TWISTED FOURIER-MUKAI FUNCTORS

ALBERTO CANONACO AND PAOLO STELLARI

Abstract. Due to a theorem by Orlov every exact fully faithful functor between the bounded derived categories of coherent sheaves on smooth projective varieties is of Fourier-Mukai type. We extend this result to the case of bounded derived categories of twisted coherent sheaves and at the same time we weaken the hypotheses on the functor. As an application we get a complete description of the exact functors between the abelian categories of twisted coherent sheaves on smooth projective varieties.

1. Introduction

If X and Y are smooth projective varieties, an exact functor $F : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ between the corresponding bounded derived categories of coherent sheaves is of *Fourier-Mukai type* if there exists $\mathcal{E} \in \mathrm{D^b}(X \times Y)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{E}}$, where, denoting by $p : X \times Y \to Y$ and $q : X \times Y \to X$ the natural projections, $\Phi_{\mathcal{E}} : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ is the exact functor defined by

$$\Phi_{\mathcal{E}} := \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(-)). \tag{1.1}$$

Such a complex $\mathcal{E}$ is called a *kernel* of F.

The importance of functors of this type in geometric contexts cannot be overestimated. Indeed, all meaningful geometric functors are of Fourier-Mukai type and conjecturally the same is true for every exact functor from $\mathrm{D^b}(X)$ to $\mathrm{D^b}(Y)$. As a first evidence for the truth of this conjecture, in the fundamental paper [14], Orlov proved that any exact fully faithful functor from $\mathrm{D^b}(X)$ to $\mathrm{D^b}(Y)$ which admits a left adjoint is of Fourier-Mukai type. Moreover its kernel is uniquely determined up to isomorphism.

Since the publication of [14], some significant improvements were obtained. The main one is due to Kawamata ([11]), who extended this result to the case of smooth quotient stacks. His proof partially follows Orlov's original one but at some crucial points new deep ideas are needed. It is also worth noticing that, due to the results in [2], every exact functor $F : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ admits a left adjoint (see Remark 2.1 below).

In recent years some attention was paid to the case of *twisted varieties* (i.e. pairs (X, α), where X is a smooth projective variety and α is an element in the Brauer group of X). Since [3] appeared, it has been proved that some results from the untwisted setting can be generalized to the case of twisted derived categories. For example, if M is a K3 surface and a moduli space of stable sheaves on a K3 surface X, then there exist α in the Brauer group of M and an equivalence between $\mathrm{D^b}(X)$ and $\mathrm{D^b}(M, \alpha)$, the bounded derived category of α-twisted coherent sheaves on M. This was first proved by Căldăraru ([3, 4]) and then generalized in [12, 13, 17] and [9, 10]. Nevertheless a question remained open:

Are all equivalences between the bounded derived categories of twisted coherent sheaves on smooth projective varieties of Fourier-Mukai type?

As before, given two twisted varieties (X, α) and (Y, β), a functor $F : \mathrm{D^b}(X, \alpha) \to \mathrm{D^b}(Y, \beta)$ is of Fourier-Mukai type if there exist $\mathcal{E} \in \mathrm{D^b}(X \times Y, \alpha^{-1} \boxtimes \beta)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{E}}$, where $\Phi_{\mathcal{E}}$ is again defined as in (1.1).

2000 *Mathematics Subject Classification.* 14F22, 18E10, 18E30.
Key words and phrases. Twisted sheaves, derived categories, Fourier-Mukai functors.

A complete answer to the previous question comes as an easy corollary of the following theorem which is the main result of this paper:

Theorem 1.1. *Let* (X,α) *and* (Y,β) *be twisted varieties and let* $F:\mathrm{D^b}(X,\alpha)\to\mathrm{D^b}(Y,\beta)$ *be an exact functor such that, for any* $\mathcal{F},\mathcal{G}\in\mathbf{Coh}(X,\alpha)$,

$$\mathrm{Hom}_{\mathrm{D^b}(Y,\beta)}(F(\mathcal{F}),F(\mathcal{G})[j])=0 \;\; \text{if } j<0. \tag{1.2}$$

Then there exist $\mathcal{E}\in\mathrm{D^b}(X\times Y,\alpha^{-1}\boxtimes\beta)$ *and an isomorphism of functors* $F\cong\Phi_{\mathcal{E}}$*. Moreover,* $\mathcal{E}$ *is uniquely determined up to isomorphism.*

Few comments about the relevance of the previous result are in order here. First of all observe that any full functor satisfies (1.2). This means that Theorem 1.1 gives a substantial improvement of Orlov's result. As a consequence, we will observe that also the hypotheses in Kawamata's result ([11]) can be weakened (see Remark 4.1).

