{1}{u}Z(H^{0}_{\mathcal{A}}(E)) = 0$. By definition of $\mathcal{F}^{\mu}_{\sigma}$, this implies that $H^{-1}_{\mathcal{A}}(E) = 0$. Therefore, $E \in \mathcal{T}^{\mu}_{\sigma} \subset \mathcal{A}$, and so $E \in \mathcal{A}_0$, proving the first claim.

We observe that $\mathcal{A}_0$ is therefore closed under subobjects, quotients, and extensions in $\mathcal{A}^{\mu}_{\sigma}$ as well.

To prove the second statement, let $B \in \mathcal{A}^{\mu}_{\sigma}$, let $K \in \mathcal{A}_0$, and assume $\mathrm{Hom}(K, B) \neq 0$. By the previous observation, we can assume that K is a subobject of B. Let $\widetilde{K} \subset H^{0}_{\mathcal{A}}(B)$ be the image of the composition $K \hookrightarrow B \twoheadrightarrow H^{0}_{\mathcal{A}}(B)$ and $\widehat{K} \subset H^{-1}_{\mathcal{A}}(B)[1]$ its kernel. Then $\widetilde{K}, \widehat{K} \in \mathcal{A}_0$. By property (b), we have $\widetilde{K} \subset \widetilde{H^{0}_{\mathcal{A}}(B)}$ in $\mathcal{A}_0$. Similarly by property (c), we have $\widehat{K} \subset \widehat{H^{-1}_{\mathcal{A}}(B)}/H^{-1}_{\mathcal{A}}(B)$ in $\mathcal{A}_0$. Given an increasing sequence of subobjects $K_n \subset B$, the corresponding sequences $\widetilde{K_n}$ and $\widehat{K_n}$ also form increasing sequences of subobjects; by noetherianity of $\mathcal{A}_0$, both terminate, and thus we obtain the existence of a maximal subobject $\widetilde{B}$ as we wanted. □

Example 2.16. Let $\sigma = (\mathrm{Coh}(X), Z_H)$ be the weak stability condition of Example 2.8. Then σ satisfies the conditions of Lemma 2.15. Here $\mathrm{Coh}(X)_0$ are the torsion sheaves supported in codimension at least 2. For $A \in \mathrm{Coh}(X)$, $\widetilde{A}$ is the torsion part in codimension at least 2 of the torsion filtration, while $\widehat{A}$ is the double-dual of a torsion-free sheaf.

The key fact is that the same holds for tilt-stability:

Proposition 2.17. *The weak stability condition* $\sigma_{\alpha,\beta} = (\mathrm{Coh}^\beta(X), Z_{\alpha,\beta})$ *satisfies the hypothesis of Lemma 2.15.*

In [PT15, Definition 2.13], condition (c) of Lemma 2.15 is part of the definition of *good very weak stability condition*. Proposition 2.17 can be deduced from the general result [PT15, Proposition 3.10]; the issue is that [PT15, Conjecture 3.8] is assumed as hypothesis. Our proof is unconditional and follows closely [BMT14, Section 5]. More precisely, on the tilted category, there is a double-dual operation as well as for coherent sheaves. This was defined in [BMT14, Proposition 5.1.3] for threefolds, and an analogous statement works in any dimension.

Let $\mathbb{D}(\_) := \mathbf{R}\mathcal{H}om(-, \mathcal{O}_X)[1]$ denote the duality functor.

Lemma 2.18. *Let* $E \in \mathrm{Coh}^\beta(X)$ *be such that* $\mu^+_{\sigma_{\alpha,\beta}}(E) < +\infty$.

(a) *We have* $H^j_{\mathrm{Coh}^{-\beta}(X)}(\mathbb{D}(E)) = 0$, *for all* $j < 0$, *and* $H^j_{\mathrm{Coh}^{-\beta}(X)}(\mathbb{D}(E))$ *is a torsion sheaf supported in codimension at least* $j+2$, *for all* $j \geq 1$. *We define* $E^\sharp$ *as* $H^0_{\mathrm{Coh}^{-\beta}(X)}(\mathbb{D}(E))$.

(b) *There exists an exact sequence in* $\mathrm{Coh}^\beta(X)$

$$0 \to E \to E^{\sharp\sharp} \to E^{\sharp\sharp}/E \to 0$$

with $E^{\sharp\sharp}/E \in \mathrm{Coh}^\beta(X)_0$ *and* $E^{\sharp\sharp}$ *is quasi-isomorphic to a two term complex* $C^{-1} \to C^0$ *with* C^{-1} *locally-free and* C^0 *reflexive.*

Part (a) of Lemma 2.18 can be rephrased by saying that there exists an exact triangle

$$E^\sharp \to \mathbb{D}(E) \to Q,$$

with $E^\sharp \in \mathrm{Coh}^{-\beta}(X)$, $H^j_{\mathrm{Coh}(X)}(Q) = 0$ for all $j \leq 0$, and $H^j_{\mathrm{Coh}(X)}(Q)$ a torsion sheaf supported in codimension at least $j+2$, for all $j \geq 1$.

Proof. In this proof, we will write $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ for the standard t-structure on $\mathrm{D}^{\mathrm{b}}(X)$, and H^i and $\tau_{\leq n}, \tau_{\geq n}$ for the associated cohomology and truncation functors.

We first recall that for a coherent sheaf F, the complex $\mathbb{D}(F)$ satisfies

$$H^j(\mathbb{D}(F)) = \begin{cases} 0 & \text{for } j < -1 \\ \mathcal{H}om(F, \mathcal{O}_X) & \text{for } j = -1 \\ \mathcal{E}xt^{j+1}(F, \mathcal{O}_X), \text{ a sheaf supported in codimension} \geq j+1 & \text{for } j \geq 0. \end{cases}$$

Moreover, if F is supported in codimension k, then $H^{k-1}(\mathbb{D}(E))$ is the smallest degree with a non-vanishing cohomology sheaf.

(a) We initially define $E^\sharp$ as the truncation $E^\sharp = \tau_{\leq 0}(\mathbb{D}(E))$, and $Q = \tau_{\geq 1}(\mathbb{D}(E))$. We dualize the triangle $H^{-1}(E)[1] \to E \to H^0(E)$ and consider the long exact cohomology sequence. We first get an isomorphism

$$\mathcal{H}om(H^0(E), \mathcal{O}_X) = H^{-1}(\mathbb{D}(H^0(E)) \cong H^{-1}(\mathbb{D}(E))$$

which is in $\mathcal{F}^{-\beta}$ as claimed. We next get a long exact sequence

$$0 \to \mathcal{E}xt^1(H^0(E), \mathcal{O}_X) \to H^0(\mathbb{D}(E)) \to \mathcal{H}om(H^{-1}(E), \mathcal{O}_X) \to \mathcal{E}xt^2(H^0(E), \mathcal{O}_X).$$

Due to our assumption $\mu^+_{\sigma_{\alpha,\beta}}(E) < +\infty$, we have $\mathcal{H}om(H^{-1}[1], \mathcal{O}_X) \in \mathcal{T}^{-\beta}$. Since $\mathcal{E}xt^2(H^0(E), \mathcal{O}_X)$ has codimension at least two, the same is true for the kernel of the right-most map. Since the sheaf $\mathcal{E}xt^1(H^0(E), \mathcal{O}_X)$ is torsion, it follows that $H^0(\mathbb{D}(E)) \in \mathcal{T}^{-\beta}$ as well.

For positive j, the long exact sequence shows that $H^j(\mathbb{D}(E)) = H^j(Q)$ is supported in codimension at least $j+1$; it remains to show that it is in fact supported in codimension at least $j+2$.

Assume otherwise. Note that $H^{-1}(\mathbb{D}(Q))$ vanishes, and that $H^0(\mathbb{D}(Q))$ is an extension of the corresponding sheaves $H^0\left(\mathbb{D}(H^j(Q)[-j])\right)$ for each of the cohomology sheaves of Q, which is non-vanishing if and only if our claim is false; it is always supported in codimension at least two.

Consider the triangle $\mathbb{D}(Q) \to \mathbb{D}(\mathbb{D}(E)) = E \to \mathbb{D}(E^\sharp)$. The composition $H^0(\mathbb{D}(Q)) \to E$ is non-vanishing, since there is no map from $H^0(\mathbb{D}(Q))$ to $H^{-1}(\mathbb{D}(E^\sharp)) \in \mathcal{F}^\beta$. This is a contradiction to $\mu^+_{\sigma_{\alpha,\beta}}(E) < +\infty$.

(b) By part (a) we have a diagram of exact triangles

$$\begin{tikzcd}
& & E^{\sharp\sharp} \arrow[d] \\
\mathbb{D}(Q) \arrow[r] & E \arrow[r] & \mathbb{D}(E^\sharp) \arrow[d] \\
& & Q'
\end{tikzcd}$$

with $\mathbb{D}(Q), Q' \in \mathcal{D}^{\geq 1}$, and all their cohomology sheaves are supported in codimension at least 3, whereas $E, E^{\sharp\sharp} \in \operatorname{Coh}^\beta X \subset \mathcal{D}^{\leq 0}$. Clearly $\operatorname{Hom}(E, Q') = 0$, so we have an induced morphism $E \to E^{\sharp\sharp}$. The cone C of this morphism fits into an exact triangle $Q'[-1] \to C \to \mathbb{D}(Q)[1]$ and therefore has only non-negative cohomologies with respect to the t-structure $\operatorname{Coh}^\beta X$. In other words, $E \to E^{\sharp\sharp}$ is injective in $\operatorname{Coh}^\beta X$, with cokernel a torsion sheaf supported in codimension at least 3.

To finish the proof, we consider a locally-free resolution $G^\bullet$ of $E^\sharp$. By taking the functor $\mathbb{D}$, we obtain a morphism

$$E^{\sharp\sharp} \to \left(G_0^\vee \xrightarrow{\phi} G_1^\vee \xrightarrow{\psi} \cdots \to G_k^\vee\right)[1].$$

By part (a), $E^{\sharp\sharp}$ is quasi-isomorphic to the complex $G_0^\vee \xrightarrow{\phi} \mathrm{Ker}(\psi)$, and $\mathrm{Ker}(\psi)$ is reflexive since it is the kernel of a morphism of locally-free sheaves. ☐

Proof of Proposition 2.17. First of all note that $\mathrm{Coh}^\beta(X)_0$ are exactly the torsion sheaves supported on codimension at least 3. This shows property (a).

Regarding (b), by Lemma 2.15 applied to $(\mathrm{Coh}(X), Z_H)$, we know that for any $A \in \mathrm{Coh}^\beta(X)$ there exists a maximal subobject $\widetilde{A}'$ which is a torsion sheaf supported in codimension at least 2. Since $\mathrm{Coh}^\beta(X)_0 \subset \mathrm{Coh}(X)_0$, for any subobject $K \subset A$ such that $K \in \mathrm{Coh}^\beta(X)_0$ we have $K \subset \widetilde{A}'$. Since $\mathrm{Coh}(X)_0$ is noetherian, we find a maximal subobject $\widetilde{A} \subset \widetilde{A}' \subset A$ satisfying property (b).

To prove property (c), by Lemma 2.18(b), we have an exact sequence in $\mathrm{Coh}^\beta(X)$

$$0 \to A \to A^{\sharp\sharp} \to A^{\sharp\sharp}/A \to 0$$

with $A^{\sharp\sharp}/A \in \mathrm{Coh}^\beta(X)_0$ and $A^{\sharp\sharp}$ is quasi-isomorphic to a two-term complex $C^{-1} \to C^0$ with $\mathrm{Ext}^1(\mathrm{Coh}^\beta(X)_0, C^0) = \mathrm{Ext}^2(\mathrm{Coh}^\beta(X)_0, C^{-1}) = 0$. Then $\widehat{A} := A^{\sharp\sharp}$ satisfies property (c). Indeed, this follows immediately by using the exact triangle

$$C^0 \to A^{\sharp\sharp} \to C^{-1}[1]. \qquad \square$$

Proof of Proposition 2.14. By Lemma 2.15 and Proposition 2.17, every $E \in \mathrm{Coh}^\mu_{\alpha,\beta}(X)$ has a subobject $\widetilde{E} \subset E$ with $\widetilde{E} \in \mathrm{Coh}^\beta(X)_0$ and $\mathrm{Hom}(\mathrm{Coh}^\beta(X)_0, E/\widetilde{E}) = 0$.

Let $(\mathcal{T}^\mu, \mathcal{F}^\mu)$ denote the torsion pair in $\mathrm{Coh}^\beta(X)$ from which we construct $\mathrm{Coh}^\mu_{\alpha,\beta}(X)$; by definition, $(\mathcal{F}^\mu[1], \mathcal{T}^\mu)$ is a torsion pair in $\mathrm{Coh}^\mu_{\alpha,\beta}(X)$. If an object $E \in \mathcal{T}^\mu$ is $\sigma_{\alpha,\beta}$-semistable, then $E/\widetilde{E}$ is $\sigma^\mu_{\alpha,\beta}$-semistable; similarly for objects $E \in \mathcal{F}^\mu$ up to the shift $[1]$. Conversely, any stable object in $\mathrm{Coh}^\mu_{\alpha,\beta}(X)$ is, up to shift, a $\sigma_{\alpha,\beta}$-stable object in $\mathrm{Coh}^\beta(X)$.

Any object E fits into a short exact sequence

$$0 \to F[1] \to E \to T \to 0$$

with $F \in \mathcal{F}^\mu_{\sigma_{\alpha,\beta}}$ and $T \in \mathcal{T}^\mu_{\sigma_{\alpha,\beta}}$. The objects F and T have Harder-Narasimhan filtrations with respect to $\sigma_{\alpha,\beta}$, such that all quotients and subobjects in the filtration lie in $\mathcal{F}^\mu_{\sigma_{\alpha,\beta}}$ and $\mathcal{T}^\mu_{\sigma_{\alpha,\beta}}$, respectively. Combined, they give a finite filtration of E. Let $E \twoheadrightarrow Q$ be the quotient of E corresponding to the last filtration step of E. Then the composition $E \twoheadrightarrow Q \twoheadrightarrow Q/\widetilde{Q}$ gives the maximal destabilizing quotient of E with respect to $\sigma^\mu_{\alpha,\beta}$; continuing this process produces the Harder-Narasimhan filtration of E. ☐

3. REVIEW ON SEMIORTHOGONAL DECOMPOSITIONS

The second main ingredient in this paper consists in semiorthogonal decompositions. We begin with a very general and quick review, by following [BO95]. To this extent, let $\mathcal{D}$ be a triangulated category.

Definition 3.1. A *semiorthogonal* decomposition of $\mathcal{D}$ is a sequence of full triangulated subcategories $\mathcal{D}_1, \dots, \mathcal{D}_m \subseteq \mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}_i, \mathcal{D}_j) = 0$, for $i > j$ and, for all $G \in \mathcal{D}$, there exists a chain of morphisms in $\mathcal{D}$

$$0 = G_m \to G_{m-1} \to \dots \to G_1 \to G_0 = G$$

with $\mathrm{cone}(G_i \to G_{i-1}) \in \mathcal{D}_i$, for all $i = 1, \dots, m$.

We will denote such a decomposition by $\mathcal{D} = \langle \mathcal{D}_1, \dots, \mathcal{D}_m \rangle$. The semiorthogonality condition implies that $G \mapsto \mathrm{cone}(G_i \to G_{i-1}) \in \mathcal{D}_i$ defines a functor $\mathrm{pr}_i : \mathcal{D} \to \mathcal{D}_i$, called the *i-th projection functor*.

Definition 3.2. (a) An object $E \in \mathcal{D}$ is *exceptional* if $\mathrm{Hom}_{\mathcal{D}}(E, E[p]) = 0$, for all $p \neq 0$, and $\mathrm{Hom}_{\mathcal{D}}(E, E) \cong \mathbb{C}$.

(b) A collection $\{E_1, \dots, E_m\}$ of objects in $\mathcal{D}$ is called an *exceptional collection* if E_i is an exceptional object, for all i, and $\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[p]) = 0$, for all p and all $i > j$.

An exceptional collection $\{E_1, \dots, E_m\}$ in $\mathcal{D}$ provides a semiorthogonal decomposition

$$\mathcal{D} = \langle \mathcal{D}', E_1, \dots, E_m \rangle,$$

where we have denoted by E_i the full triangulated subcategory of $\mathcal{D}$ generated by E_i and

$$\mathcal{D}' = \langle E_1, \dots, E_m \rangle^{\perp} = \{G \in \mathcal{D}\colon\ \mathrm{Hom}(E_i, G[p]) = 0,\ \text{for all } p \text{ and } i\}.$$

Similarly, one can define ${}^{\perp}\langle F_1, \dots, F_m \rangle = \{G \in \mathcal{D} : \mathrm{Hom}(G, F_i[p]) = 0,\ \text{for all } p \text{ and } i\}$.

Let $E \in \mathcal{D}$ be an exceptional object. We can define the *left and right mutation functors*, $\mathbf{L}_E, \mathbf{R}_E : \mathcal{D} \to \mathcal{D}$ in the following way

$$\mathbf{L}_E(G) := \mathrm{cone}\left(\mathrm{ev}\colon \bigoplus_p \mathrm{Hom}_{\mathcal{D}}(E, G[p]) \otimes E \to G\right)$$

$$\mathbf{R}_E(G) := \mathrm{cone}\left(\mathrm{ev}^{\vee}\colon G \to \bigoplus_p \mathrm{Hom}_{\mathcal{D}}(G, E[p])^{\vee} \otimes E\right)[-1].$$

We will use the following properties of mutations and semiorthogonal decompositions, where E, F are exceptional objects, and S is a Serre functor of $\mathcal{D}$.

(a) Given a semiorthogonal decomposition

$$\mathcal{D} = \langle \mathcal{D}_1, \dots, \mathcal{D}_k, E, \mathcal{D}_{k+1}, \dots, \mathcal{D}_n \rangle,$$

with E exceptional, we can apply left and right mutations and get

$$\mathcal{D} = \langle \mathcal{D}_1, \dots, \mathcal{D}_k, \mathbf{L}_E(\mathcal{D}_{k+1}), E, \mathcal{D}_{k+2}, \dots, \mathcal{D}_n \rangle = \langle \mathcal{D}_1, \dots, \mathcal{D}_{k-1}, E, \mathbf{R}_E(\mathcal{D}_k), \mathcal{D}_{k+1}, \dots, \mathcal{D}_n \rangle.$$

(b) If (E, F) is an exceptional pair, then $\mathbf{R}_E \mathbf{L}_E F = F$.

(c) $\mathbf{R}_{S(E)}$ is right adjoint to $\mathbf{L}_E$ while $\mathbf{R}_E$ is left adjoint to $\mathbf{L}_E$.