Our proof of Theorem 1.1 was inspired by [11] and [14] although different approaches are needed in many crucial points. In particular, the idea to use extensively convolutions of bounded complexes comes from [14].

In Section 5 we apply Theorem 1.1 to describe exact functors between the abelian categories of twisted coherent sheaves. In particular we deduce a Gabriel-type result for twisted varieties.

Notations. We will work over a fixed field K. All triangulated and abelian categories and all exact functors will be assumed to be K-linear. For an abelian category $\mathbf{A}$ we will denote by $\mathrm{D}(\mathbf{A})$ the derived category of $\mathbf{A}$. An object $C^\bullet$ of $\mathrm{D}(\mathbf{A})$ is a complex in $\mathbf{A}$, i.e. it is given by a collection of objects C^i and morphisms $d^i:C^i\to C^{i+1}$ of $\mathbf{A}$ such that $d^{i+1}\circ d^i=0$. The bounded derived category of $\mathbf{A}$ is the full subcategory $\mathrm{D^b}(\mathbf{A})$ of $\mathrm{D}(\mathbf{A})$ with objects the complexes $C^\bullet$ such that $C^i=0$ for $|i|\gg 0$. If there is no ambiguity, we will usually write C instead of $C^\bullet$. If $\mathbf{B}$ is another abelian category, every exact functor $G:\mathbf{A}\to\mathbf{B}$ trivially induces exact functors of triangulated categories $\mathrm{D}(G):\mathrm{D}(\mathbf{A})\to\mathrm{D}(\mathbf{B})$ and $\mathrm{D^b}(G):\mathrm{D^b}(\mathbf{A})\to\mathrm{D^b}(\mathbf{B})$. Recall that an abelian category $\mathbf{A}$ is of finite homological dimension if there exists an integer l such that, for any $i>l$ and any $A,B\in\mathrm{Ob}(\mathbf{A})$, $\mathrm{Hom}_{\mathrm{D^b}(\mathbf{A})}(A,B[i])=0$; if $N\in\mathbb{N}$ is the least such integer l, then $\mathbf{A}$ is said to be of homological dimension N.

2. Boundedness and ample sequences

For a smooth projective variety X consider the cohomology group $H^2_{\text{ét}}(X,\mathcal{O}_X^*)$ in the étale topology. Any $\alpha\in H^2_{\text{ét}}(X,\mathcal{O}_X^*)$ can be represented by a Čech 2-cocycle on an étale cover $\{U_i\}_{i\in I}$ of X using sections $\alpha_{ijk}\in\Gamma(U_i\cap U_j\cap U_k,\mathcal{O}_X^*)$. An *$\alpha$-twisted quasi-coherent sheaf* $\mathcal{F}$ consists of a pair $(\{\mathcal{F}_i\}_{i\in I},\{\varphi_{ij}\}_{i,j\in I})$, where $\mathcal{F}_i$ is a quasi-coherent sheaf on U_i and $\varphi_{ij}:\mathcal{F}_j|_{U_i\cap U_j}\to\mathcal{F}_i|_{U_i\cap U_j}$ is an isomorphism such that $\varphi_{ii}=\mathrm{id}$, $\varphi_{ji}=\varphi_{ij}^{-1}$ and $\varphi_{ij}\circ\varphi_{jk}\circ\varphi_{ki}=\alpha_{ijk}\cdot\mathrm{id}$.

The category of α-twisted quasi-coherent sheaves on X will be denoted by $\mathbf{QCoh}(X,\alpha)$. An α-twisted quasi-coherent sheaf $(\{\mathcal{F}_i\}_{i\in I},\{\varphi_{ij}\}_{i,j\in I})$ is an *α-twisted coherent sheaf* if $\mathcal{F}_i$ is coherent for any $i\in I$. We write $\mathbf{Coh}(X,\alpha)$ for the abelian category of α-twisted coherent sheaves and $\mathrm{D^b}(X,\alpha):=\mathrm{D^b}(\mathbf{Coh}(X,\alpha))$ for the bounded derived category of $\mathbf{Coh}(X,\alpha)$. The *Brauer group* of X is the group $\mathrm{Br}(X)$ consisting of all $\alpha\in H^2_{\text{ét}}(X,\mathcal{O}_X^*)$ such that $\mathbf{Coh}(X,\alpha)$ contains a locally free α-twisted coherent sheaf (actually, due to [5], $\mathrm{Br}(X)$ coincides with $H^2_{\text{ét}}(X,\mathcal{O}_X^*)$).

Let X and Y be smooth projective varieties and let $f:X\to Y$ be a morphism. The following derived functors are defined: $-\overset{\mathbf{L}}{\otimes}-:\mathrm{D^b}(X,\alpha)\times\mathrm{D^b}(X,\alpha')\to\mathrm{D^b}(X,\alpha\cdot\alpha')$, $\mathbf{R}f_*:\mathrm{D^b}(X,f^*(\beta))\to\mathrm{D^b}(Y,\beta)$ and $\mathbf{L}f^*:\mathrm{D^b}(Y,\beta)\to\mathrm{D^b}(X,f^*(\beta))$, where $\alpha,\alpha'\in\mathrm{Br}(X)$ and $\beta\in\mathrm{Br}(Y)$ (see [3, Thm. 2.2.4, Thm. 2.2.6]). For the rest of this paper (X,α) and (Y,β) will denote two twisted varieties as in Theorem 1.1.

Remark 2.1. (i) If (X,α) is a twisted variety and X has dimension n, then $\mathbf{Coh}(X,\alpha)$ has homological dimension n. To prove this claim, one can proceed as in the untwisted case (see [8, Prop. 3.12]), using the fact that the functor $S(-)=(-)\otimes\omega_X[n]$ is the Serre functor of $\mathrm{D^b}(X,\alpha)$.

(ii) If (X,α) and (Y,β) are twisted varieties, then any exact functor $G:\mathrm{D^b}(X,\alpha)\to\mathrm{D^b}(Y,\beta)$ has a left adjoint $G^*:\mathrm{D^b}(Y,\beta)\to\mathrm{D^b}(X,\alpha)$. Indeed, it is proved in [15] (generalizing ideas from [2] and [16]) that any cohomological functor of finite type is representable. Hence, for any $\mathcal{F}\in\mathrm{D^b}(Y,\beta)$ the functor $\mathrm{Hom}_{\mathrm{D^b}(Y,\beta)}(G(-),\mathcal{F})$ is representable by a unique $\mathcal{E}\in\mathrm{D^b}(X,\alpha)$. Setting $G'(\mathcal{F}):=\mathcal{E}$, by the Yoneda Lemma we get a functor which is right adjoint to G. Since $\mathrm{D^b}(X,\alpha)$ and $\mathrm{D^b}(Y,\beta)$ have Serre functors, G has also a left adjoint G^*.

Definition 2.2. Given an abelian category $\mathbf{A}$ with finite dimensional Hom's, a subset $\{P_i\}_{i\in\mathbb{Z}}\subset\mathrm{Ob}(\mathbf{A})$ is an *ample sequence* if, for any $B\in\mathrm{Ob}(\mathbf{A})$, there exists an integer $i(B)$ such that, for any $i\le i(B)$,

(1) the natural morphism $\mathrm{Hom}_{\mathbf{A}}(P_i,B)\otimes P_i\to B$ is surjective;
(2) if $j\neq 0$ then $\mathrm{Hom}_{\mathrm{D^b}(\mathbf{A})}(P_i,B[j])=0$;
(3) $\mathrm{Hom}_{\mathbf{A}}(B,P_i)=0$.