(d) If $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is a semiorthogonal decomposition, then so are

$$\mathcal{D} = \langle S(\mathcal{D}_2), \mathcal{D}_1 \rangle = \langle \mathcal{D}_2, S^{-1}(\mathcal{D}_1) \rangle.$$

4. INDUCING T-STRUCTURES

Let $\mathcal{D}$ be a triangulated category admitting a Serre functor S, and with a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. In this section, we give a general criterion for inducing a bounded t-structure on $\mathcal{D}_1$ from a bounded t-structure on $\mathcal{D}$. While in this paper, we are only interested in the case where $\mathcal{D}_2$ is generated by an exceptional collection in $\mathcal{D}$, we state our criterion in a more general setting in terms of a spanning class of $\mathcal{D}_2$.

Definition 4.1. A spanning class of a triangulated category $\mathcal{D}$ is a set of objects $\mathcal{G}$ such that if $F \in \mathcal{D}$ satisfies $\mathrm{Hom}(G, F[p]) = 0$ for all $G \in \mathcal{G}$ and all $p \in \mathbb{Z}$, then $F = 0$.

The following observation is immediate:

Lemma 4.2. *Let $\mathcal{G}$ be a spanning class of $\mathcal{D}_2$. Then for an object $F \in \mathcal{D}$ we have $F \in \mathcal{D}_1$ if and only if $\mathrm{Hom}(G, F[p]) = 0$ for all $G \in \mathcal{G}$ and all $p \in \mathbb{Z}$.*

The key ingredient of our entire construction is the following observation, slightly generalizing [VdB00, Theorem 4.1] and [BMMS12, Lemma 3.4].

Lemma 4.3. *Let $\mathcal{A} \subset \mathcal{D}$ be the heart of a bounded t-structure. Assume that the spanning class $\mathcal{G}$ of $\mathcal{D}_2$ satisfies $\mathcal{G} \subset \mathcal{A}$, and $\mathrm{Hom}(G, F[p]) = 0$ for all $G \in \mathcal{G}, F \in \mathcal{A}$, and all $p > 1$. Then $\mathcal{A}_1 := \mathcal{D}_1 \cap \mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}_1$.*

Proof. Clearly $\mathcal{A}_1$ satisfies the first condition of Definition 2.1, and we only need to verify the second.

Consider $F \in \mathcal{D}_1$. For every $G \in \mathcal{G}$ there is a spectral sequence (see e.g., [Oka06, Proposition 2.4])

$$E_2^{p,q} = \mathrm{Hom}(G, H^q_{\mathcal{A}}(F)[p]) \Rightarrow \mathrm{Hom}(G, F[p+q]).$$

By the assumptions, these terms vanish except for $p = 0, 1$, and thus the spectral sequence degenerates at E_2. On the other hand, since $G \in \mathcal{D}_2$ and $F \in \mathcal{D}_1$ we have $\mathrm{Hom}(G, F[p+q]) = 0$. Therefore, $\mathrm{Hom}(G, H^q_{\mathcal{A}}(F)[p]) = 0$ for all $G \in \mathcal{G}$ and all $p \in \mathbb{Z}$; by Lemma 4.2 we conclude $H^q_{\mathcal{A}}(F) \in \mathcal{A} \cap \mathcal{D}_1 = \mathcal{A}_1$. This proves the claim. □

We will always apply this lemma via the following consequence:

Corollary 4.4. *Let $\mathcal{A} \subset \mathcal{D}$ be the heart of a bounded t-structure such that $\mathcal{G} \subset \mathcal{A}$ and $S(\mathcal{G}) \subset \mathcal{A}[1]$. Then $\mathcal{A}_1 := \mathcal{A} \cap \mathcal{D}_1 \subset \mathcal{D}_1$ is the heart of a bounded t-structure.*

Proof. Given $G \in \mathcal{G}$ and $F \in \mathcal{A}$, as well as $p > 1$ we have

$$\mathrm{Hom}(G, F[p]) = \mathrm{Hom}(F, S(G)[-p])^\vee = 0$$

as $S(G)[-p] \in \mathcal{A}[1-p]$. Therefore, the assumptions of Lemma 4.3 are satisfied. □

Example 4.5. Let X be a smooth projective surface with canonical divisor K_X, and let H be an ample divisor. Assume that there exists a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$$

and a set of generators $\mathcal{G}$ of $\mathcal{D}_2$ such that $\mathcal{G}$ consists of slope-semistable torsion-free sheaves with

$$\mu_H(G \otimes K_X) \leq \beta < \mu_H(G),$$

for $\beta \in \mathbb{Q}$ and for all $G \in \mathcal{G}$. Then there exists a Bridgeland stability condition on $\mathcal{D}_1$.

Indeed, let $G \in \mathcal{G}$, and consider the tilted heart $\mathcal{A}^\sharp = \mathrm{Coh}_H^\beta(X)$ of Definition 2.10. Then by construction, we have $G \in \mathrm{Coh}_H^\beta(X)$ and $S(G) = G \otimes K_X[2] \in \mathcal{A}^\sharp[1]$. By Corollary 4.4, we obtain an induced heart of a bounded t-structure $\mathcal{A}_1 = \mathcal{A}^\sharp \cap \mathcal{D}_1$ on $\mathcal{D}_1$.

Let $Z_H(E) := \mathrm{i}\, H^2 \operatorname{rk}(E) - H \operatorname{ch}_1(E)$ be the weak central charge on $\mathrm{Coh}(X)$ inducing slope-stability for coherent sheaves, and let u_β be the unit vector in the upper half plane with slope $-\frac{\Re u_\beta}{\Im u_\beta} = \beta$. We claim that with

$$Z_1(E) := \frac{1}{u_\beta} Z_H(E),$$

the pair $(\mathcal{A}_1, Z_1)$ defines a stability condition on $\mathcal{D}_1$.

Indeed, it is clear that Z_1 is a weak stability condition on $\mathrm{Coh}_H^\beta(X)$, with $Z_1(E) = 0$ for $E \in \mathrm{Coh}_H^\beta(X)$ if and only if $E = H^0(E)$ is a 0-dimensional torsion sheaf. However, if $E = H^0(E)$ is a 0-dimensional torsion sheaf, then $\mathrm{Hom}(G, E) \neq 0$ for any $G \in \mathcal{G}$; therefore $E \notin \mathcal{D}_1$. This shows that Z_1 is a stability function for $\mathcal{A}_1$.

Since $\mathrm{Coh}_H^\beta(X)$ is Noetherian, the same holds for its subcategory $\mathcal{A}_1$; since $\beta \in \mathbb{Q}$, the stability function is discrete; by [BM11, Proposition B.2] this shows that $(\mathcal{A}_1, Z_1)$ satisfies the Harder-Narasimhan property. It remains to establish the support property; this is part of the results of the following section.

5. INDUCING STABILITY CONDITIONS

The goal of this section is to enhance the method of the previous section, and show that when $\mathcal{D}_2$ is generated by an exceptional collection, then we can use the same procedure to induce a stability condition on $\mathcal{D}_1$ from a stability condition on $\mathcal{D}$, such that the underlying hearts are related by the construction of Lemma 4.3 and Corollary 4.4.

Result. Let $E_1, \dots, E_m$ be an exceptional collection in a triangulated category $\mathcal{D}$. We let $\mathcal{D}_2 = \langle E_1, \dots, E_m \rangle$ be the category generated by the exceptional objects, and we write

$$\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$$

for the resulting semiorthogonal decomposition of $\mathcal{D}$. We continue to write S for the Serre functor on $\mathcal{D}$. The main result of this section is the following:

Proposition 5.1. *Let $\sigma = (\mathcal{A}, Z)$ be a weak stability condition on $\mathcal{D}$ with the following properties for all $i = 1, \dots, m$:*

(a) $E_i \in \mathcal{A}$,
(b) $S(E_i) \in \mathcal{A}[1]$, *and*
(c) $Z(E_i) \neq 0$.

Assume moreover that there are no objects $0 \neq F \in \mathcal{A}_1 := \mathcal{A} \cap \mathcal{D}_1$ *with* $Z(F) = 0$ *(i.e.,* $Z_1 := Z|_{K(\mathcal{A}_1)}$ *is a stability function on* $\mathcal{A}_1$*). Then the pair* $\sigma_1 = (\mathcal{A}_1, Z_1)$ *is a stability condition on* $\mathcal{D}_1$*.*

Inducing Harder-Narasimhan filtrations. We start with some easy observations about the category $\mathcal{A}_1$. If $F, G \in \mathcal{A}_1$ are objects with a morphism $f \colon F \to G$, then f is injective (resp. surjective) as a morphism in $\mathcal{A}_1$ if and only if it is injective (resp. surjective) as a morphism in $\mathcal{A}$. In other words, the inclusion $\mathcal{A}_1 \hookrightarrow \mathcal{A}$ is an exact functor.

We will prove the following slightly more general statement:

Lemma 5.2. *Let* $(\mathcal{A}, Z)$ *be a weak stability condition. Let* $\mathcal{A}_1 \subset \mathcal{A}$ *be an abelian subcategory such that the inclusion functor is exact. Assume moreover that* Z *restricted to* $K(\mathcal{A}_1)$ *is a stability function. Then Harder-Narasimhan filtrations exist in* $\mathcal{A}_1$ *for the stability function* Z*.*

The existence of Harder-Narasimhan filtrations for objects $F \in \mathcal{A}$ follows directly from the existence of Harder-Narasimhan polygons and the concept of mass. We recall all the necessary definitions and basic facts here; see [Bay16, Section 3] for some context.

Let $\mathcal{B}$ be an abelian category, and $Z \colon K(\mathcal{B}) \to \mathbb{C}$ a weak stability function on $\mathcal{B}$ (see Definition 2.2).

Definition 5.3. The Harder-Narasimhan polygon $\mathrm{HN}^Z_{\mathcal{B}}(F)$ of an object $F \in \mathcal{B}$ is the convex hull in the complex plane of the set

$$\{Z(A) : \ A \subset F\}.$$

We say that the Harder-Narasimhan polygon $\mathrm{HN}^Z_{\mathcal{B}}(F)$ is *finite polyhedral on the left* if the intersection of $\mathrm{HN}^Z_{\mathcal{B}}(F)$ with the closed half-plane to the left of the line through 0 and $Z(F)$ is a polygon. The following Proposition is a variant of a well-known statement.

Proposition 5.4. *If* $F \in \mathcal{B}$ *admits a Harder-Narasimhan filtration, then* $\mathrm{HN}^Z_{\mathcal{B}}(F)$ *is finite polyhedral on the left. If moreover* Z *is a stability function, then the converse holds.*

Proof. When Z is a stability function, then both directions are well-known, see e.g., [Bay16, Proposition 3.3]. When Z is only a weak stability function, the first statement is proved easily with the same arguments as in the case of a stability function. □

We also recall that the support property as defined here is equivalent to the one originally appearing in [Bri07]:

Proposition 5.5 ([BMS16, Lemma A.4])**.** *A pair* $(\mathcal{B}, Z)$ *of an abelian category with a weak stability function satisfies the support property if and only if there is a metric* $\|\cdot\|$ *on* $\Lambda \otimes \mathbb{R}$*, and a constant* $C > 0$ *such that for all* Z*-semistable objects* $F \in \mathcal{A}$*, we have*

$$|Z(F)| \geq C\|v(F)\|.$$

We now return to the setting of Lemma 5.2, and assume that $(\mathcal{A}, Z)$ is a weak stability condition.

Definition 5.6. The *mass* $m^Z_{\mathcal{A}}(F)$ of an object $F \in \mathcal{A}$ is the length of the boundary of the Harder-Narasimhan polygon $\mathrm{HN}^Z_{\mathcal{A}}(F)$ on the left, between 0 and $Z(F)$.

The triangle inequality gives:

Proposition 5.7. *With a metric* $\|\cdot\|$ *and* $C>0$ *as in Proposition 5.5, any object* F *satisfies*

$$m^Z_{\mathcal{B}}(F) \geq C\|v(F)\|.$$

Lemma 5.8. *Let* $F \in \mathcal{A}_1$. *Then the Harder-Narasimhan polygon* $\mathrm{HN}^Z_{\mathcal{A}_1}(F)$ *(with respect to subobjects in* $\mathcal{A}_1$*) is finite polyhedral on the left.*

Proof. Consider a subobject $A \in F$ with $A \in \mathcal{A}_1$ with $\mu_Z(A) > \mu_Z(F)$. Then $\mathrm{HN}^Z_{\mathcal{A}}(A) \subset \mathrm{HN}^Z_{\mathcal{A}}(F)$. It follows from a simple picture that the mass of A (as an object of $\mathcal{A}$) is bounded above; therefore, by Proposition 5.7, so is $\|v(A)\|$.

It follows that there are only finitely many classes $v(A) \in \Lambda$ of subobjects $A \subset F$ with $\mu_Z(A) > \mu_Z(F)$. □

By Proposition 5.4, this concludes the proof of Lemma 5.2.

Inducing support property. It remains to show that $(\mathcal{A}_1, Z_1)$ satisfies the support property. By induction, we may assume for the rest of this section that $\mathcal{D}_2$ is generated by a single exceptional object E.

Lemma 5.9. *Any* $G \in \mathcal{A}$ *fits into a four-term short exact sequence in* $\mathcal{A}$

$$0 \to E \otimes \mathrm{Hom}(E, G) \to G \to G_1 \to E \otimes \mathrm{Ext}^1(E, G) \to 0$$

with $G_1 \in \mathcal{A}_1$.

Proof. This is the long exact cohomology sequence with respect to $\mathcal{A}$ applied to the exact triangle

$$E \otimes \mathbf{R}\mathrm{Hom}(E, G) \xrightarrow{\mathrm{ev}} G \to G_1. \qquad \square$$

Lemma 5.10. *Consider a short exact sequence*

$$0 \to A \to F \to B \to 0$$

in $\mathcal{A}$ *with* $F \in \mathcal{A}_1$. *Then there exists a short exact sequences*

$$0 \to A_1 \to F \to B_1 \to 0$$

in $\mathcal{A}_1$, *together with short exact sequences*

$$0 \to A \to A_1 \to E \otimes V \to 0, \quad 0 \to E \otimes V \to B \to B_1 \to 0$$

in $\mathcal{A}$ *where* $V = \mathrm{Hom}(E, B)$.

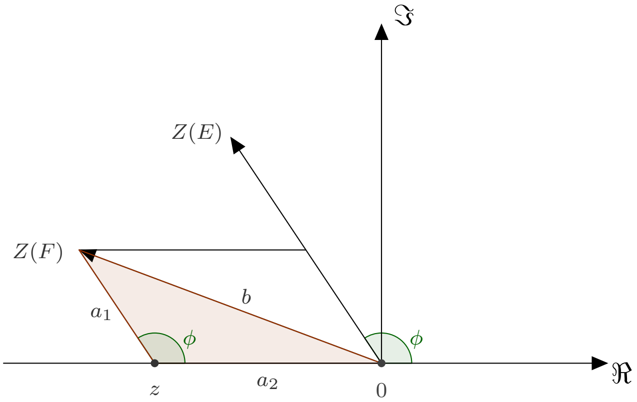

FIGURE 2. The left-hand side of $\mathrm{HN}^Z_{\mathcal{A}}(F)$ is contained in the triangle $0, z, Z(F)$.

Proof. The long exact Hom-sequence shows $\mathrm{Hom}(E, A) = 0 = \mathrm{Ext}^1(E, B)$ and $V = \mathrm{Hom}(E, B) = \mathrm{Ext}^1(E, A)$. Therefore, the four-term exact sequences of Lemma 5.9 for A and B become short exact sequences as claimed; the short exact sequence involving F follows from the octahedral axiom. □

Proof of Proposition 5.1. We now fix a norm $\|\cdot\|$ on $\Lambda\otimes\mathbb{R}$ and a constant C such that the weak stability conditions $(\mathcal{A}, Z)$ satisfies the support property in the formulation given in Proposition 5.5. Assume that $F \in \mathcal{A}_1$ is semistable within the category $\mathcal{A}_1$, and consider a quotient $F \twoheadrightarrow B$ as in Lemma 5.10. Since $\mu_Z(B_1) \geq \mu_Z(F)$ (by semistability of F in $\mathcal{A}_1$), it follows that $\mu_Z(B) \geq \min\{\mu_Z(F), \mu_Z(E)\}$. In particular, if $\mu_Z(F) \leq \mu_Z(E)$, then F is also semistable as an object of $\mathcal{A}$, and thus satisfies the support property with respect to the same constant C. Otherwise, the left-hand side of the Harder-Narasimhan polygon $\mathrm{HN}^Z_{\mathcal{A}}(F)$ is contained in the triangle with vertexes $0, z, Z(F)$ where z is the point on the negative real line such that $Z(F) - z$ is proportional to $Z(E)$ (see Figure 2); therefore $m^Z_{\mathcal{A}}(F) \leq |z| + |Z(F) - z|$.

By Lemma 5.11, there is a constant $D > 0$, depending only on $Z(E)$, such that

$$m^Z_{\mathcal{A}}(F) \leq |z| + |Z(F) - z| \leq D\,|Z(F)|\,.$$

Combined with Proposition 5.7, it follows that $\|v(F)\| \leq CD\,|Z(F)|$ for all semistable objects $F \in \mathcal{A}_1$, i.e., the pair $(\mathcal{A}_1, Z)$ satisfies the support property. □

Lemma 5.11. *Let $0 < \phi < \pi$ be a fixed angle. Then there is a constant $D > 0$ such that for all triangles with one angle given by ϕ, and with adjacent side lengths a_1, a_2 and b the side length opposite of ϕ, we have $a_1 + a_2 \leq Db$.*

Proof. With $D = \sqrt{\frac{1+\cos\phi}{2}}$, this follows from

$$b^2 = a_1^2 + a_2^2 - 2a_1a_2\cos\phi = \frac{1+\cos\phi}{2}(a_1+a_2)^2 + \frac{1-\cos\phi}{2}(a_1-a_2)^2. \qquad \square$$

We note the following observation made in the proof:

Remark 5.12. An object $F \in \mathcal{A}_1$ with $\mu_Z(F) \leq \mu_Z(E)$ is semistable as an object in $\mathcal{A}_1$ if and only if it is semistable as an object in $\mathcal{A}$. More generally, any object $F \in \mathcal{A}_1$ satisfies $\mu_Z^-(F) \geq \min\{\mu_{Z_1}^-(F), \mu_Z(E)\}$, where $\mu_Z^-(F)$ is computed by the HN filtration of F as an object of $\mathcal{A}$, and $\mu_{Z_1}^-(F)$ by HN filtration in $\mathcal{A}_1$.

When combined with the results in [CP10], Proposition 5.1 has the following consequence:

Proposition 5.13. *Given the same assumptions as in Proposition 5.1, there exists a stability conditions $\sigma' = (\mathcal{A}', Z')$ on $\mathcal{D}$ with*

$$\mathcal{A}' = \langle \mathcal{A}_1, E_1[1], E_2[2], \dots, E_m[m] \rangle$$

and Z' determined by

$$Z'|_{\mathcal{D}_1} = Z|_{\mathcal{D}_1}, \quad Z'(E_i) = (-1)^{i+1}$$

for $i = 1, \dots, m$.