Lemma 2.3. *Let $E\in\mathbf{Coh}(X,\alpha)$ be a locally free sheaf. If $\{\mathcal{A}_k\}_{k\in\mathbb{Z}}$ is an ample sequence in $\mathbf{Coh}(X)$, then $\{E\otimes\mathcal{A}_k\}_{k\in\mathbb{Z}}$ is an ample sequence in $\mathbf{Coh}(X,\alpha)$. In particular, if $L\in\mathbf{Coh}(X)$ is an ample line bundle, then $\{E\otimes L^{\otimes k}\}_{k\in\mathbb{Z}}$ is an ample sequence.*

Proof. Observe that since $\{\mathcal{A}_k\}_{k\in\mathbb{Z}}$ is an ample sequence, for any $\mathcal{E}\in\mathbf{Coh}(X,\alpha)$ and for $i\ll 0$, there exists a surjective map $\mathrm{Hom}_{\mathbf{Coh}(X)}(\mathcal{A}_i,E^\vee\otimes\mathcal{E})\otimes\mathcal{A}_i\twoheadrightarrow E^\vee\otimes\mathcal{E}$. Then (1) in the previous definition follows from the fact that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Coh}(X,\alpha)}(E\otimes\mathcal{A}_i,\mathcal{E})\otimes E\otimes\mathcal{A}_i & \longrightarrow & \mathcal{E}\\
\cong\downarrow & & \uparrow\!\uparrow\\
\mathrm{Hom}_{\mathbf{Coh}(X)}(\mathcal{A}_i,E^\vee\otimes\mathcal{E})\otimes E\otimes\mathcal{A}_i & \twoheadrightarrow & E\otimes E^\vee\otimes\mathcal{E}
\end{array}$$

commutes. Analogously, $\mathrm{Hom}_{\mathrm{D^b}(X,\alpha)}(E\otimes\mathcal{A}_i,\mathcal{E}[j])\cong\mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{A}_i,E^\vee\otimes\mathcal{E}[j])=0$ and

$$\mathrm{Hom}_{\mathbf{Coh}(X,\alpha)}(\mathcal{E},E\otimes\mathcal{A}_i)\cong\mathrm{Hom}_{\mathbf{Coh}(X)}(\mathcal{E}\otimes E^\vee,\mathcal{A}_i)=0,$$

for $i\ll 0$ and $j\neq 0$. This proves that (2) and (3) hold true. The second part of the lemma follows from the easy fact that $\{L^{\otimes k}\}_{k\in\mathbb{Z}}$ is an ample sequence in $\mathbf{Coh}(X)$. □

Recall that, given two abelian categories $\mathbf{A}$ and $\mathbf{B}$, a functor $G:\mathrm{D^b}(\mathbf{A})\to\mathrm{D^b}(\mathbf{B})$ is *bounded* if there exist $a\in\mathbb{Z}$ and $n\in\mathbb{N}$ such that $H^i(G(A))=0$ for any $A\in\mathrm{Ob}(\mathbf{A})$ and any $i\notin[a,a+n]$.

Proposition 2.4. *Let (X,α) and (Y,β) be twisted varieties and assume that $G:\mathrm{D^b}(X,\alpha)\to\mathrm{D^b}(Y,\beta)$ is an exact functor. Then G is bounded.*

Proof. Due to Lemma 2.3, given a locally free sheaf $E\in\mathbf{Coh}(Y,\beta)$ and a very ample line bundle $L\in\mathbf{Coh}(Y)$ (defining an embedding $Y\hookrightarrow\mathbb{P}^N$), the set $\{E\otimes L^{\otimes k}\}_{k\in\mathbb{Z}}$ is an ample sequence in $\mathbf{Coh}(Y,\beta)$. For $k<0$, Beilinson's resolution ([1]), pulled back to Y, yields an isomorphism in $\mathrm{D^b}(Y)$

$$L^{\otimes k}\cong\{V_N^k\otimes\mathcal{O}_Y\to V_{N-1}^k\otimes L\to\ldots\to V_0^k\otimes L^{\otimes N}\},\tag{2.1}$$

where $V_i^k:=H^N(\mathbb{P}^N,\Omega^i_{\mathbb{P}^N}(i+k-N))$. In particular, $E\otimes L^{\otimes k}\cong C_k^\bullet$ in $\mathrm{D^b}(Y,\beta)$ where $C_k^i=0$, for $|i|>N$, and each C_k^i is a finite direct sum of terms of the form $E\otimes L^j$ for $0\le j\le N$. This implies that $\{G^*(E\otimes L^{\otimes k})\}_{k<0}$ is bounded in $\mathrm{D^b}(X,\alpha)$, where G^* is the left adjoint of G (see Remark 2.1(ii)).

This is enough to conclude that G is bounded. Indeed we can reason in the following rather standard way. Given $\mathcal{A}\in\mathbf{Coh}(X,\alpha)$ and $i\in\mathbb{Z}$, it is easy to see that $H^i(G(\mathcal{A}))=0$ is implied by

$\mathrm{Hom}_{\mathrm{D^b}(Y,\beta)}(E\otimes L^{\otimes k}, G(\mathcal{A})[i]) = 0$ for $k \ll 0$. Choosing m such that $H^j(G^*(E\otimes L^{\otimes k})) = 0$ for $|j| \geq m$ and for $k < 0$ and denoting by n the homological dimension of $\mathbf{Coh}(X,\alpha)$ (see Remark 2.1(i)), it is clear that

$$\mathrm{Hom}_{\mathrm{D^b}(Y,\beta)}(E\otimes L^{\otimes k}, G(\mathcal{A})[i]) \cong \mathrm{Hom}_{\mathrm{D^b}(X,\alpha)}(G^*(E\otimes L^{\otimes k}), \mathcal{A}[i]) = 0$$

for $|i| > n+m$ and for $k < 0$. □

The following easy lemma will be used in the forthcoming sections.

Lemma 2.5. *Let (X,α) and (Y,β) be twisted varieties, let $\mathcal{E}\in \mathrm{D^b}(X\times Y, \alpha^{-1}\boxtimes\beta)$ and let $l\in\mathbb{Z}$. If $E\in\mathbf{Coh}(X,\alpha)$ is locally free and $L\in\mathbf{Coh}(X)$ is ample, then $H^l(\mathcal{E})=0$ if $H^l(\Phi_{\mathcal{E}}(E\otimes L^{\otimes k}))=0$ for any $k\gg 0$.*

Proof. Fix a locally free sheaf $F\in\mathbf{Coh}(Y,\beta)$ and define $\mathcal{A} := \mathcal{E}\otimes p^*(F^\vee)\otimes q^*(E)$, where $p : X\times Y\to Y$ and $q : X\times Y\to X$ are the natural projections. If $H^j(\mathcal{A})\neq 0$, $\mathbf{R}^i p_*(H^j(\mathcal{A})\otimes q^*(L^{\otimes k})) = 0$ for any $k\gg 0$ if and only if $i\neq 0$ (for a proof of this well-known fact see, for example, [7], Chapter III, Theorem 8.8). Hence, using the spectral sequence

$$\mathbf{R}^i p_*(H^j(\mathcal{A})\otimes q^*(L^{\otimes k})) \Longrightarrow H^{i+j}(\Phi_{\mathcal{A}}(L^{\otimes k}))$$

we deduce that $H^l(\mathcal{A}) = 0$ if $H^l(\Phi_{\mathcal{A}}(L^{\otimes k})) = 0$, for any $k\gg 0$.