Proof. Let $\mathcal{A}_2 := \langle E_1[1], \dots, E_m[m] \rangle$; by [CP10, Section 2], $\mathcal{A}_2$ is the heart of a bounded t-structure in $\mathcal{D}_2$, that together with $\mathcal{A}_1$ produce a heart $\mathcal{A}' \subset \mathcal{D}$ with description as in the claim; moreover, the pair $(\mathcal{A}_2, \mathcal{A}_1)$ is a torsion pair in $\mathcal{A}$.

By construction, the objects $\mathcal{A}_2 \subset \mathcal{A}'$ have maximal slope; thus, the existence of the torsion pair, combined with the existence of HN-filtrations in $\mathcal{A}_1$ with respect to Z_1, gives HN filtrations in $\mathcal{A}'$ (similar to the proof of [CP10, Proposition 3.3]). Finally, σ'-stable objects are either σ_1-stable objects of $\mathcal{A}_1$, or of the form $E_i[i]$. By Proposition 5.5, this shows that the support property for σ_1 implies the support property for σ'. □

In other words, a *weak* stability condition on $\mathcal{D}$ with the assumptions of Proposition 5.1 automatically produces an actual stability condition on $\mathcal{D}$—at the cost of making the description of the associated heart more implicit.

6. FANO THREEFOLDS

In this section we apply the results of the previous sections in order to construct stability conditions on the Kuznetsov component of all but one deformation type of Fano threefolds of Picard rank one.

Review. We begin by reviewing Kuznetsov's semiorthogonal decompositions for the Fano threefolds appearing in Theorem 1.1, following [Kuz09a]. Recall that a Fano variety X has index i_X if $K_X = -i_X H$, where K_X is the canonical divisor, and H is a primitive ample divisor.

In the case of index two, the definition of the Kuznetsov component is straightforward:

Definition 6.1 ([Kuz09a])**.** Let X be a Fano threefold of index two, and let $H = -\frac{1}{2}K_X$. Then the Kuznetsov component $\mathcal{K}u(X)$ is defined by the semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(H) \rangle.$$

For index one and Picard rank one, we additionally need the existence of an exceptional vector bundle due to Mukai. The deformation classes of such threefolds are paremeterised by the genus g defined by $2g-2 = H^3$.

Theorem 6.2 ([Muk92, Kuz09a]). *Let X be a Fano threefold of Picard rank one, index one, and even genus $g = 2s > 2$. Then there exists a stable vector bundle $\mathcal{E}_2$ on X of rank 2, with $c_1(\mathcal{E}_2) = -H$ and $\mathrm{ch}_2(\mathcal{E}_2) = (s-2)L$, where L is the class of a line on X.*

Proposition and Definition 6.3 ([Kuz09a]). *Let X be a Fano threefold of Picard rank one, index one, and even genus $g > 2$. Then the pair $(\mathcal{E}_2, \mathcal{O})$ is exceptional, and the Kuznetsov component of X is defined by the semiorthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{K}u(X), \mathcal{E}_2, \mathcal{O} \rangle.$$

Proposition and Definition 6.4 ([Muk92, Kuz06]). *Let X be a Fano threefold of Picard rank one, index one, and genus 7 (resp. 9). Then there exists a rank 5 (resp. 3) vector bundle $\mathcal{E}_5$ (resp. $\mathcal{E}_3$) such that the pair $(\mathcal{E}_5, \mathcal{O})$ (resp. $(\mathcal{E}_3, \mathcal{O})$) is exceptional. The Kuznetsov component of X is defined to be its right orthogonal component.*

Definition 6.5. Let X be a Fano threefold of Picard rank one, index one, and genus 2, 3 or 5. Then we define the Kuznetsov component of X as the right orthogonal to $\langle \mathcal{O}_X \rangle$.

Let X be a Fano threefold. We consider the lattice $\Lambda_H^2 \cong \mathbb{Z}^3$ as in Example 2.8 and the natural map $v_H^2 \colon K(X) \to \Lambda_H^2$. For an admissible subcategory $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$, we denote by

$$\Lambda_{H,\mathcal{D}}^2 := \mathrm{Im}\left(K(\mathcal{D}) \to K(X) \to \Lambda_H^2 \right). \tag{2}$$

the image of the composite map and, by abuse of notation, the induced morphism $v_H^2 \colon K(\mathcal{D}) \to \Lambda_{H,\mathcal{D}}^2$.

Result and context. The goal of this section is to prove Theorem 1.1. We will split the statement into various cases and we summarize the results in the following two tables of Fano threefolds of Picard rank one:

$\rho_X = 1$ & $i_X = 2$			
deg	Semiorthogonal decomposition		$\exists$ stability conditions
5	$\mathrm{D}^{\mathrm{b}}(Y_5) = \langle \mathcal{F}_2(-H), \mathcal{O}(-H), \mathcal{F}_2, \mathcal{O} \rangle$	[Orl91]	expl. descr.[2] or Thm. 6.8
4	$\mathrm{D}^{\mathrm{b}}(Y_4) = \left\langle \mathrm{D}^{\mathrm{b}}(C_2), \mathcal{O}(-H), \mathcal{O} \right\rangle$ [3]	[BO95, Thm. 2.9]	expl. descr. or Thm. 6.8
3	$\mathrm{D}^{\mathrm{b}}(Y_3) = \langle \mathcal{K}u(Y_3), \mathcal{O}(-H), \mathcal{O} \rangle$		[BMMS12] or Thm. 6.8
2	$\mathrm{D}^{\mathrm{b}}(Y_2) = \langle \mathcal{K}u(Y_2), \mathcal{O}(-H), \mathcal{O} \rangle$		Thm. 6.8
1	$\mathrm{D}^{\mathrm{b}}(Y_1) = \langle \mathcal{K}u(Y_1), \mathcal{O}(-H), \mathcal{O} \rangle$		Thm. 6.8

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[2] By *explicit description (expl. descr.)*, we mean that the explicit description of the Kuznetsov component, combined with the construction of stability conditions for curves and categories of quiver representations, implies the existence of stability conditions.

[3] We denote by C_g a smooth genus g curve.

$\rho_X = 1$ & $i_X = 1$			
g_X	Semiorthogonal decomposition		$\exists$ stability conditions
12	$\mathrm{D}^{\mathrm{b}}(X_{22}) = \langle \mathcal{E}_4, \mathcal{E}_3, \mathcal{E}_2, \mathcal{O} \rangle$	[Kuz09a, Thm. 4.1]	expl. descr. or Thm. 6.9
10	$\mathrm{D}^{\mathrm{b}}(X_{18}) = \left\langle \mathrm{D}^{\mathrm{b}}(C_2), \mathcal{E}_2, \mathcal{O} \right\rangle$	[Kuz06, §6.4]	expl. descr. or Thm. 6.9
9	$\mathrm{D}^{\mathrm{b}}(X_{16}) = \left\langle \mathrm{D}^{\mathrm{b}}(C_3), \mathcal{E}_3, \mathcal{O} \right\rangle$	[Kuz06, §6.3]	expl. descr.
8	$\mathrm{D}^{\mathrm{b}}(X_{14}) = \langle \mathcal{K}u(X_{14}), \mathcal{E}_2, \mathcal{O} \rangle$	[Kuz04]	[BMMS12] or Thm. 6.9
7	$\mathrm{D}^{\mathrm{b}}(X_{12}) = \left\langle \mathrm{D}^{\mathrm{b}}(C_7), \mathcal{E}_5, \mathcal{O} \right\rangle$	[Kuz06, §6.2]	expl. descr.
6	$\mathrm{D}^{\mathrm{b}}(X_{10}) = \langle \mathcal{K}u(X_{10}), \mathcal{E}_2, \mathcal{O} \rangle$	[Kuz09a, Lem. 3.6]	Thm. 6.9
5	$\mathrm{D}^{\mathrm{b}}(X_8) = \langle \mathcal{K}u(X_8), \mathcal{O} \rangle$		Thm. 6.7
4	$\mathrm{D}^{\mathrm{b}}(X_6) = \langle \mathcal{K}u(X_6), \mathcal{E}_2, \mathcal{O} \rangle$	[Kuz09a, Lem. 3.6]	Unknown[4]
3	$\mathrm{D}^{\mathrm{b}}(X_4) = \langle \mathcal{K}u(X_4), \mathcal{O} \rangle$		Thm. 6.7
2	$\mathrm{D}^{\mathrm{b}}(X_2) = \langle \mathcal{K}u(X_2), \mathcal{O} \rangle$		Thm. 6.7

For Fano threefolds of Picard rank one, there is a conjectural relation between the index two case of degree d and the index one and genus $2d + 2$, due to Kuznetsov (see [Kuz09a, Conjecture 3.7][5]). This is proved in the case $d = 3, 4, 5$ and asserts there the equivalence between the respective Kuznetsov components. In fact, our result may turn useful in understanding this conjecture, as explained in the following example.

Example 6.6 ($d = 4$)**.** The space of Bridgeland stability conditions on $\mathcal{K}u(X_{18}) \cong \mathrm{D}^{\mathrm{b}}(C_2)$ consists of a unique orbit, with respect to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action, containing $(\mathrm{Coh}(C_2), \mathfrak{i}\,\mathrm{rk} - \deg)$ (see [Mac07]). In particular, the stability condition σ constructed in Theorem 1.1 lies in the same orbit.

The curve C_2 can be reconstructed as moduli space of skyscraper sheaves, which are stable with respect to any stability condition on $\mathrm{D}^{\mathrm{b}}(C_2)$. Hence, C_2 can be identified with the moduli space of σ-stable objects in $\mathcal{K}u(X_{18})$ with Chern character $3 - 2H + 9L - \frac{1}{2}\mathrm{pt}$, which is the image of the Chern character of a skyscraper sheaf via the inclusion $K_{\mathrm{num}}(C_2) \cong K_{\mathrm{num}}(\mathcal{K}u(X_{18})) \subset K_{\mathrm{num}}(X_{18})$ (see [Kuz09a, Proposition 3.9]). The equivalence $\mathcal{K}u(X_{18}) \cong \mathrm{D}^{\mathrm{b}}(C_2)$ can then be reinterpreted as the one coming from the universal family on such a moduli space.

The Fano threefold Y_4 can then be reconstructed as the moduli space of rank 2 vector bundles on C_2 with fixed determinant of odd degree (see [New68, Theorem 1] or [NR69, Theorem 4]). The equivalence $\mathrm{D}^{\mathrm{b}}(C_2) \cong \mathcal{K}u(Y_4)$ can again be reinterpreted as the one coming from the universal family (see [BO95, Theorem 2.7] and [Nar17, Remark 5]).

Proof of Theorem 1.1, case index one and low genus. We divide the proof of the Theorem 1.1 in three cases, according to the index and the genus. We begin with the easiest case. We will prove the following more general statement, which holds for all Fano threefolds, of any Picard number and index.

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[4] Chunyi Li has informed us that he will be able to prove this remaining case in [Li17]; see Remark 6.12.

[5] For $d = 1, 2$ the conjecture in *loc. cit.* needs to be modified as remarked in [BT16, Theorem 7.2]. A modified version does not ask the maps to be dominant, so the two Kuznetsov components would only be deformation equivalent [Kuz15].

Theorem 6.7. *Let X be a Fano threefold. Consider the semiorthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{O}_X^{\perp}, \mathcal{O}_X \rangle.$$

Then $\mathcal{O}_X^{\perp}$ has a Bridgeland stability condition with respect to the lattice $\Lambda_H^2 \cong \mathbb{Z}^3$.[6]

In particular, if X has index 1 and genus $g \in \{2, 3, 5\}$, then $\mathcal{K}u(X)$ has a stability condition.

Proof. We want to apply Proposition 5.1 to the exceptional collection of length one given by $\mathcal{O}_X$.

For $\mathrm{Coh}(X)$, we have $\mathcal{O}_X \in \mathrm{Coh}(X)$, but $S(\mathcal{O}_X) = \mathcal{O}_X(K_X)[3] \in \mathrm{Coh}(X)[3]$. We need a weak stability condition whose heart still contains $\mathcal{O}_X$, but also $\mathcal{O}_X(K_X)[2] = \mathcal{O}_X(-i_X H)[2]$ instead of $\mathcal{O}_X(K_X)$, i.e., we will need to tilt $\mathrm{Coh}(X)$ twice.

Let $\beta = -\frac{1}{2} i_X$. Then clearly $\mathcal{O}_X, \mathcal{O}_X(K_X)[1] \in \mathrm{Coh}_H^{\beta}(X)$. Now consider the weak stability condition $\sigma_{\alpha,\beta} = \left(\mathrm{Coh}_H^{\beta}(X), Z_{\alpha,\beta}\right)$ of Proposition 2.11, for β as above and for α sufficiently small. By Proposition 2.13, both $\mathcal{O}_X$ and $\mathcal{O}_X(K_X)[1]$ are $\sigma_{\alpha,\beta}$-stable. Since α is sufficiently small, we have

$$\Re Z_{\alpha,\beta}(\mathcal{O}_X) = \frac{\alpha^2}{2} H^3 - H \cdot \frac{1}{2}\left(\frac{K_X}{2}\right)^2 \quad < 0 < \quad \Re Z_{\alpha,\beta}(\mathcal{O}_X(K_X)[1]) = -\frac{\alpha^2}{2} H^3 + H \cdot \frac{1}{2}\left(\frac{K_X}{2}\right)^2$$

$$\text{and so} \quad \mu_{\alpha,\beta}(\mathcal{O}_X(K_X)[1]) \quad < 0 < \quad \mu_{\alpha,\beta}(\mathcal{O}_X). \tag{3}$$

Therefore, if we tilt a second time to obtain the weak stability condition $\sigma_{\alpha,\beta}^0$ of Proposition 2.14, then its heart $\mathrm{Coh}_{\alpha,\beta}^0(X)$ contains both $\mathcal{O}_X$ and $\mathcal{O}_X(K_X)[2]$.

By Lemma 2.15, there cannot be any objects $F \in \mathrm{Coh}_{\alpha,\beta}^0(X)_0$ that are also in $\mathcal{O}_X^{\perp}$ (indeed, any zero-dimensional torsion sheaf has global sections). Therefore, all assumptions of Proposition 5.1 are satisfied, and we obtain a stability condition on $\mathcal{O}_X^{\perp}$. □

Proof of Theorem 1.1, case index two. In this section we prove the following case of Theorem 1.1 without the Picard rank one assumption.

Theorem 6.8. *Let X be a Fano threefold of index two. Then the Kuznetsov component $\mathcal{K}u(X)$ has a Bridgeland stability condition with respect to the lattice $\Lambda_{H,\mathcal{K}u(X)}^2 \cong \mathbb{Z}^2$.*

Proof. The proof goes along the exact same lines as the proof of Theorem 6.7, by applying Proposition 5.1 to the exceptional collection $\langle \mathcal{O}_X, \mathcal{O}_X(H) \rangle$. We set $\beta = -\frac{1}{2}$; then again $G, G(K_X)[1] \in \mathrm{Coh}_H^{\beta}(X)$ for $G \in \{\mathcal{O}_X, \mathcal{O}_X(H)\}$. The exact same computation leading to (3) shows that, for α small, we have

$$\mu_{\alpha,-\frac{1}{2}}(G(K_X)[1]) < 0 < \mu_{\alpha,-\frac{1}{2}}(G)$$

for the same G. With the same arguments as before, we apply Proposition 5.1 to show that the weak stability condition $\sigma_{\alpha,-\frac{1}{2}}^0$ of Proposition 2.14 induces a stability condition on $\mathcal{K}u(X)$. □

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[6] Note that in this case the lattice $\Lambda_{H,\mathcal{O}_X^{\perp}}^2$ defined in (2) coincides with Λ_H^2.

Proof of Theorem 1.1, case index one and high genus. The remaining non-trivial cases of Theorem 1.1 are covered by the following result:

Theorem 6.9. *Let X be a Fano threefold of Picard rank one, index one, and even genus $g \geq 6$. Then the Kuznetsov component $\mathcal{K}u(X)$ has a Bridgeland stability condition with respect to the lattice $\Lambda^2_{H,\mathcal{K}u(X)} \cong \mathbb{Z}^2$.*

We want to apply the same proof as in the previous cases. Note that $\mathcal{E}_2$ is slope-stable with $\mu_H(\mathcal{E}_2) = -\frac{1}{2}$, whereas $S(\mathcal{O}_X) = \mathcal{O}_X(-H)[3]$, $S(\mathcal{E}_2) = \mathcal{E}_2(-H)[3]$ are shifts of slope-stable sheaves of slope -1 and $-\frac{3}{2}$, respectively. Therefore, the first step works exactly as before: for any β with $-1 < \beta < -\frac{1}{2}$, the abelian category $\mathrm{Coh}^\beta(X)$ contains $\mathcal{O}_X, \mathcal{E}_2$ as well as $\mathcal{O}_X(-H)[1]$ and $\mathcal{E}_2(-H)[1]$.

To continue as before, we need to show tilt-stability of $\mathcal{E}_2$; the corresponding statement was automatic in the previous cases by Proposition 2.13. We start with an auxiliary observation. Recall from Section 2 that we have a quadratic form Δ_H on $\Lambda^2_H \otimes \mathbb{R} \cong \mathbb{R}^3$.

Lemma 6.10. *Consider the tangent planes to the quadric $\Delta_H = 0$ in $\mathbb{R}^3$ at $v^2_H(\mathcal{O}_X)$ and $v^2_H(\mathcal{O}_X(-H))$. If $g \geq 6$ is an even genus, then $v^2_H(\mathcal{E}_2)$ lies on the interior of these two tangent planes.*

By *interior* we mean the open half-space containing the part of the negative cone with classes of positive rank, see also Figure 3.

Proof. By symmetry, the intersection of these two tangent planes is also contained in the plane $\mu_H(\_) = \frac{1}{2}$ containing $v^2_H(\mathcal{E}_2)$; therefore, it is enough to prove the claim for one of the two planes.

The tangent plane to $\Delta_H = 0$ at $v^2_H(\mathcal{O}_X)$ is given by $H\,\mathrm{ch}_2(\_) = 0$, and thus the claim is immediate from $\mathrm{ch}_2(\mathcal{E}_2) = \left(\frac{g}{2} - 2\right) L$, see Theorem 6.2. □

Lemma 6.11. *For $\epsilon > 0$ sufficiently small, the objects $\mathcal{E}_2$ and $\mathcal{E}_2(-H)[1]$ are $\sigma_{\alpha,-1+\epsilon}$-stable for all $\alpha > 0$.*

Proof. We first observe that both objects are $\sigma_{\alpha,-1}$-stable for all $\alpha > 0$. Indeed, for $\alpha \gg 0$, this follows from slope-stability of $\mathcal{E}_2$ and Proposition 2.12. Moreover, since

$$\Im Z_{\alpha,-1}(\mathcal{E}_2) = H^2\,\mathrm{ch}_1^{-1}(\mathcal{E}_2) = H^3 = -H^2\,\mathrm{ch}_1^{-1}(\mathcal{E}_2(-H)) = \Im Z_{\alpha,-1}(\mathcal{E}_2(-H)[1]),$$

and since $\Im Z_{\alpha,-1}(F) \in \mathbb{Z}_{\geq 0} \cdot H^3$ for all objects $F \in \mathrm{Coh}^{-1}(X)$, neither object can be strictly $\sigma_{\alpha,-1}$-semistable for any $\alpha > 0$; by the existence of the wall-and-chamber structure for tilt-stability, this means they must be $\sigma_{\alpha,-1}$-stable for all $\alpha > 0$.

In the case of $\mathcal{E}_2(-H)[1]$, combined with the previous Lemma 6.10, this already suffices to prove the statement: any wall intersecting the line segment of slope $\mu_H = -1 + \epsilon$ would also intersect the line segment $\mu_H = -1$.

Now consider the location of possible walls for $\sigma_{\alpha,\beta}$-semistability of $\mathcal{E}_2$, as in Figure 4; in this picture, they are given as the intersection of lines through $v^2_H(\mathcal{E}_2)$ with the interior of the negative cone $\Delta_H(\_) < 0$. By the argument in the previous paragraph, no such wall can be in the interior of the triangle with

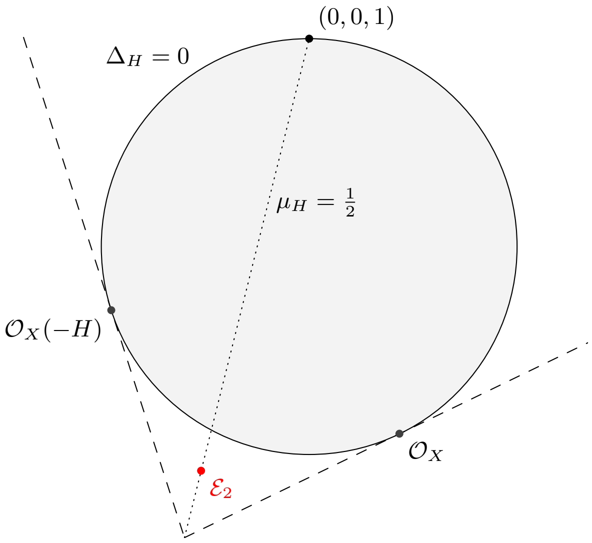

FIGURE 3. The location of $v_H(\mathcal{E}_2)$

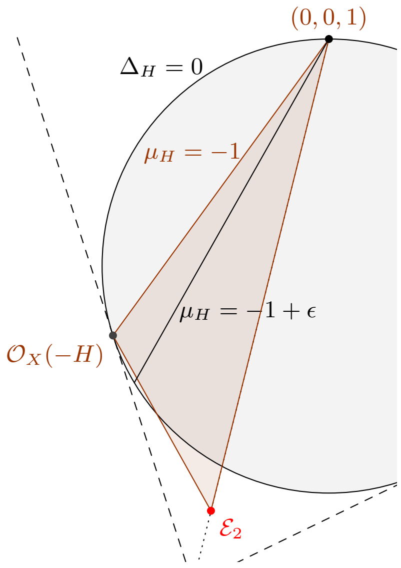

FIGURE 4. (Lack of) Walls for $\mathcal{E}_2$

vertices $v_H^2(\mathcal{E}_2)$, $v_H^2(\mathcal{O}_X(-H))$ and $(0,0,1)$. By the local finiteness of walls, it suffices to prove that the line segment connecting $v_H^2(\mathcal{E}_2)$ and $v_H^2(\mathcal{O}_X(-H))$ is not a wall.

Assume otherwise. Then there is a short exact sequence $A \hookrightarrow \mathcal{E}_2 \twoheadrightarrow B$ such that when (α,β) lies on the wall, then $Z_{\alpha,\beta}(A)$ and $Z_{\alpha,\beta}(B)$ lie on the open line segment connecting 0 and $Z_{\alpha,\beta}(E)$ in the complex plane. By continuity, this still holds at the end point $(\alpha,\beta) = (0,-1)$; with the same integrality argument as before, we conclude either $Z_{0,-1}(A) = 0$ or $Z_{0,-1}(B) = 0$; in particular, $v_H^2(A)$ or $v_H^2(B)$ are proportional to $v_H^2(\mathcal{O}_X(-H))$, respectively. By Proposition 2.13, we must have $A \cong \mathcal{O}_X(-H)[1]$ or $B \cong \mathcal{O}_X(-H)[1]$. But both of these are impossible: we clearly have $\mathrm{Hom}(\mathcal{O}_X(-H)[1], \mathcal{E}_2) = 0$, and, by Serre duality and the fact that $(\mathcal{O}_X, \mathcal{E}_2)$ is an exceptional pair, also $\mathrm{Hom}(\mathcal{E}_2, \mathcal{O}_X(-H)[1]) = \mathrm{Hom}(\mathcal{O}_X, \mathcal{E}_2[2])^\vee = 0$. □

Proof of Theorem 6.9. We have all the ingredients in place to apply Proposition 5.1. Indeed, for $\beta = -1+\epsilon$ and $\alpha > 0$ sufficiently small, the objects $\mathcal{O}_X, \mathcal{E}_2, \mathcal{O}_X(-H)[1], \mathcal{E}_2(-H)[1]$ of $\mathrm{Coh}^\beta(X)$ are $\sigma_{\alpha,\beta}$-stable; one also easily checks with a computation, or a picture using Lemma 6.10, that

$$\mu_{\sigma_{\alpha,\beta}}(\mathcal{E}_2(-H)[1]) < \mu_{\sigma_{\alpha,\beta}}(\mathcal{O}_X(-H)[1]) < \mu_{\sigma_{\alpha,\beta}}(\mathcal{E}_2) < \mu_{\sigma_{\alpha,\beta}}(\mathcal{O}_X).$$

Therefore, for μ in between the second and the third slope in these inequalities, the tilted category $\mathrm{Coh}^\mu_{\alpha,\beta}(X)$ contains all of $\mathcal{O}_X, \mathcal{E}_2, \mathcal{O}_X(-H)[2]$ and $\mathcal{E}_2(-H)[2]$. Thus the weak stability condition $(\mathrm{Coh}^\mu_{\alpha,\beta}(X), Z^\mu_{\alpha,\beta})$ of Proposition 2.14 satisfies all the assumptions of Proposition 5.1. □

Remark 6.12. In the remaining case of index one and genus 4, Lemma 6.10 fails to hold: in fact, $v_H(\mathcal{E}_2)$ lies exactly on the intersection of the two tangent planes. Therefore, our methods are insufficient to treat this case without an improvement on the classical Bogomolov-Gieseker inequality (for both slope-stable sheaves and tilt-stable complexes). The key result of [Li17], where this case will be treated, is exactly such an improvement: it allows for a weak stability condition such that the kernel of the associated central charge lies on the line segment connecting $v_H(\mathcal{E}_2)$ and $v_H(\mathcal{O}_X(-H))$.

Relation to Bridgeland stability on $\mathrm{D^b}(X)$. In [Li15], stability conditions have been constructed on the whole category $\mathrm{D^b}(X)$, when X is a Fano threefold of Picard rank one (and in general in [Piy16, BMSZ16]). In particular, the category $\mathrm{Coh}^{\mu}_{\alpha,\beta}(X)$ in Theorems 6.7, 6.8, and 6.9 is the heart of a Bridgeland stability condition on $\mathrm{D^b}(X)$. While this much stronger result is not needed for our construction, it may be useful to compare stable objects in $\mathrm{D^b}(X)$ versus stable objects in $\mathcal{K}u(X)$, in a similar fashion as what has been done in [LMS15, Section 3]. More precisely, in [BMMS12], the Kuznetsov component $\mathcal{K}u(Y_3)$ is realized as an admissible subcategory in $\mathrm{D^b}(\mathbb{P}^2, \mathcal{B}_0)$ orthogonal to the right of an exceptional object (see also Section 7 below). In [LMS15] the comparison is between stable objects in $\mathcal{K}u(Y_3)$ and Bridgeland stable objects in $\mathrm{D^b}(\mathbb{P}^2, \mathcal{B}_0)$.

7. CONIC FIBRATIONS ASSOCIATED TO CUBIC FOURFOLDS

In this section, we start the study of the Kuznetsov component $\mathcal{K}u(X)$ of a cubic fourfold X. In principle, we would like to apply a similar argument as in the Fano threefold case above. To this end, we would need to tilt three times starting from $\mathrm{Coh}(X)$. The issue is the lack of a positivity result, generalizing Bogomolov inequality for stable sheaves to tilt-stable objects, which prevents us to tilt a third time. The key idea then is to realize $\mathcal{K}u(X)$ as an admissible subcategory of a derived category of modules over $\mathbb{P}^3$ with respect to an algebra vector bundle $\mathcal{B}_0$. By choosing a line in X, the induced conic fibration provides $\mathcal{B}_0$ as the even part of the associated Clifford algebra vector bundle.

After a brief recall on Kuznetsov's result on semiorthogonal decompositions for quadric fibrations, the goal of this section is to describe such an embedding (see Proposition 7.7).

Modules over algebra vector bundles. Let Y be a smooth projective variety, and let $\mathcal{B}$ be a sheaf of $\mathcal{O}_Y$-algebras over Y; we will always assume that $\mathcal{B}$ is a locally free sheaf over Y, and call such $\mathcal{B}$ an *algebra vector bundle*. We denote by $\mathrm{Coh}(Y, \mathcal{B})$ the category of coherent sheaves on Y with a right $\mathcal{B}$-module structure, and denote its derived category by $\mathrm{D^b}(Y, \mathcal{B})$. The forgetful functor is denoted by $\mathrm{Forg}\colon \mathrm{D^b}(Y, \mathcal{B}) \to \mathrm{D^b}(Y)$.

Now consider a morphism $f\colon Y' \to Y$, and let $\mathcal{B}' := f^*\mathcal{B}$. Then the usual pull-back and push-forward for coherent sheaves directly induce functors $f^*\colon \mathrm{D^b}(Y, \mathcal{B}) \to \mathrm{D^b}(Y', \mathcal{B}')$ and, when f is projective, $f_*\colon \mathrm{D^b}(Y', \mathcal{B}') \to \mathrm{D^b}(Y, \mathcal{B})$; in other words, these functors commute with the forgetful functors on Y and Y', and the ordinary pull-back and push-forward for coherent sheaves, respectively.

Let $E \in \mathrm{D^b}(Y, \mathcal{B})$. By abuse of notation and language, we will write $\mathrm{ch}(E) = \mathrm{ch}(\mathrm{Forg}(E))$ for the Chern character of the underlying complex of coherent sheaves, and call it the Chern character of E. By the observation in the previous paragraph, the behavior of this Chern character behaves exactly as the Chern character of coherent sheaves under pull-backs and push-forwards whenever we are in the situation above (in particular, with $\mathcal{B}' = f^*\mathcal{B}$).

Since we assume Y to be smooth and $\mathcal{B}$ to be a vector bundle, we can also write the Serre functor on $\mathrm{D}^{\mathrm{b}}(Y,\mathcal{B})$ explicitly as $S(\_) = \omega_Y \otimes_{\mathcal{O}_Y} (\_) \otimes_{\mathcal{B}} \mathcal{B}^{\vee}[\dim(Y)]$, where $\mathcal{B}^{\vee}$ denotes the dual of $\mathcal{B}$ as a coherent sheaf, together with its canonical structure as a $\mathcal{B}$-bimodule. This follows from Serre duality on Y together with the standard adjunctions for the forgetful functor.

Conic fibrations. We now recall Kuznetsov's description of the derived category of quadric fibrations, specialized to the case of relative dimension one. So let $\pi\colon X \to Y$ be a fibration in conics over a smooth projective variety Y. There is a rank three vector bundle $\mathcal{F}$ on Y and a line bundle $\mathcal{L}$ such that X embeds into the $\mathbb{P}^2$-bundle $\mathbb{P}_Y(\mathcal{F})$ as the zero locus of a section $s_X \in H^0(Y, \operatorname{Sym}^2 \mathcal{F}^{\vee} \otimes \mathcal{L}^{\vee}) = H^0(\mathbb{P}_Y(\mathcal{F}), \mathcal{O}_{\mathbb{P}_Y(\mathcal{F})}(2) \otimes \mathcal{L}^{\vee})$.

In this setting, the even part of the Clifford algebra of π as a sheaf is

$$\operatorname{Forg}(\mathcal{B}_0) = \mathcal{O}_Y \oplus \textstyle\bigwedge^2 \mathcal{F} \otimes \mathcal{L}. \tag{4}$$

Its algebra structure is determined by

$$v_i \wedge v_k \cdot v_k \wedge v_j = s_X(v_k \otimes v_k) v_i \wedge v_j, \quad v_i \wedge v_k \cdot v_i \wedge v_k = s_X(v_i \otimes v_i) s_X(v_k \otimes v_k),$$

for an orthogonal basis (v_1, v_2, v_3) and $i \neq j \neq k \neq i$.

The odd part of the Clifford algebra of π is denoted by $\mathcal{B}_1$. Furthermore, we define the following $\mathcal{B}_0$-bimodules, for $j \in \mathbb{Z}$:

$$\mathcal{B}_{2j} := \mathcal{B}_0 \otimes \mathcal{L}^{-j} \text{ and } \mathcal{B}_{2j+1} := \mathcal{B}_1 \otimes \mathcal{L}^{-j}.$$

The fundamental result on the derived categories of quadric fibrations is the following:

Theorem 7.1 ([Kuz08])**.** *There is a semiorthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \Phi(\mathrm{D}^{\mathrm{b}}(Y, \mathcal{B}_0)), \pi^* \mathrm{D}^{\mathrm{b}}(Y) \rangle.$$

We will describe the fully faithful functor $\Phi\colon \mathrm{D}^{\mathrm{b}}(Y,\mathcal{B}_0) \to \mathrm{D}^{\mathrm{b}}(X)$ and its left adjoint Ψ explicitly below.

Consider a base change $f\colon Y' \to Y$. Then there are two algebras on Y': the Clifford algebra $\mathcal{B}'_0$ of the conic fibration $\pi'\colon Y' \times_Y X \to Y'$, and the pull-back $f^*\mathcal{B}_0$.

Lemma 7.2. *In the above situation, we have a natural isomorphism $\mathcal{B}'_0 \cong f^*\mathcal{B}_0$.*

Proof. This is observed in the proof of [Kuz08, Lemma 3.2]. □

Cubic hypersurfaces and conic fibrations. Let $N \geq 1$ and let $X \subset \mathbb{P}^{N+2}$ be a smooth cubic hypersurface of dimension $N+1$. We can associate to X a conic fibration as follows. Let $L_0 \subset X$ be a line. Consider the blow-up $\sigma\colon \widetilde{X} \to X$ along L_0, and denote by $i\colon D \hookrightarrow \widetilde{X}$ its exceptional divisor. Then the projection from L_0 onto the projective space $\mathbb{P}^N$ induces a conic fibration $\pi\colon \widetilde{X} \to \mathbb{P}^N$ whose discriminant locus is a hypersurface of degree 5. We denote by $\alpha\colon \widetilde{X} \hookrightarrow \widetilde{\mathbb{P}}$ the embedding into the

$\mathbb{P}^2$-bundle $q\colon \widetilde{\mathbb{P}} \to \mathbb{P}^N$, where $\widetilde{\mathbb{P}}$ is the blow-up of $\mathbb{P}^{N+2}$ along L_0. Summarizing, we have the following diagram:

(5)
$$\begin{tikzcd}
 & D \arrow[r, "i"] \arrow[dl, "p"'] & \widetilde{X} \arrow[r, "\alpha"] \arrow[dl, "\sigma"'] \arrow[drr, "\pi"] & \widetilde{\mathbb{P}} \arrow[dl] \arrow[dr, "q"] & \\
L_0 \arrow[r] & X \arrow[r] & \mathbb{P}^{N+2} & & \mathbb{P}^N.
\end{tikzcd}$$

Remark 7.3. Take a generic hyperplane $\mathbb{P}^{N-1} \hookrightarrow \mathbb{P}^N$. The restriction of the conic fibration π to $\mathbb{P}^{N-1}$ is the conic fibration obtained by blowing up along L_0 the smooth cubic hypersurface of dimension N obtained by intersecting X with the $\mathbb{P}^{N+1}$ spanned by L_0 and $\mathbb{P}^{N-1}$.

We will abuse notation and denote by H (resp. by h) both the class of the hyperplane in $\mathbb{P}^{N+2}$ (resp. $\mathbb{P}^N$) and the pull-back to $\widetilde{X}$ and to $\widetilde{\mathbb{P}}$.

In the notation of the previous section, we then have $\widetilde{\mathbb{P}} = \mathbb{P}(\mathcal{F})$, where $\mathcal{F} = \mathcal{O}_{\mathbb{P}^N}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^N}(-h)$; the line bundle $\mathcal{L}$ is $\mathcal{O}_{\mathbb{P}^N}(-h)$. The forgetful sheaf $\mathrm{Forg}(\mathcal{B}_0)$ in (4) of the even part $\mathcal{B}_0$ of the Clifford algebra of π is

$$\mathrm{Forg}(\mathcal{B}_0) \cong \mathcal{O}_{\mathbb{P}^N} \oplus \mathcal{O}_{\mathbb{P}^N}(-h) \oplus \mathcal{O}_{\mathbb{P}^N}(-2h)^{\oplus 2},$$

while the odd part $\mathcal{B}_1$ is

$$\mathrm{Forg}(\mathcal{B}_1) \cong \mathcal{F} \oplus \textstyle\bigwedge^3 \mathcal{F} \otimes \mathcal{L} \cong \mathcal{O}_{\mathbb{P}^N}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^N}(-h) \oplus \mathcal{O}_{\mathbb{P}^N}(-2h).$$

We can now define the functors Φ and Ψ of Theorem 7.1. There is a canonical map of left $q^*\mathcal{B}_0$-modules $q^*\mathcal{B}_0 \to q^*\mathcal{B}_1(H)$, which is injective and its cokernel is supported on $\widetilde{X}$. Twisting by $\mathcal{O}_{\widetilde{\mathbb{P}}}(-2H)$, we obtain an exact sequence

$$0 \to q^*\mathcal{B}_0(-2H) \to q^*\mathcal{B}_1(-H) \to \alpha_*\mathcal{E}' \to 0,$$

where $\mathcal{E}'$ is a sheaf of left $\pi^*\mathcal{B}_0$-modules on $\widetilde{X}$ and $\mathrm{Forg}(\mathcal{E}')$ is a vector bundle of rank 2. The functor $\Phi\colon \mathrm{D}^{\mathrm{b}}(\mathbb{P}^N, \mathcal{B}_0) \to \mathrm{D}^{\mathrm{b}}(\widetilde{X})$ is defined as:

$$\Phi(\_) = \pi^*(\_) \otimes_{\pi^*\mathcal{B}_0} \mathcal{E}'.$$

The left adjoint functor of Φ is

$$\Psi(\_) := \pi_*(\_ \otimes \mathcal{O}_{\widetilde{X}}(h) \otimes \mathcal{E}[1]),$$

where $\mathcal{E}$ is a sheaf of right $\pi^*\mathcal{B}_0$-modules on $\widetilde{X}$ and $\mathrm{Forg}(\mathcal{E})$ is a vector bundle of rank 2, defined by the following short exact sequence of $q^*\mathcal{B}_0$-modules:

$$0 \to q^*\mathcal{B}_{-1}(-2H) \to q^*\mathcal{B}_0(-H) \to \alpha_*\mathcal{E} \to 0. \tag{6}$$

The Kuznetsov component of a cubic fourfold. We can now describe the Kuznetsov component of a cubic fourfold as an admissible subcategory in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$.

Definition 7.4. Let X be a cubic fourfold. The Kuznetsov component $\mathcal{K}u(X)$ of X is defined by the semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle .$$

We fix a line $L_0 \subset X$ and keep the notation as in the previous section. We start by describing a fully faithful functor $\Xi \colon \mathcal{K}u(X) \to \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ in Lemma 7.6 below. The semiorthogonal complement is then described in Proposition 7.7. The functor Ξ will depend on the choice of L_0.

In the proof of Lemma 7.6 we will use several times the following elementary lemma, whose statement and proof are analogous to [Kuz10, Lemma 4.1].

Lemma 7.5. *We have linear equivalences in $\widetilde{X}$:*

$$D = H - h, \qquad K_{\widetilde{X}} = -3H + 2D = -H - 2h.$$

Theorem 7.1 gives the following semiorthogonal decomposition:

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle \Phi(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)), \underbrace{\mathcal{O}_{\widetilde{X}}(-h), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(2h)}_{\pi^* \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3)} \rangle. \tag{7}$$

Set $\Phi' := \mathbf{R}_{\mathcal{O}_{\widetilde{X}}(-h)} \circ \Phi$.

Lemma 7.6. *The admissible subcategory $\Phi'(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)) \subset \mathrm{D}^{\mathrm{b}}(\widetilde{X})$ has a semiorthogonal decomposition*

$$\Phi'(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)) = \left\langle \sigma^* \mathcal{K}u(X), \mathcal{O}_{\widetilde{X}}(h - H), \mathbf{L}_{\mathcal{O}} \mathbf{L}_{\mathcal{O}(h)} \mathcal{O}_{\widetilde{X}}(H), \mathbf{L}_{\mathcal{O}} \mathbf{L}_{\mathcal{O}(h)} i_* \mathcal{O}_D(h) \right\rangle ,$$

which induces a fully faithful embedding:

$$\Xi = \Phi^{-1} \circ \mathbf{L}_{\mathcal{O}_{\widetilde{X}}(-h)} \circ \sigma^* \colon \mathcal{K}u(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0).$$

Proof. In view of (7), the derived category $\mathrm{D}^{\mathrm{b}}(\widetilde{X})$ has the following semiorthogonal decompositions

$$\begin{aligned} \mathrm{D}^{\mathrm{b}}(\widetilde{X}) &= \langle \mathcal{O}_{\widetilde{X}}(-h), \Phi'(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(2h) \rangle \\ &= \langle \Phi'(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(2h), \mathcal{O}_{\widetilde{X}}(H + h) \rangle, \end{aligned} \tag{8}$$

where the second one is obtained via Serre duality.

Since $\widetilde{X}$ is the blow-up of X along L_0, we can apply [Orl92] and get the following semiorthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(\widetilde{X})$ (here we use the notation in (5))

$$\langle \overbrace{\mathcal{O}_{\widetilde{X}}(-H), \sigma^* \mathcal{K}u(X), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(H)}^{\sigma^* \mathrm{D}^{\mathrm{b}}(X)}, \overbrace{i_* \mathcal{O}_D, i_* \mathcal{O}_D(H)}^{\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1) \rangle}, \overbrace{i_* \mathcal{O}_D(H - D), i_* \mathcal{O}_D(2H - D)}^{\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1) = \langle \mathcal{O}(1), \mathcal{O}(2) \rangle} \rangle \tag{9}$$

The pair $(\mathcal{O}_{\widetilde{X}}, i_* \mathcal{O}_D(-H + mh))$ is completely orthogonal, for $m = -1, 0$. Indeed, first of all we observe that $\sigma_* \mathcal{O}_{\widetilde{X}}(-D) = I_{L_0}$ and $\sigma_* \mathcal{O}_{\widetilde{X}} = \sigma_* \mathcal{O}_{\widetilde{X}}(D) = \mathcal{O}_X$. Since $D = H - h$, we then have

$\mathrm{Ext}^\bullet(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(-2H+mh)) = 0$, for $m = 0, 1$, and $\mathrm{Ext}^\bullet(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(-H+mh)) = 0$, for $m = -1, 0, 1$. This gives the orthogonality property, as we wanted. As a consequence, from (9) we get

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle \mathcal{O}_{\widetilde{X}}(-H), \sigma^*\mathcal{K}u(X), \mathcal{O}_{\widetilde{X}}, i_*\mathcal{O}_D, \mathcal{O}_{\widetilde{X}}(H), i_*\mathcal{O}_D(H), i_*\mathcal{O}_D(h), i_*\mathcal{O}_D(H+h)\rangle. \tag{10}$$

Observe now that we have the following equalities

$$\mathbf{L}_{\mathcal{O}_{\widetilde{X}}}(i_*\mathcal{O}_D) \cong \mathcal{O}_{\widetilde{X}}(h-H)[1] \qquad \text{and} \qquad \mathbf{R}_{\mathcal{O}_{\widetilde{X}}(h-H)}(i_*\mathcal{O}_D) \cong \mathcal{O}_{\widetilde{X}}. \tag{11}$$

Indeed, note that $\mathrm{Ext}^k(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(H-h)) = 0$ if $k \neq 0$ and $\mathrm{Hom}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(H-h)) \cong \mathbb{C}$. By definition of right mutation, we have the following distinguished triangle

$$\mathbf{R}_{\mathcal{O}_{\widetilde{X}}}(\mathcal{O}_{\widetilde{X}}(h-H)) \to \mathcal{O}_{\widetilde{X}}(h-H) \to \mathcal{O}_{\widetilde{X}}.$$

Since $H-h = D$, the last map in this triangle is given by the equation of D. Thus $\mathbf{R}_{\mathcal{O}_{\widetilde{X}}}(\mathcal{O}_{\widetilde{X}}(h-H)) \cong i_*\mathcal{O}_D[-1]$. Equivalently, $\mathbf{L}_{\mathcal{O}_{\widetilde{X}}}(i_*\mathcal{O}_D) \simeq \mathcal{O}_{\widetilde{X}}(h-H)[1]$. For the second isomorphism in (11), note that $\mathrm{Ext}^k(i_*\mathcal{O}_D, \mathcal{O}_{\widetilde{X}}(h-H)) = 0$ if $k \neq 1$ and $\mathrm{Ext}^1(i_*\mathcal{O}_D, \mathcal{O}_{\widetilde{X}}(h-H)) \cong \mathbb{C}$. Again, we consider the distinguished triangle

$$\mathbf{R}_{\mathcal{O}_{\widetilde{X}}(h-H)}(i_*\mathcal{O}_D) \to i_*\mathcal{O}_D \to \mathcal{O}_{\widetilde{X}}(h-H)[1].$$

Since $H-h = D$, we can argue as above and conclude that $\mathbf{R}_{\mathcal{O}_{\widetilde{X}}(h-H)}(i_*\mathcal{O}_D) \cong \mathcal{O}_{\widetilde{X}}$.

Thus, applying twice (11) to (10), we get the following

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle \mathcal{O}_{\widetilde{X}}(-H), \sigma^*\mathcal{K}u(X), \mathcal{O}_{\widetilde{X}}(h-H), \underbrace{\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(H), i_*\mathcal{O}_D(h)}_{\mathcal{D}}, i_*\mathcal{O}_D(H+h)\rangle. \tag{12}$$

By applying mutations in $\mathcal{D}$, we get the semiorthogonal decomposition

$$\mathcal{D} = \langle \mathbf{L}_{\mathcal{O}_{\widetilde{X}}}\mathbf{L}_{\mathcal{O}_{\widetilde{X}}(h)}\mathcal{O}_{\widetilde{X}}(H), \mathbf{L}_{\mathcal{O}_{\widetilde{X}}}\mathbf{L}_{\mathcal{O}_{\widetilde{X}}(h)}i_*\mathcal{O}_D(h), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(h)\rangle. \tag{13}$$

And plugging (13) into (12), we get

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle \mathcal{O}_{\widetilde{X}}(-H), \sigma^*\mathcal{K}u(X), \mathcal{O}_{\widetilde{X}}(h-H), \mathcal{D}, i_*\mathcal{O}_D(H+h)\rangle.$$

We apply Serre duality, and we can rewrite it as

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle \sigma^*\mathcal{K}u(X), \mathcal{O}_{\widetilde{X}}(h-H), \mathcal{D}, i_*\mathcal{O}_D(H+h), \mathcal{O}_{\widetilde{X}}(2h)\rangle. \tag{14}$$

Finally, by applying again (11), we get from (14) the semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle \sigma^*\mathcal{K}u(X), \mathcal{O}_{\widetilde{X}}(h-H), \mathcal{D}, \mathcal{O}_{\widetilde{X}}(2h), \mathcal{O}_{\widetilde{X}}(H+h)\rangle. \tag{15}$$

Comparing the two semiorthogonal decompositions (8) and (15), i.e., comparing

$$\langle \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(2h), \mathcal{O}_{\widetilde{X}}(H+h)\rangle^\perp$$

inside them, we get the desired equivalence. □

We are now ready to prove the main result of this section.

Proposition 7.7. *If $F \in \mathcal{K}u(X)$, then $\Xi(F) = \Psi(\sigma^* F)$. Moreover,*

$$\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0) = \langle \Psi(\sigma^* \mathcal{K}u(X)), \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle .$$

Proof. The projection formula, the fact that $\pi = q \circ \alpha$, and (6) show that

$$\Psi(\mathcal{O}_{\widetilde{X}}(mh)) = 0.$$

This implies immediately the first part of the statement.

Again, a direct computation based on relative Grothendieck-Serre duality shows that

$$\begin{aligned} \Psi(\mathcal{O}_{\widetilde{X}}(mh - H)) &= \mathcal{B}_{-1}(mh) = \mathcal{B}_{-1+2m} \\ \Psi(\mathcal{O}_{\widetilde{X}}(mh + H)) &= \mathcal{B}_2(mh)[1] = \mathcal{B}_{2+2m}[1]. \end{aligned}$$

Hence, as an application of Lemma 7.6, we get the second part of the statement. □

8. A BOGOMOLOV INEQUALITY

To construct Bridgeland stability conditions on the Kuznetsov component of a cubic fourfold, we need to be able to define tilt-stability on the category $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ introduced in the previous section. To this end, we need a Bogomolov inequality for slope-stable torsion-free sheaves in $\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$. This is the content of this section.

Slope-stability. Let Y be a smooth projective variety of dimension n, and let $\mathcal{B}$ be an algebra vector bundle on Y. Let $\underline{D} = \{D_1, \dots, D_{n-1}\}$ be nef divisor classes on Y such that $D_1^2 D_2 \cdots D_{n-1} > 0$, and consider the lattice

$$\Lambda^1_{\underline{D}} = \langle D_1^2 D_2 \cdots D_{n-1} \operatorname{ch}_0,\ D_1 D_2 \cdots D_{n-1} \operatorname{ch}_1 \rangle \cong \mathbb{Z}^2.$$

By Remark 2.5, since $\mathrm{Coh}(Y, \mathcal{B})$ is noetherian, the following slight generalization of Example 2.8 holds:

Proposition and Definition 8.1. *The pair $\sigma_{\underline{D}} = (\mathrm{Coh}(Y, \mathcal{B}), Z_{\underline{D}})$, where*

$$Z_{\underline{D}} = \mathfrak{i}\, D_1^2 D_2 \cdots D_{n-1} \operatorname{ch}_0 - D_1 D_2 \cdots D_{n-1} \operatorname{ch}_1$$

defines a weak stability condition on $\mathrm{D}^{\mathrm{b}}(Y, \mathcal{B})$, which we still call slope stability. *We write $\mu_{\underline{D}}$ for the associated slope function.*

When $D_1 = \cdots = D_{n-2} = D$, we use the notation $Z_{D, D_{n-1}}$ and $\mu_{D, D_{n-1}}$. When moreover $D_{n-1} = D$, we also use the notation Z_D and μ_D, compatibly with Example 2.8.

The main theorem. Let $N \geq 2$. Let X be a smooth cubic hypersurface in $\mathbb{P}^{N+2}$ and let $L \subseteq X$ be a line. Consider the blow-up $\widetilde{X}$ of X along L and the natural conic bundle $\pi\colon \widetilde{X} \to \mathbb{P}^N$. According to the discussion in Section 7, this yields an algebra vector bundle $\mathcal{B}_0$ on $\mathbb{P}^N$.

Definition 8.2. Let $\mathcal{V} = (V_1, \dots, V_m)$ be an ordered configuration of linear subspaces of codimension 2 in $\mathbb{P}^N$. We say that $\psi\colon Y \to \mathbb{P}^N$ is a blow-up along $\mathcal{V}$ if ψ is the iterated blow-up along the strict transforms of the V_j's.

Theorem 8.3. *Let $\psi\colon Y \to \mathbb{P}^N$ be a blow-up along an ordered configuration of codimension 2 linear subspaces. Let $h = \psi^*\mathcal{O}_{\mathbb{P}^N}(1)$. Assume that $E \in \mathrm{Coh}(Y, \psi^*\mathcal{B}_0)$ is a μ_h-slope semistable torsion-free sheaf. Then*

$$\Delta_{\psi^*\mathcal{B}_0}(E) := h^{N-2}\left(\mathrm{ch}_1(E)^2 - 2\,\mathrm{rk}(E)\left(\mathrm{ch}_2(E) - \frac{11}{32}\,\mathrm{rk}(E)\right)\right) \geq 0.$$

The result will be proved in the rest of this section. It is mainly based on the induction on the rank of E, which is a variant of an argument by Langer (see [Lan04, Section 3]). This basically allows a reduction to the case $N = 2$ where we provide the estimate using Grothendieck-Riemann-Roch.

While a general Bogomolov inequality can be proved along the lines of [Lie07, Section 3.2.3] (after reinterpreting $\mathcal{B}_0$-modules as modules over an Azumaya algebra on a root stack over Y, see [Kuz08, Section 3.6]), this will not be strong enough for our argument. In particular, we will need that our inequality is sharp for $\mathcal{B}_j$, for all $j \in \mathbb{Z}$:

Remark 8.4. The rank of an object in $\mathrm{Coh}(Y, \psi^*\mathcal{B}_0)$ is always a multiple of 4: this is part of [BMMS12, Proposition 2.12] for $Y = \mathbb{P}^2$; using Remark 7.3, the general case follows by pushing forward along ψ, followed by restricting to a generic plane $\mathbb{P}^2 \subset \mathbb{P}^N$. In particular $\psi^*\mathcal{B}_j$ is μ_h-stable, for all $j \in \mathbb{Z}$. Moreover, observe that $\Delta_{\psi^*\mathcal{B}_0}(\psi^*\mathcal{B}_j) = 0$.

Blow-ups and the surface case. Let Y be a smooth projective variety of dimension n, and let $\mathcal{B}$ be an algebra vector bundle on Y. Let h be a big and nef divisor class on Y.

Lemma 8.5. *Let $q\colon \widetilde{Y} \to Y$ be the blow up of Y at a smooth codimension 2 subvariety S. Let $E \in \mathrm{Coh}(\widetilde{Y}, q^*\mathcal{B})$ be a torsion-free object.*

(a) *The complex $q_*E \in \mathrm{D}^{\mathrm{b}}(Y, \mathcal{B})$ has two cohomology objects; the sheaf R^0q_*E is torsion-free while R^1q_*E is topologically supported on S.*
(b) $\mathrm{ch}_0(q_*E) = \mathrm{ch}_0(R^0q_*E) = \mathrm{ch}_0(E)$.
(c) $h^{n-1}\,\mathrm{ch}_1(q_*E) = h^{n-1}\,\mathrm{ch}_1(R^0q_*E) = (q^*h)^{n-1}\,\mathrm{ch}_1(E)$.
(d) *If E is μ_{q^*h}-semistable, then R^0q_*E is μ_h-semistable.*

Proof. This follows immediately from the case $\mathcal{B} \cong \mathcal{O}_Y$, since the forgetful functor commutes with push-forward and pull-back. ☐

Assume now that Y has dimension 2.

Lemma 8.6. *Assume that, for any* $F \in \mathrm{Coh}(Y, \mathcal{B})$ *which is* μ_h*-semistable, we have* $\Delta_{\mathcal{B}}(F) \geq 0$ *and let* $q\colon \widetilde{Y} \to Y$ *be the blow-up at a point. Then, for any* $E \in \mathrm{Coh}(\widetilde{Y}, q^*\mathcal{B})$ *which is* μ_{q^*h}*-semistable, we have* $\Delta_{q^*\mathcal{B}}(E) \geq 0$.

Proof. Consider a μ_{q^*h}-semistable $q^*\mathcal{B}$-module E on $\widetilde{Y}$. We want to apply Lemma 8.5(d). We write $\mathrm{ch}_1(E) = q^*l + ae$, where e is the class of the exceptional divisor, and l a divisor class on Y. After tensoring with an appropriate power of $\mathcal{O}_{\widetilde{Y}}(e)$, we may assume that $0 \leq a < \mathrm{rk}(E)$. The relative Todd class of ψ is given by $1 - \frac{e}{2}$. We obtain

$$\begin{aligned} 0 &\leq \Delta_{\mathcal{B}}(R^0q_*E) \leq \Delta_{q^*\mathcal{B}}(q_*E) \\ &= l^2 - 2\,\mathrm{rk}(E)\left(\mathrm{ch}_2(E) + \frac{a}{2} - \frac{11}{32}\,\mathrm{rk}(E)\right) \\ &\leq l^2 - a^2 - 2\,\mathrm{rk}(E)\left(\mathrm{ch}_2(E) - \frac{11}{32}\,\mathrm{rk}(E)\right) = \Delta_{q^*\mathcal{B}}(E), \end{aligned}$$

where the first inequality used the assumption; the second inequality follows from Lemma 8.5(a), the next equality follows from Grothendieck-Riemann-Roch, and the last inequality is due to $0 \leq a < \mathrm{rk}(E)$. □

We can now prove Theorem 8.3 for $N = 2$. Let $\mathcal{B}_0$ be the Clifford algebra bundle associated to a cubic threefold, as in the statement.

Proposition 8.7. *Let* Y *be a smooth projective surface with a birational morphism* $\psi\colon Y \to \mathbb{P}^2$. *Let* E *be a* μ_{ψ^*h}*-semistable* $\psi^*\mathcal{B}_0$*-module on* Y. *Then* $\Delta_{\psi^*\mathcal{B}_0}(E) \geq 0$.

Proof. In view of Lemma 8.6, it is enough to prove the result for $\psi = \mathrm{id}$. In this situation, we have (see [LMS15, Equation (2.2.2)])

$$\chi(E, E) = -\frac{7}{64}\,\mathrm{rk}(E)^2 - \frac{1}{4}\,\mathrm{ch}_1(E)^2 + \frac{1}{2}\,\mathrm{rk}(E)\,\mathrm{ch}_2(E).$$

Since E is μ_h-semistable, $\mathrm{Ext}^2(E, E) = 0$ and we have $\chi(E, E) \leq 1 \leq \frac{r^2}{16}$. Here we have used that the rank of E is always divisible by 4, as already observed in Remark 8.4 (see [BMMS12, Proposition 2.12]).

Thus, we have

$$-\frac{7}{64}\,\mathrm{rk}(F)^2 - \frac{1}{4}\,\mathrm{ch}_1(F)^2 + \frac{1}{2}\,\mathrm{rk}(F)\,\mathrm{ch}_2(F) \leq \frac{r^2}{16},$$

which is what we claimed. □

Deformation of stability. Let $\psi\colon Y \to \mathbb{P}^N$ be a blow-up along an ordered configuration of codimension 2 linear subspaces, and let $h = \psi^*\mathcal{O}_{\mathbb{P}^N}(1)$. Set $\Pi \subset |h|$ a general pencil and let

$$\widetilde{Y} = \{(D, y) \in \Pi \times Y \mid y \in D\} \tag{16}$$

be the incidence variety with projections $p\colon \widetilde{Y} \to \Pi$ and $q\colon \widetilde{Y} \to Y$. Note that q is the blow-up of Y along the base locus of Π which is a smooth codimension 2 subvariety. Moreover, since Π is a general pencil, the composition $\widetilde{\psi} := \psi \circ q \colon \widetilde{Y} \to \mathbb{P}^N$ is again a blow-up along an ordered configuration of codimension-2 linear subspaces.

Let f be the class of a fiber of p and, by abuse of notation, we also denote by h the class of $\widetilde{\psi}^*\mathcal{O}_{\mathbb{P}^N}(1)$ in $\widetilde{Y}$. We consider slope-stability on $\mathrm{Coh}(\widetilde{Y}, \widetilde{\psi}^*\mathcal{B}_0)$ with respect to the divisor classes $h, \overset{N-2}{\dots}, h, h_t$, where $\mathcal{B}_0$ is the Clifford algebra in Theorem 8.3, and $h_t := th + f$, for $t \in \mathbb{R}_{\geq 0}$.

To apply Langer's argument [Lan04, Section 3], we want deduce the positivity of the discriminant of $\mu_{h,f}$-stable sheaves from the analogous positivity for μ_h-stable objects with smaller rank. We prove this by deforming the slope function μ_{h,h_t}.

Lemma 8.8. *Let $t_0 \in \mathbb{R}_{\geq 0}$. Let $E \in \mathrm{Coh}(\widetilde{Y}, \widetilde{\psi}^*\mathcal{B}_0)$ be a $\mu_{h,h_{t_0}}$-semistable torsion-free sheaf and let $E_1, \dots, E_m$ be its Jordan-Hölder $\mu_{h,h_{t_0}}$-stable factors. If $\Delta_{\widetilde{\psi}^*\mathcal{B}_0}(E_j) \geq 0$, for all $j = 1, \dots, m$, then $\Delta_{\widetilde{\psi}^*\mathcal{B}_0}(E) \geq 0$.*

Proof. This follows immediately from [BMS16, Lemma A.6]. Indeed, first of all we observe that the lemma in *loc. cit.* still holds for weak stability conditions, with the same proof. Then, the only thing to check is that $\mathrm{Ker}\, Z_{h,h_{t_0}}$ is negative semi-definite with respect to the quadratic form $\Delta_{\widetilde{\psi}^*\mathcal{B}_0}$. Explicitly, this means the following. Let D be a divisors class on $\widetilde{Y}$ such that $h^{N-2}h_tD = 0$. We need to show that $h^{N-2}D \leq 0$, which follows immediately from the Hodge Index Theorem. □

Proposition 8.9. *Let $E \in \mathrm{Coh}(\widetilde{Y}, \widetilde{\psi}^*\mathcal{B}_0)$ be a $\mu_{h,f}$-semistable torsion-free sheaf. If $\Delta_{\widetilde{\psi}^*\mathcal{B}_0}(A) \geq 0$ for all μ_h-semistable torsion-free sheaves $A \in \mathrm{Coh}(\widetilde{Y}, \widetilde{\psi}^*\mathcal{B}_0)$ with $\mathrm{ch}_0(A) \leq \mathrm{ch}_0(E)$, then $\Delta_{\widetilde{\psi}^*\mathcal{B}_0}(E) \geq 0$.*

Proof. To start with, by Lemma 8.8, we can assume that E is $\mu_{h,f}$-stable. By arguing as in [Lan04, Section 3.6], if a sheaf $F \in \mathrm{Coh}(\widetilde{Y}, \widetilde{\psi}^*\mathcal{B}_0)$ is $\mu_{h,h_{t_0}}$-stable, then it is μ_{h,h_t}-stable, for $t \in \mathbb{R}_{\geq 0}$ sufficiently close to t_0.

We now have two possible situations. If E is μ_{h,h_t}-stable for all $t \geq 0$, then E is μ_h-stable. Hence we can take $A = E$ and apply the assumption, getting the desired inequality.

Let $E_1, \dots, E_m$ be its Jordan-Hölder factors. For each of them we can apply the same argument. Since whenever we replace E with its Jordan-Hölder factors the rank drops, in a finite number of steps we get to a situation where all Jordan-Hölder factors are μ_{h,h_t}-stable, for all t large. Hence they are μ_h-stable and, by assumption, they all satisfy the inequality $\Delta_{\widetilde{\psi}^*\mathcal{B}_0} \geq 0$. By applying Lemma 8.8 again, we conclude the proof. □

Induction on the rank. Let $\psi \colon Y \to \mathbb{P}^N$ be a blow-up along an ordered configuration of codimension 2 linear subspaces, let $h = \psi^*\mathcal{O}_{\mathbb{P}^N}(1)$. We want to prove Theorem 8.3 by induction on the rank. To this end, as in [Lan04], we consider the following version of Theorem 8.3 with fixed rank:

Theorem 8.3 ($\mathrm{rk}(E) = r$)**.** *Let $E \in \mathrm{Coh}(Y, \psi^*\mathcal{B}_0)$ be a μ_h-slope semistable torsion-free sheaf of rank r. Then $\Delta_{\psi^*\mathcal{B}_0}(E) \geq 0$.*

We also consider the following statement, again with fixed rank:

Theorem 8.10 ($\mathrm{rk}(E) = r$)**.** *Let $E \in \mathrm{Coh}(Y, \psi^*\mathcal{B}_0)$ be a rank r μ_h-slope semistable torsion-free sheaf. Assume that the restriction $E|_D \in \mathrm{Coh}(D, \psi^*\mathcal{B}_0|_D)$ of E to a general divisor $D \in |h|$ is not slope semistable with respect to $h|_D$. Then*

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq \Delta_{\psi^*\mathcal{B}_0}(E),$$

where μ_i (resp., r_i) denote the slopes (resp., the ranks) of the Harder-Narasimhan factors of $E|_D$.

Note that, since h is a linear hyperplane section, $\psi|_D : D \to \mathbb{P}^{N-1}$ is again a blow-up along an ordered configuration of codimension 2 linear subspaces in $\mathbb{P}^{N-1}$, and $\psi^*\mathcal{B}_0|_D = \psi|_D^*(\mathcal{B}_0|_h)$. By Remark 7.3, $\mathcal{B}_0|_h$ on $\mathbb{P}^{N-1}$ is still the sheaf of even parts of the Clifford algebra associated to smooth cubic hypersurface of dimension N.

We will use induction on N, and, in each induction step, induction on the rank (which is divisible by 4 as observed in Remark 8.4). The idea is then to show that Theorem 8.10(r) implies Theorem 8.3(r), and Theorem 8.3($\leq r-4$) implies Theorem 8.10($\leq r$), for all Y at once. The case in which Y is a surface corresponds to Proposition 8.7, while Theorem 8.10($r = 4$) is clear.

Theorem 8.10(r) implies Theorem 8.3(r). Let us assume that E is μ_h-semistable but $\Delta_{\psi^*\mathcal{B}_0}(E) < 0$. Theorem 8.10 implies that the restriction of $E|_D$ is semistable, for any general divisor $D \in |h|$.

By induction on dimension, the restriction of E to a very general complete intersection $Y' = |h| \cap \overset{N-2}{\cdots} \cap |h|$ of dimension 2 is semistable. Then, Proposition 8.7 implies the result. □

The second implication follows line-by-line the argument in [Lan04, Section 3.9]. The only difference is that we cannot do a complete induction as in *loc. cit.*, since such a strong inequality is not necessarily true for arbitrary surfaces. Therefore, we have to use the deformation argument in Proposition 8.9 to reduce to blow-ups of $\mathbb{P}^2$.

Theorem 8.3($\leq r-4$) implies Theorem 8.10($\leq r$). As in the previous section, let Π denote a general pencil in $|h|$ and consider the incidence variety $\widetilde{Y}$ in (16) with projections $p\colon \widetilde{Y} \to \Pi$ and $q\colon \widetilde{Y} \to Y$. We denote by e the class of the exceptional divisor of q and, as before, f the class of the fiber of p. Note that the center of the blow-up q is smooth and connected (for $N = 2$ we use $h^N = 1$).

Note that the Harder-Narasimhan filtration of q^*E with respect to $\mu_{h,f}$ corresponds to the relative Harder-Narasimhan filtration ([HL10, Theorem 2.3.2], generalized to $\mathcal{B}$-modules, with a similar proof) of E with respect to p. Hence, since $E|_D$ is not $\mu_{h|_D}$-semistable, q^*E is not $\mu_{h,f}$-semistable. Therefore, we consider $0 \subset E_0 \subset E_1 \subset \cdots \subset E_m = q^*E$ the (non-trivial) Harder-Narasimhan filtration with respect to $\mu_{h,f}$ and let $F_i = E_i/E_{i-1}$ be the corresponding $\mu_{h,f}$-semistable factors.

There exist integers a_i and divisor classes l_i in Y such that $\mathrm{ch}_1(F_i) = q^*l_i + a_i e$. Then, since $f = h - e$ and $h^{N-2}e^2 = -1$, we have

$$\mu_i = \mu_{h,f}(F_i) = \frac{h^{N-1}l_i + a_i}{r_i}. \tag{17}$$

On the other hand, since $R^0q_*E_i \subset E$ and E is μ_h-semistable, by Lemma 8.5(b),(c), we have

$$\frac{\sum_{j\leq i} h^{N-1}l_j}{\sum_{j\leq i} r_j} \leq \mu_h(E). \tag{18}$$

Hence, by (17) and (18), we deduce

$$\sum_{j\leq i} r_i\left(\mu_j - \mu_h(E)\right) \leq \sum_{j\leq i} a_j. \tag{19}$$

Since $\mathrm{rk}(F_j) \leq r-4$, by Theorem 8.3($\leq r-4$) and Proposition 8.9, we have $\Delta_{\widetilde{\psi}^*\mathcal{B}_0}(F_j) \geq 0$ for all j. Therefore,

$$\begin{aligned}
\frac{\Delta_{\psi^*\mathcal{B}_0}(E)}{r} &= \sum_i \frac{\Delta_{\widetilde{\psi}^*\mathcal{B}_0}(F_i)}{r_i} - \frac{1}{r}\sum_{i<j} r_ir_j\, h^{N-2}\left(\frac{\mathrm{ch}_1(F_i)}{r_i} - \frac{\mathrm{ch}_1(F_j)}{r_j}\right)^2 \\
&\geq \frac{1}{r}\sum_{i<j} r_ir_j\left(\left(\frac{a_i}{r_i} - \frac{a_j}{r_j}\right)^2 - h^{N-2}\left(\frac{l_i}{r_i} - \frac{l_j}{r_j}\right)^2\right) \\
&\geq \frac{1}{r}\sum_{i<j} r_ir_j\left(\left(\frac{a_i}{r_i} - \frac{a_j}{r_j}\right)^2 - \left(\frac{h^{N-1}l_i}{r_i} - \frac{h^{N-1}l_j}{r_j}\right)^2\right),
\end{aligned}$$

where the last inequality follows from the Hodge Index Theorem. By using (17) and simplifying, we see that the last expression in the above inequality is equal to

$$2\sum_i a_i\mu_i - \frac{1}{r}\sum_{i<j} r_ir_j(\mu_i - \mu_j)^2.$$

By (19), we have

$$\begin{aligned}
\sum_i a_i\mu_i &= \sum_i\left(\sum_{j\leq i} a_j\right)(\mu_i - \mu_{i+1}) \geq \sum_i\left(\sum_{j\leq i} r_j(\mu_j - \mu_h(E))\right)(\mu_i - \mu_{i+1}) \\
&= \sum_i r_i\mu_i^2 - r\mu_h(E)^2 = \sum_{i<j}\frac{r_ir_j}{r}(\mu_i - \mu_j)^2.
\end{aligned}$$

Therefore we obtain,

$$\frac{\Delta_{\sigma^*\mathcal{B}_0}(E)}{r} \geq \sum_{i<j}\frac{r_ir_j}{r}(\mu_i - \mu_j)^2,$$

as we wanted. □

9. CUBIC FOURFOLDS

The goal of this section is to prove the existence of Bridgeland stability conditions on the Kuznetsov component $\mathcal{K}u(X)$ of a cubic fourfold X.

We first use the Bogomolov inequality proved in Section 8, to extend the notion of tilt-stability to $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$. Then, Theorem 1.2 will follow by the general method described in Sections 4 and 5 as in the case of Fano threefolds. Finally, we identify the central charge of the associated stability condition with the natural A_2-lattice associated to any cubic fourfold. This will allow us to prove a stronger version of the support property, and so to obtain an open subset of the whole space of full numerical stability conditions on $\mathcal{K}u(X)$.

Weak stability conditions on the twisted projective space. Let $N \geq 2$. Let X be a smooth cubic hypersurface in $\mathbb{P}^{N+2}$ and let $L_0 \subset X$ be a line. Let $\mathcal{B}_0$ be the sheaf of even parts of the Clifford algebra on $\mathbb{P}^N$ associated to the conic fibration induced by projection from L_0, as in Section 7. Let H be an hyperplane section

We modify the Chern character as follows.

Definition 9.1. For $E \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^N, \mathcal{B}_0)$, we set

$$\mathrm{ch}_{\mathcal{B}_0}(E) := \mathrm{ch}(\mathrm{Forg}(E)) \left(1 - \frac{11}{32} H^2\right),$$

where H^2 denotes the class of a codimension 2 linear subspace in $\mathbb{P}^N$.

In particular, note that $\mathrm{ch}_{\mathcal{B}_0}$ differs from the usual Chern character only in degree ≥ 2. We will only care about $\mathrm{ch}_{\mathcal{B}_0,2}$. The Bogomolov inequality in Theorem 8.3 assumes the following more familiar form. For any μ-semistable object $E \in \mathrm{Coh}(\mathbb{P}^N, \mathcal{B}_0)$, we have

$$\Delta_{\mathcal{B}_0}(E) = \mathrm{ch}_{\mathcal{B}_0,1}(E)^2 - 2\,\mathrm{rk}(E)\,\mathrm{ch}_{\mathcal{B}_0,2}(E) \geq 0,$$

where, as usual, we used h to identify the Chern characters on $\mathbb{P}^N$ with rational numbers.

Definition 9.2. We write $\mathrm{Coh}^\beta(\mathbb{P}^N, \mathcal{B}_0)$ for the heart of a bounded t-structure obtained by tilting $\mathrm{Coh}(\mathbb{P}^N, \mathcal{B}_0)$ with respect to slope-stability at the slope $\mu = \beta$.

The following result, generalizing Proposition 2.11, can be proved analogously by using Theorem 8.3. We define first a twisted Chern character $\mathrm{ch}^\beta_{\mathcal{B}_0} := e^{-\beta}\,\mathrm{ch}_{\mathcal{B}_0}$ and a lattice $\Lambda^2_{\mathcal{B}_0} \cong \mathbb{Z}^3$, as in Example 2.8.

Proposition 9.3. *Given $\alpha > 0, \beta \in \mathbb{R}$, the pair $\sigma_{\alpha,\beta} = (\mathrm{Coh}^\beta(\mathbb{P}^N, \mathcal{B}_0), Z_{\alpha,\beta})$ with*

$$Z_{\alpha,\beta}(E) := \mathfrak{i}\,\mathrm{ch}^\beta_{\mathcal{B}_0,1}(E) + \frac{1}{2}\alpha^2\,\mathrm{ch}^\beta_{\mathcal{B}_0,0}(E) - \mathrm{ch}^\beta_{\mathcal{B}_0,2}(E)$$

defines a weak stability condition on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ with respect to $\Lambda^2_{\mathcal{B}_0}$. The quadratic form Q can be given by the discriminant $\Delta_{\mathcal{B}_0}$; these stability conditions vary continuously as $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ varies.

Remark 9.4. By Remark 8.4, for all $j \in \mathbb{Z}$, $\mathcal{B}_j$ is slope-stable with slope $\mu(\mathcal{B}_j) = \frac{-5+2j}{4}$ and $\Delta_{\mathcal{B}_0}(\mathcal{B}_j) = 0$. Hence, $\mathcal{B}_j \in \mathrm{Coh}^\beta(\mathbb{P}^N, \mathcal{B}_0)$ (resp. $\mathcal{B}_j[1] \in \mathrm{Coh}^\beta(\mathbb{P}^N, \mathcal{B}_0)$), for $\beta < \frac{-5+2j}{4}$ (resp. $\beta \geq \frac{-5+2j}{4}$), and by Proposition 2.13, it is $\sigma_{\alpha,\beta}$-stable, for all $\alpha > 0$.

Proof of Theorem 1.2. Let X be a cubic fourfold and let us fix a line $L_0 \subset X$. By Proposition 7.7, we can realize its Kuznetsov component in the following semiorthogonal decomposition:

$$\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0) = \langle \mathcal{K}u(X), \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle .$$

Since we have defined tilt-stability for $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ in Proposition 9.3, the proof of Theorem 1.2 now goes along the exact same lines as the proof of Theorem 6.7, by applying Proposition 5.1 to the exceptional collection $\langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle$.

We set $\beta = -1$. The Serre functor S on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ is given by $S(\_) = \_ \otimes_{\mathcal{B}_0} \mathcal{B}_{-3}[3]$. By [Kuz08, Corollary 3.9], $\mathcal{B}_i \otimes_{\mathcal{B}_0} \mathcal{B}_j \cong \mathcal{B}_{i+j}$, and so $S(\mathcal{B}_j) = \mathcal{B}_{j-3}[3]$. Hence, by Remark 9.4, $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_{-2}[1], \mathcal{B}_{-1}[1], \mathcal{B}_0[1] \in \mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$, and they are $\sigma_{\alpha,-1}$-stable for all $\alpha > 0$.

An easy computation shows that, for α sufficiently small,

$$\mu_{\alpha,-1}(\mathcal{B}_{-2}[1]) < \mu_{\alpha,-1}(\mathcal{B}_{-1}[1]) < \mu_{\alpha,-1}(\mathcal{B}_0[1]) < 0 < \mu_{\alpha,-1}(\mathcal{B}_1) < \mu_{\alpha,-1}(\mathcal{B}_2) < \mu_{\alpha,-1}(\mathcal{B}_3).$$

Therefore, if we tilt a second time to obtain the weak stability condition $\sigma^0_{\alpha,-1}$ (exactly in the same way as in Proposition 2.14), then its heart $\mathrm{Coh}^0_{\alpha,-1}(\mathbb{P}^3, \mathcal{B}_0)$ contains $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_{-2}[2], \mathcal{B}_{-1}[2], \mathcal{B}_0[2]$, for α sufficiently small.

By Lemma 2.15, $F \in (\mathrm{Coh}^0_{\alpha,-1}(\mathbb{P}^3, \mathcal{B}_0))_0$ implies that $\mathrm{Forg}(F)$ is a torsion sheaf supported in dimension 0. But then $F \notin \mathcal{K}u(X)$: indeed, $\mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_j, F) = \mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_0, F) = \mathrm{Hom}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}, \mathrm{Forg}(F))$. Therefore, all assumptions of Proposition 5.1 are satisfied, and we obtain a stability condition $\sigma_{\mathcal{K}u(X)}$ on $\mathcal{K}u(X)$.

This proves Theorem 1.2. We still need to discuss though in detail the support property. Indeed, by Proposition 5.1, we know that the stability condition $\sigma_{\mathcal{K}u(X)}$ in $\mathcal{K}u(X)$ just constructed only satisfies the support property with respect to the lattice $\Lambda^2_{\mathcal{B}_0,\mathcal{K}u(X)} \cong \mathbb{Z}^2$ (defined analogously as in (2)). The numerical Grothendieck group of the Kuznetsov component is larger for any special cubic fourfold, and we will prove that $\sigma_{\mathcal{K}u(X)}$ satisfies the support property for the full numerical Grothendieck group.

The Mukai lattice of the Kuznetsov component of a cubic fourfold. Let X be a cubic fourfold. The Kuznetsov component $\mathcal{K}u(X)$ can be considered as a non-commutative K3 surface. Its Serre functor is equal to the double shift functor [2] (see [Kuz04, Lemma 4.2]) and there is an analogue of "singular cohomology" (and Hochschild (co)homology) for $\mathcal{K}u(X)$ which is isomorphic to the one of a K3 surface (e.g., [Kuz09b, Kuz10, AT14]).

We summarize the basic properties of the Mukai structure on $\mathcal{K}u(X)$ in the statement below. The proofs and some context can be found in [AT14, Section 2].

Proposition and Definition 9.5. *Let X be a cubic fourfold.*

(a) *The* singular cohomology *of $\mathcal{K}u(X)$ is defined as*

$$H^*(\mathcal{K}u(X), \mathbb{Z}) := \{\kappa \in K_{\mathrm{top}}(X) \,:\, \chi_{\mathrm{top}}([\mathcal{O}_X(i)], \kappa) = 0, \textit{ for } i = 0, 1, 2\},$$

where $K_{\text{top}}(X)$ *denotes the topological K-theory of* X *and* χ_{top} *the topological Euler characteristic*[7]*. It is endowed with a bilinear symmetric non-degenerate unimodular even form* $(\_,\_) = -\chi_{\text{top}}(\_,\_)$[8] *called the* Mukai pairing*, and with a weight-*2 *Hodge structure*

$$H^*(\mathcal{K}u(X),\mathbb{C}) := H^*(\mathcal{K}u(X),\mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=2} \widetilde{H}^{p,q}(\mathcal{K}u(X)).$$

As a lattice $H^*(\mathcal{K}u(X),\mathbb{Z})$ *is abstractly isomorphic to the extended K3 lattice* $U^4 \oplus E_8(-1)^2$.

(b) *The* Mukai vector $\mathbf{v}\colon K(\mathcal{K}u(X)) \to H^*(\mathcal{K}u(X),\mathbb{Z})$ *is the morphism induced by the natural morphism* $K(X) \to K_{\text{top}}(X)$.

(c) *The* algebraic Mukai lattice *of* $\mathcal{K}u(X)$ *is defined as*

$$H^*_{\text{alg}}(\mathcal{K}u(X),\mathbb{Z}) := H^*(\mathcal{K}u(X),\mathbb{Z}) \cap \widetilde{H}^{1,1}(\mathcal{K}u(X)).$$

It coincides with the image of $\mathbf{v}$*, and it is isomorphic with* $K_{\text{num}}(\mathcal{K}u(X))$*. Moreover, given* $E, F \in \mathcal{K}u(X)$*, we have*

$$\chi(E,F) = -(\mathbf{v}(E),\mathbf{v}(F)). \tag{20}$$

The signature of the Mukai pairing on $H^*_{\text{alg}}(\mathcal{K}u(X),\mathbb{Z})$ is $(2,\rho)$, where $0 \leq \rho \leq 20$.

When $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$, where S is a K3 surface, then this coincides with the usual Mukai structure on S. For a very general cubic fourfold ($\rho = 0$) its Kuznetsov component is never equivalent to the derived category of a K3 surface.

The algebraic Mukai lattice always contains two special classes:

$$\boldsymbol{\lambda}_1 := \mathbf{v}(\mathrm{pr}\,\mathcal{O}_L(H)) \quad \text{and} \quad \boldsymbol{\lambda}_2 := \mathbf{v}(\mathrm{pr}\,\mathcal{O}_L(2H)),$$

where $L \subset X$ denotes a line and $\mathrm{pr} : \mathrm{D}^{\mathrm{b}}(X) \to \mathcal{K}u(X)$ is the natural projection functor. They generate the primitive positive definite sublattice

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \subset H^*_{\text{alg}}(\mathcal{K}u(X),\mathbb{Z})$$

which agrees with the image of the restriction $K_{\text{top}}(\mathbb{P}^5) \to K_{\text{top}}(X)$ followed by projection. We think of this primitive sublattice as the choice of a polarization on $\mathcal{K}u(X)$. We will make this precise, since we will see in Proposition 9.10 that the central charge of the stability condition constructed in Theorem 1.2 will exactly correspond to this sublattice (see also [KS08, Section 1.2]). A first instance of this is given by the following result.

Let $F(X)$ denote the Fano variety of lines contained in X endowed with the natural polarization g coming from the Plücker embedding $F(X) \hookrightarrow \mathrm{Gr}(2,6)$.

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[7] See, e.g., [AT14, Section 2.1] for the basic definitions; but note that our notation $H^*(\mathcal{K}u(X),\mathbb{Z})$ is different.

[8] Our pairing is the opposite of the one in [AT14], to be coherent with the usual Mukai structure on K3 surfaces.

Proposition 9.6 ([AT14, Proposition 2.3])**.** *There exist Hodge isometries*

$$\langle \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \rangle^\perp \cong H^4_{\mathrm{prim}}(X,\mathbb{Z})(-1) \cong H^2_{\mathrm{prim}}(F(X),\mathbb{Z})$$

where $\langle \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \rangle^\perp \subset H^*(\mathcal{K}u(X),\mathbb{Z})$.

Bridgeland stability conditions on the Kuznetsov component. Let X be a cubic fourfold. In this section we review the basic theory of Bridgeland stability conditions for K3 categories [Bri08] applied to $\mathcal{K}u(X)$.

Let Λ be a lattice together with a surjective map $v\colon K(\mathcal{K}u(X)) \twoheadrightarrow \Lambda$. Assume that v factors via the surjections $K(\mathcal{K}u(X)) \xrightarrow{\mathbf{v}} H^*_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{Z}) \xrightarrow{u} \Lambda$. Let $\sigma = (\mathcal{A}, Z)$ be a stability condition with respect to such Λ and let $\eta(\sigma) \in H^*_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{C})$ be such that

$$(Z \circ u)(\_) = (\eta(\sigma), \_).$$

As in [Bri08], we define $\mathcal{P} \subset H^*_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{C})$ as the open subset consisting of those vectors whose real and imaginary parts span positive-definite two-planes in $H^*_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{R})$, and $\mathcal{P}_0$ as

$$\mathcal{P}_0 := \mathcal{P} \setminus \bigcup_{\delta\in\Delta} \delta^\perp,$$

where $\Delta := \left\{ \delta \in H^*_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{Z}) \,:\, \chi(\delta,\delta) = 2 \right\}$.

Lemma 9.7. *Let* $\sigma = (\mathcal{A}, Z)$ *be a stability condition with respect to such* Λ*. If* $\eta(\sigma) \in \mathcal{P}_0$*, then* $\sigma' := (\mathcal{A}, Z \circ u)$ *is a stability condition with respect to* $H^*_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{Z})$*.*

Proof. By Proposition 5.5, this follows immediately from [Bri08, Lemma 8.1]: the generalization to the case of the Kuznetsov component is straight-forward, since the Mukai pairing has the correct signature $(2,\rho)$ and, by Serre duality and (20), for any σ-stable object E in $\mathcal{K}u(X)$, we have $\mathbf{v}(E)^2 \geq -2$. □

Motivated by the above lemma, we can then make the following definition.

Definition 9.8. A *full numerical stability condition* on $\mathcal{K}u(X)$ is a Bridgeland stability condition on $\mathcal{K}u(X)$ whose lattice Λ is given by the Mukai lattice $H^*_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{Z})$ and the map v is given by the Mukai vector $\mathbf{v}$.

We denote by $\mathrm{Stab}(\mathcal{K}u(X))$ the space of full numerical stability conditions on $\mathcal{K}u(X)$. The map $\eta\colon \mathrm{Stab}(\mathcal{K}u(X)) \to H^*_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{C})$ defined above is then a local homeomorphism, by Bridgeland's Deformation Theorem [Bri07, Theorem 2.1]. If $\eta(\sigma) \in \mathcal{P}_0$, then we have a more precise result.

Proposition 9.9. *The map* $\eta\colon \eta^{-1}(\mathcal{P}_0) \subset \mathrm{Stab}(\mathcal{K}u(X)) \to \mathcal{P}_0$ *is a covering map.*

Proof. This can be proved exactly as in [Bri08, Proposition 8.3] (see also [Bay16, Corollary 1.3]). □

The support property. Let X be a cubic fourfold and let us fix a line $L_0 \subset X$. We can now show that the stability condition in Theorem 1.2 is a full numerical stability condition on $\mathcal{K}u(X)$.

Consider the fully faithful functor $\Xi\colon \mathcal{K}u(X) \to \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ of Lemma 7.6, and the composition $\mathrm{Forg} \circ \Xi$. The composition of the induced morphism $(\mathrm{Forg} \circ \Xi)_*$ at level of numerical Grothendieck groups and the truncated Chern character $\mathrm{ch}_{\mathcal{B}_0, \leq 2}$ gives a surjective morphism

$$H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z}) \twoheadrightarrow \Lambda^2_{\mathcal{B}_0, \mathcal{K}u(X)},$$

where $\Lambda^2_{\mathcal{B}_0}$ is the lattice generated by $\mathrm{ch}_{\mathcal{B}_0,0}, \mathrm{ch}_{\mathcal{B}_0,1}, \mathrm{ch}_{\mathcal{B}_0,2}$ (see Proposition 9.3) and $\Lambda^2_{\mathcal{B}_0,\mathcal{K}u(X)} \subset \Lambda^2_{\mathcal{B}_0}$ is nothing but the image of $K(\mathcal{K}u(X))$ (see (2)).

Consider the stability condition $\sigma_{\mathcal{K}u(X)} = (\mathcal{A}, Z)$ defined in the proof of Theorem 1.2. By the above discussion, we can then define $\eta(\sigma_{\mathcal{K}u(X)}) \in H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$. To prove that $\sigma_{\mathcal{K}u(X)}$ is a full numerical stability condition on $\mathcal{K}u(X)$ we only need to check the condition of Lemma 9.7, namely the following proposition:

Proposition 9.10. *We have* $\eta(\sigma_{\mathcal{K}u(X)}) \in (A_2)_{\mathbb{C}} \cap \mathcal{P} \subset \mathcal{P}_0$.

Proof. We consider the subspace V in $H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{R})$ generated by the real and imaginary part of $\eta(\sigma_{\mathcal{K}u(X)})$. We claim that $V = A_2$.

To prove the claim, we freely use the notation from Section 7. First of all, by definition of $Z_\alpha = Z^0_{\alpha,-1}$, it is straightforward to check that V has real dimension 2. Indeed, $\mathrm{ch}_{\mathcal{B}_0,\leq 2}(\Xi(\mathrm{pr}(\mathcal{O}_L(H)))) = (4, -1, -\frac{15}{8})$ and $\mathrm{ch}_{\mathcal{B}_0,\leq 2}(\Xi(\mathrm{pr}(\mathcal{O}_L(2H)))) = (-8, 8, -\frac{18}{8})$. Since

$$Z_\alpha = \mathrm{ch}^{-1}_{\mathcal{B}_0,1} + \mathrm{i}\left(-\tfrac{1}{2}\alpha^2 \,\mathrm{ch}^{-1}_{\mathcal{B}_0,0} + \mathrm{ch}^{-1}_{\mathcal{B}_0,2}\right),$$

we have $Z_\alpha(\mathrm{pr}(\mathcal{O}_L(H))) = 3 + \mathrm{i}(-2\alpha^2 - \frac{7}{8})$ and $Z_\alpha(\mathrm{pr}(\mathcal{O}_L(2H))) = \mathrm{i}(4\alpha^2 + \frac{7}{4})$, and they are linearly independent.

Hence, to prove the claim $V = A_2$ it remains to show that $\eta(\sigma_{\mathcal{K}u(X)}) \in (A_2)_{\mathbb{C}}$; equivalently, we have to show that for F with $\mathbf{v}(F) \in A_2^{\perp} = H^4_{\mathrm{prim}}(X, \mathbb{Z})(-1)$ (see Proposition 9.6), we have $Z_\alpha(F) = 0$.

Let $j\colon \mathbb{P}^2 \hookrightarrow \mathbb{P}^3$ be the inclusion of a hyperplane, and let $j_X\colon X_H \hookrightarrow X$ be the inclusion of the corresponding hyperplane section of X containing L_0. Let $\Xi_H\colon \mathrm{D}^{\mathrm{b}}(X_H) \to \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0|_{\mathbb{P}^2})$ be the restriction of the functor Ξ.

The assumption $\mathbf{v}(F) \in H^4_{\mathrm{prim}}(X, \mathbb{Z})$ implies that $\mathrm{ch}\,(j_{X*} j_X^* F) = 0$, since the class $[F] - [F \otimes \mathcal{O}_X(-H)]$ in the topological K-group is zero. On the other hand, since our formula for the central charge only depends on $\mathrm{ch}_i(\mathrm{Forg}(\Xi(F)))$ for $0 \leq i \leq 2$, it is determined by the Chern character of

$$j_* j^* \mathrm{Forg}\big(\Xi(F)\big) = j_* \mathrm{Forg}\,\big(\Xi_H(j_X^* F)\big) = \mathrm{Forg} \circ \Xi\big(j_{X*} j_X^* F\big),$$

where we used base change in the first equality, and projection formula in the second. By the observation above, the class of this object in the K-group of $\mathbb{P}^3$ vanishes, and thus its central charge $Z_\alpha(\_)$ is zero. Therefore $\eta(\sigma_{\mathcal{K}u(X)}) \in (A_2)_{\mathbb{C}}$ as claimed.

By [Voi86, Proposition 1, page 596] the primitive cohomology $H^4_{\mathrm{prim}}(X, Z)(-1)$ cannot contain algebraic classes δ with square $\delta^2 = -2$; in other words, $(A_2)_{\mathbb{C}} \cap \mathcal{P} \subset \mathcal{P}_0$ completing the proof. □

Remark 9.11. Combining Propositions 9.9 and 9.10, we obtain an open subset $\mathrm{Stab}^{\dagger}(\mathcal{K}u(X)) \subset \mathrm{Stab}(\mathcal{K}u(X))$ with a covering to $\mathcal{P}_0$. However, at this point we cannot prove that it forms a connected component. A proof would follow from the existence of semistable objects for every primitive class $\mathbf{v} \in H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ and for every stability condition σ in this open subset.

Finally, let us point out that Proposition 5.13 and its proof give:

Corollary 9.12. *There are stability conditions σ' on $\mathrm{D}^{\mathrm{b}}(X)$. They satisfy the support property with respect to the image $\Lambda \subset H^*(X, \mathbb{Q})$ of the Chern character.*

APPENDIX A. THE TORELLI THEOREM FOR CUBIC FOURFOLDS

by A. BAYER, M. LAHOZ, E. MACRÌ, P. STELLARI, X. ZHAO

In the recent inspiring paper [HR16], Huybrechts and Rennemo gave a new proof of the Torelli Theorem for cubic fourfolds by first proving a categorical version of it. The aim of this appendix is to present a different proof, based on Theorem 1.2, of their categorical Torelli Theorem; our proof works for *very general* cubic fourfolds.

Theorem A.1. *Let X and Y be smooth cubic fourfolds. Assume that $H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ has no (-2)-classes. Then $X \cong Y$ if and only if there is an equivalence $\Phi\colon \mathcal{K}u(X) \to \mathcal{K}u(Y)$ whose induced map $H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z}) \to H^*_{\mathrm{alg}}(\mathcal{K}u(Y), \mathbb{Z})$ commutes with the action of (1).*

In the above statement, we denote by (1) the natural autoequivalence of the Kuznetsov component induced by $\_ \otimes \mathcal{O}_X(1)$ followed by projection; it is called the *degree shift functor*. Note that the algebraic Mukai lattice is isomorphic to the numerical Grothendieck group, hence the action induced by Φ can be defined without assuming that it is of Fourier-Mukai type.

Theorem A.1 is a very general version of [HR16, Corollary 2.10] in the cubic fourfold case (with the minor improvement that we do not need to assume that the equivalence is given by a Fourier-Mukai functor, and that we only need compatibility with (1) on the level of algebraic Mukai lattice). It is still enough to deduce the classical Torelli Theorem, as we briefly sketch in Section A.3. Particular cases of it also appeared as [BMMS12, Proposition 6.3] (for generic cubics containing a plane) and [Huy15, Theorem 1.5, (iii)] (for cubics such that A_2 is the entire algebraic Mukai lattice); however, both proofs rely on the classical Torelli Theorem for cubic fourfolds.

The main idea of the proof of Theorem A.1 is to use the existence of Bridgeland stability conditions on $\mathcal{K}u(X)$. As observed in [KM09, Section 5], the Fano variety of lines $F(X)$ of a cubic fourfold X is isomorphic to a moduli space of torsion-free stable sheaves on X which belong to $\mathcal{K}u(X)$. Given a line $L_X \subset X$, we denote by $F_{X,L_X} \in \mathcal{K}u(X)$ the corresponding sheaf. For a cubic fourfold X for which $H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ has no (-2)-classes, the objects F_{X,L_X} are also Bridgeland stable in $\mathcal{K}u(X)$ for any stability condition. Moreover, any object with the same numerical class and the same Ext-groups must be one of them (up to shift). Given an equivalence of triangulated categories $\Phi\colon \mathcal{K}u(X) \xrightarrow{\simeq} \mathcal{K}u(Y)$, we can look at the images $\Phi(F_{X,L_X})$. If the induced action at the level of the algebraic Mukai lattice commutes with the degree shift functor (1), then we can assume that all objects $\Phi(F_{X,L_X})$ and F_{Y,L_Y} have the same numerical class up to composing Φ with the shift functor $[1]$. Hence, we deduce an isomorphism between $F(X)$ and $F(Y)$. Finally, by [BM14a], moduli spaces of Bridgeland stable objects come equipped with a natural line bundle. If we choose a stability condition with central charge in the A_2-lattice, as the one we construct in Theorem 1.2, the induced line bundle is exactly the Plücker polarization on the Fano variety of lines (up to constant). Hence, the isomorphism between $F(X)$ and $F(Y)$ preserves the Plücker polarization. This is enough to recover an isomorphism $X \cong Y$, by an elementary argument by Chow [Cho49], as observed in [Cha12, Proposition 4].

This argument should hold without the assumption on (-2)-classes, and so prove Theorem A.1 without assumptions, as originally stated in [HR16]. This would also directly imply the strong version of the classical Torelli theorem, as originally stated in [Voi86]. The two issues are to prove that the objects F_{X,L_X} are Bridgeland stable with respect to the stability conditions we constructed in Theorem 1.2, and that we can change the equivalence Φ by autoequivalences of $\mathcal{K}u(Y)$ until it does preserve such stability conditions.

A.1. **Classification of stable objects.** Let X be a cubic fourfold. We freely use the notation in Section 9. We start by recalling the following elementary but very useful result due to Mukai, which will allow us to control (in)stability of objects with small Ext^1. It first appeared in [Muk87]; see [BB17, Lemma 2.5] for the version stated here:

Lemma A.2 (Mukai). *Let $A \to E \to B$ be an exact triangle in $\mathcal{K}u(X)$. Assume that* $\mathrm{Hom}(A, B) = 0$. *Then*

$$\dim_{\mathbb{C}} \mathrm{Ext}^1(A, A) + \dim_{\mathbb{C}} \mathrm{Ext}^1(B, B) \leq \dim_{\mathbb{C}} \mathrm{Ext}^1(E, E).$$

As first corollary, we show that under our assumption there are no objects with $\mathrm{Ext}^1 = 0$.

Lemma A.3. *Assume that $H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ has no (-2)-classes. Then there exists no non-zero object $E \in \mathcal{K}u(X)$ with* $\mathrm{Ext}^1(E, E) = 0$.

Proof. Let $E \in \mathcal{K}u(X)$ be a non-zero object such that $\mathrm{Ext}^1(E, E) = 0$. Let $\sigma \in \mathrm{Stab}(\mathcal{K}u(X))$. By Lemma A.2, we can assume that E is σ-semistable and that it has a unique σ-stable factor E_0. Therefore, $\mathbf{v}(E)^2 < 0$. But then $\mathbf{v}(E_0)^2 < 0$ as well. Since $\mathrm{Hom}(E_0, E_0) \cong \mathbb{C}$, we have $\mathrm{Ext}^1(E_0, E_0) = 0$, and so $\mathbf{v}(E_0)^2 = -2$, a contradiction. □

By using Lemma A.3, we can show that objects with $\mathrm{Ext}^1 \cong \mathbb{C}^2$ are always stable.

Lemma A.4. *Assume that $H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ has no (-2)-classes. Let $E \in \mathcal{K}u(X)$ be an object with* $\mathrm{Ext}^1(E, E) \cong \mathbb{C}^2$. *Then, for all $\sigma \in \mathrm{Stab}(\mathcal{K}u(X))$, E is σ-stable. In particular,* $\mathbf{v}(E)^2 = 0$.

Proof. Let $E \in \mathcal{K}u(X)$ be an object with $\mathrm{Ext}^1(E, E) \cong \mathbb{C}^2$. Let $\sigma \in \mathrm{Stab}(\mathcal{K}u(X))$. By Lemma A.2 and Lemma A.3, we deduce that E is σ-semistable with a unique σ-stable object E_0. Therefore, $\mathbf{v}(E)^2 \leq 0$. But then $-2 \leq \mathbf{v}(E_0)^2 \leq 0$, and so $\mathbf{v}(E_0)^2 = 0$, by assumption. We deduce that $\mathbf{v}(E)^2 = 0$ and so that $\mathrm{Hom}(E, E) \cong \mathbb{C}$. This implies that $E = E_0$, as we wanted. □

Finally, we can study stability of objects with $\mathrm{Ext}^1 \cong \mathbb{C}^4$; this is the case which will be of interest for us, since we will apply this to reconstruct the Fano variety of lines.

Lemma A.5. *Assume that $H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ has no (-2)-classes. Let $E \in \mathcal{K}u(X)$ be an object with $\mathrm{Ext}^{<0}(E, E) = 0$, $\mathrm{Hom}(E, E) \cong \mathbb{C}$, and $\mathrm{Ext}^1(E, E) \cong \mathbb{C}^4$. Then, for all $\sigma \in \mathrm{Stab}(\mathcal{K}u(X))$, E is σ-stable.*

Proof. By assumption, $\mathbf{v}(E)^2 = 2$. Therefore, $\mathbf{v}(E)$ is a primitive vector in $H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$. By Lemma A.2, Lemma A.3, and Lemma A.4, if E is not σ-stable, then there exists a triangle $A \to E \to B$, where A and B are both σ-stable with $\mathrm{ext}^1 = 2$. Hence, $\mathbf{v}(A)^2 = \mathbf{v}(B)^2 = 0$ and $(\mathbf{v}(A) + \mathbf{v}(B))^2 = 2$. But then $(\mathbf{v}(A) - \mathbf{v}(B))^2 = -2$, a contradiction. ☐

Before stating the main result, we recall a construction by Kuznetsov and Markushevich. Given a line $L \subset X$, we define a torsion-free sheaf F_L as the kernel of the evaluation map

$$F_L := \mathrm{Ker}\left(\mathcal{O}_X^{\oplus 4} \twoheadrightarrow I_L(1)\right).$$

Then by [KM09, Section 5], F_L is a torsion-free Gieseker-stable sheaf on X which has the same Ext-groups as I_L, and which belongs to $\mathcal{K}u(X)$. By definition of $\boldsymbol{\lambda}_1$, one easily verifies $\mathbf{v}(F_L) = \boldsymbol{\lambda}_1$. By letting L vary, the sheaves F_L span a connected component of the moduli space of Gieseker-stable sheaves which is isomorphic to $F(X)$ [KM09, Proposition 5.5]. We denote by $\mathcal{F}_{\mathcal{L}}$ the universal family.

Proposition A.6. *Let $L \subset X$ be a line. Then, for all $\sigma \in \mathrm{Stab}(\mathcal{K}u(X))$, the sheaf F_L is σ-stable.*

Proof. This is now immediate from Lemma A.5. ☐

The last result we need is about moduli spaces, by generalizing an argument by Mukai [Muk87] (see [KLS06, Theorem 4.1]). Let $\sigma = (Z, \mathcal{A})$ be a Bridgeland stability condition on $\mathcal{K}u(X)$. Let us also fix a numerical class $\mathbf{v} \in H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$. We denote by $M^{\mathrm{spl}}(\mathcal{K}u(X))$ the space parameterizing simple objects in $\mathcal{K}u(X)$, which is an algebraic space locally of finite-type over $\mathbb{C}$ by [Ina02]. We also denote by $M^{\mathrm{st}}_\sigma(\mathbf{v}) \subset M^{\mathrm{spl}}(\mathcal{K}u(X))$ the subset parameterizing σ-stable objects in $\mathcal{A}$ with Mukai vector $\pm\mathbf{v}$.

Proposition A.7. *Assume there exists a smooth integral projective variety $M \subset M^{\mathrm{st}}_\sigma(\mathbf{v})$ of dimension $\mathbf{v}^2 + 2$. Then $M = M^{\mathrm{st}}_\sigma(\mathbf{v})$.*

Proof. The proof works exactly as in [KLS06, Theorem 4.1]. For simplicity, we assume the existence of a universal family $\mathcal{F}$ on M. Note that this is the case in our situation where $M = F(X)$. The general case, in which only a quasi-universal family exists, can be treated similarly as in [KLS06, Lemma 4.2].

Suppose that $M^{\mathrm{st}}_\sigma(\mathbf{v}) \neq M$, and consider objects $F \in M$ and $G \in M^{\mathrm{st}}_\sigma(\mathbf{v}) \setminus M$. We consider the product $M \times X$, and we denote by $p\colon M \times X \to M$ and $q\colon M \times X \to X$ the two projections. We will implicitly treat all objects in $\mathcal{K}u(X)$ as objects in $\mathrm{D^b}(X)$ without mentioning. The idea is to look at the following objects in $\mathrm{D^b}(M)$:

$$\mathbb{F} := p_* \mathcal{H}om(q^*F, \mathcal{F}), \qquad \mathbb{G} := p_* \mathcal{H}om(q^*G, \mathcal{F}).$$

To apply the argument in [KLS06, Theorem 4.1], we need to show that $\mathbb{F}$ is quasi-isomorphic to a complex of locally free sheaves on M of the form $A^0 \to A^1 \to A^2$ and $\mathbb{G}[-1]$ is a locally-free sheaf. By [BM02, Proposition 5.4], it suffices to show that, for all closed points $y \in M$, the complex $\mathbb{F} \otimes k(y)$ is supported in degrees $0, 1, 2$. Since the projection p is flat, by the Projection Formula and Cohomology and Base Change (see [BO95, Lemma 1.3]), we have

$$\mathrm{Tor}_{-j}(\mathbb{F}, k(y)) \cong \mathrm{Ext}^j(F, (i_y \times \mathrm{id})^* \mathcal{F}),$$

where $i_y \times \mathrm{id}\colon \{y\} \times X \to M \times X$ denotes the inclusion. But F and $(i_y \times \mathrm{id})^*\mathcal{F}$ both belong to a heart of a bounded t-structure in $\mathcal{K}u(X)$. Hence, $\mathrm{Ext}^j(F, (i_y \times \mathrm{id})^*\mathcal{F}) = 0$, for all $j \neq 0, 1, 2$, as we wanted.

A similar computation gives that $\mathbb{G} \otimes k(y)$ is supported only in degree 1, and so that $\mathbb{G}[-1]$ is quasi-isomorphic to a locally-free sheaf on M. The rest of the argument can be carried out line-by-line following [KLS06]. □

Recall that, when $M^{\mathrm{st}}_\sigma(\mathbf{v})$ is a proper algebraic space over $\mathbb{C}$, by [BM14a, Section 4], we can define a nef divisor class ℓ_σ on $M^{\mathrm{st}}_\sigma(\mathbf{v})$. By using Proposition A.6 and Proposition A.7, we can describe completely the moduli space $M^{\mathrm{st}}_\sigma(\boldsymbol{\lambda}_1)$. When $\eta(\sigma) \in (A_2)_{\mathbb{C}} \cap \mathcal{P} \subset H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$, we can also describe ℓ_σ.

Theorem A.8. *Let X be a cubic fourfold such that $H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ has no (-2)-classes. Then*

$$M^{\mathrm{st}}_\sigma(\boldsymbol{\lambda}_1) \cong F(X)$$

is a fine moduli space, for any $\sigma \in \mathrm{Stab}(\mathcal{K}u(X))$. Moreover, if $\eta(\sigma) \in (A_2)_{\mathbb{C}} \cap \mathcal{P}$, then ℓ_σ is a positive multiple of the divisor class $g = \boldsymbol{\lambda}_1 + 2\boldsymbol{\lambda}_2$ of the Plücker embedding[9].

Proof. By Proposition A.6, the Fano variety of lines is a smooth integral projective variety of dimension $4 = \boldsymbol{\lambda}_1^2 + 2$ which is contained in $M^{\mathrm{st}}_\sigma(\boldsymbol{\lambda}_1)$. Hence, by Proposition A.7, we have $F(X) = M^{\mathrm{st}}_\sigma(\boldsymbol{\lambda}_1)$. The universal family $\mathcal{F}_{\mathcal{L}}$ is also a universal family for objects in $\mathcal{K}u(X)$. Hence $F(X)$ is a fine moduli space of Bridgeland stable objects in $\mathcal{K}u(X)$.

Finally, it is a straightforward computation, as in [BM14a, Lemma 9.2], to see that ℓ_σ is proportional to $\boldsymbol{\lambda}_1 + 2\boldsymbol{\lambda}_2$. The fact that this is the Plücker polarization on $F(X)$ can be found, for example, in [Add16, Equation (6)]. □

A.2. **Proof of Theorem A.1.** Clearly only one implication is non-trivial. Let us pick any stability condition σ on $\mathcal{K}u(X)$ such that $\eta(\sigma) \in (A_2)_{\mathbb{C}} \cap \mathcal{P}$ and consider the fine moduli space $M^{\mathrm{st}}_\sigma(\boldsymbol{\lambda}_1)$. By Theorem A.8, $M^{\mathrm{st}}_\sigma(\boldsymbol{\lambda}_1)$ is isomorphic to $F(X)$ and carries a universal family $\mathcal{F}_{X,\mathcal{L}_X}$.

We recall from [Huy15, Proposition 3.12] that the action of (1) on the cohomology $H^*(\mathcal{K}u(X), \mathbb{Z})$ leaves $A_2^\perp$ invariant, whereas it cyclically permutes the roots $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$ and $-\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2$ in A_2.

Now consider an equivalence $\Phi\colon \mathcal{K}u(X) \to \mathcal{K}u(Y)$ as in Theorem A.1, and let $\sigma' := \Phi(\sigma)$. By the previous paragraph, it sends the distinguished sublattice $A_2 \subset H^*_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ to the corresponding

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[9] Here we use [Add16, Proposition 7] to identify $\mathrm{NS}(F(X))$ with $\boldsymbol{\lambda}_1^\perp$; see also Proposition 9.6.

sublattice for Y. Since the group generated by (1) and $[1]$ acts transitively on the roots of A_2, we can replace Φ by a functor with $\Phi_*(\boldsymbol{\lambda}_1) = \boldsymbol{\lambda}_1$ and $\Phi_*(\boldsymbol{\lambda}_2) = \boldsymbol{\lambda}_2$. Then it automatically induces a bijection between $M^{\mathrm{st}}_\sigma(\boldsymbol{\lambda}_1)$ and $M^{\mathrm{st}}_{\sigma'}(\boldsymbol{\lambda}_1)$. We need to show that this bijection is actually an isomorphism.

Consider the composition $\Psi\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathcal{K}u(X) \xrightarrow{\Phi} \mathcal{K}u(Y) \hookrightarrow \mathrm{D}^{\mathrm{b}}(Y)$, of Φ with the natural projection and inclusion. If this is of Fourier-Mukai type, then it makes sense to consider the functor $\Psi \times \mathrm{id}\colon \mathrm{D}^{\mathrm{b}}(X \times M^{\mathrm{st}}_\sigma(\boldsymbol{\lambda}_1)) \to \mathrm{D}^{\mathrm{b}}(Y \times M^{\mathrm{st}}_\sigma(\boldsymbol{\lambda}_1))$. Then the object $(\Psi \times \mathrm{id})(\mathcal{F}_{X,\mathcal{L}_X})$ provides a universal family on $Y \times M^{\mathrm{st}}_\sigma(\boldsymbol{\lambda}_1)$. Arguing as in [BMMS12, Section 5.3], this gives a morphism $f\colon M^{\mathrm{st}}_{\sigma'}(\boldsymbol{\lambda}_1) \to M^{\mathrm{st}}_\sigma(\boldsymbol{\lambda}_1)$. Since it is induced by Φ, this is an isomorphism.

If Ψ is not of Fourier-Mukai type, we can then proceed as in [BMMS12, Section 5.2]. While the functor $\Psi \times \mathrm{id}$ may not be well-defined, it still makes sense to define $(\Psi \times \mathrm{id})(\mathcal{F}_{X,\mathcal{L}_X})$, and then argue as before.

By construction, we have $f^*(\ell_{\sigma'}) = \ell_\sigma$. By Theorem A.8, ℓ_σ is the Plücker polarization; by the compatibility with Φ and the distinguished sublattice A_2, the same holds for $\ell_{\sigma'}$. Hence we get an isomorphism $F(X) \to F(Y)$ which preserves the Plücker polarization. By [Cha12, Proposition 4], we get an isomorphism $X \cong Y$.

A.3. **The classical Torelli theorem.** We are now ready to deduce the classical Torelli theorem for cubic fourfolds, by using the same argument as in [HR16].

Theorem A.9 (Voisin)**.** *Two smooth complex cubic fourfolds X and Y are isomorphic if and only if there exists a Hodge isometry $H^4_{\mathrm{prim}}(X,\mathbb{Z}) \cong H^4_{\mathrm{prim}}(Y,\mathbb{Z})$ between the primitive cohomologies.*

This result was originally proved in [Voi86]. Later Loojienga provided another proof in [Loo09] while describing the image of the period map. Charles [Cha12] gave an elementary proof relying on the Torelli theorem for hyperkähler manifolds [Ver13].

Proof. We briefly sketch the argument in [HR16, Section 4.2]. Let $\phi\colon H^4_{\mathrm{prim}}(X,\mathbb{Z}) \xrightarrow{\simeq} H^4_{\mathrm{prim}}(Y,\mathbb{Z})$ be a Hodge isometry. By [HR16, Proposition 3.2], it induces a Hodge isometry $\phi'\colon H^*(\mathcal{K}u(X),\mathbb{Z}) \xrightarrow{\simeq} H^*(\mathcal{K}u(Y),\mathbb{Z})$ that preserves the natural orientation.

A general deformation argument based on [Huy15] shows that ϕ' extends over a local deformation $\mathrm{Def}(X) \cong \mathrm{Def}(Y)$. The set $D \subset \mathrm{Def}(X)$ of points corresponding to cubic fourfolds X' such that $\mathcal{K}u(X') \cong \mathrm{D}^{\mathrm{b}}(S,\alpha)$, where S is a smooth projective K3 surface and α is an element in the Brauer group $\mathrm{Br}(S)$, and $H^*_{\mathrm{alg}}(\mathcal{K}u(X'),\mathbb{Z})$ has no (-2)-classes is dense (see [HMS08, Lemma 3.22]). Moreover, as argued in [HR16, Section 4.2], for any $t \in D$ there is an orientation preserving Hodge isometry $\phi_t\colon H^*(\mathcal{K}u(X_t),\mathbb{Z}) \xrightarrow{\simeq} H^*(\mathcal{K}u(Y_t),\mathbb{Z})$ which commutes with the action on cohomology of the degree shift functor (1) and which lifts to an equivalence $\Phi_t\colon \mathcal{K}u(X_t) \to \mathcal{K}u(Y_t)$. Now we can apply Theorem A.1 and get an isomorphism $X_t \cong Y_t$, for any $t \in D$. Since the moduli space of cubic fourfolds is separated, this yields $X \cong Y$. □

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