It is obvious that $H^l(\mathcal{A})=0$ if and only if $H^l(\mathcal{E})=0$. Hence the result is proved once we show that $H^l(\Phi_{\mathcal{A}}(L^{\otimes k}))=0$ if and only if $H^l(\Phi_{\mathcal{E}}(E\otimes L^{\otimes k}))=0$. By the Projection Formula

$$\Phi_{\mathcal{A}}(L^{\otimes k}) \cong \mathbf{R}p_*(\mathcal{A}\otimes q^*(L^{\otimes k})) \cong \mathbf{R}p_*(\mathcal{E}\otimes p^*(F^\vee)\otimes q^*(E)\otimes q^*(L^{\otimes k})) \cong \Phi_{\mathcal{E}}(E\otimes L^{\otimes k})\otimes F^\vee$$

which yields the desired conclusion. □

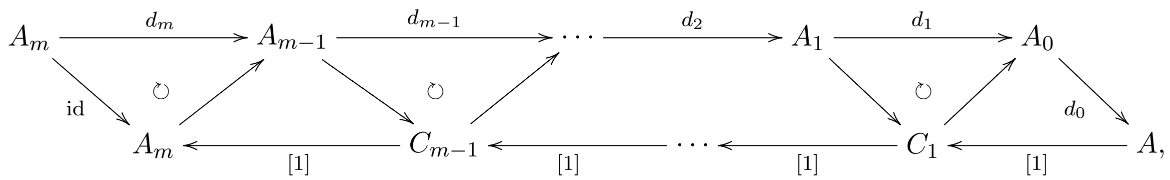
3. Convolutions and isomorphisms of functors

In this section we recall few results about convolutions of bounded complexes and we use them to study the existence of isomorphisms of exact functors.

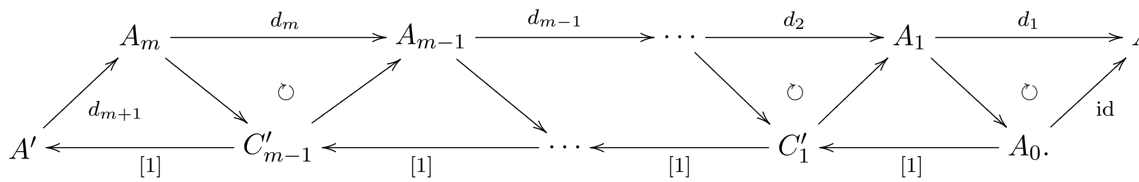
3.1. **Convolutions.** Recall that a bounded complex in a triangulated category $\mathbf{D}$ is a sequence of objects and morphisms in $\mathbf{D}$

$$A_m \xrightarrow{d_m} A_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} A_0 \tag{3.1}$$

such that $d_j\circ d_{j+1} = 0$ for $0 < j < m$. Following the terminology of [11], a *right convolution* of (3.1) is an object A together with a morphism $d_0 : A_0 \to A$ such that there exists a diagram in $\mathbf{D}$



where the triangles marked with a ↻ are commutative and the other triangles are distinguished (such an object A is called instead a left convolution of (3.1) in [14]). In a completely dual way, a *left convolution* of (3.1) is an object A' together with a morphism $d_{m+1} : A' \to A_m$ such that there exists a diagram in $\mathbf{D}$



Remark 3.1. Assume that $d_0 : A_0 \to A$ (respectively $d_{m+1} : A' \to A_m$) is a right (respectively left) convolution of (3.1). If $\mathbf{D}'$ is another triangulated category and $G : \mathbf{D} \to \mathbf{D}'$ is an exact functor, then it is obvious from the definitions that $G(d_0) : G(A_0) \to G(A)$ (respectively $G(d_{m+1}) : G(A') \to G(A_m)$) is a right (respectively left) convolution of

$$G(A_m) \xrightarrow{G(d_m)} G(A_{m-1}) \xrightarrow{G(d_{m-1})} \cdots \xrightarrow{G(d_1)} G(A_0).$$

In general a (right or left) convolution of a complex need not exist, and it need not be unique up to isomorphism when it exists, but we have the following two results, which will be constantly used in the rest of this paper:

Lemma 3.2. ([11], Lemmas 2.1 and 2.4.) *Let* (3.1) *be a complex in* $\mathbf{D}$ *satisfying*

$$\mathrm{Hom}_{\mathbf{D}}(A_a, A_b[r]) = 0 \textit{ for any } a > b \textit{ and } r < 0. \tag{3.2}$$

Then (3.1) *has right and left convolutions and they are uniquely determined up to isomorphism (in general non canonical).*

Lemma 3.3. *Let*

$$\begin{array}{ccccccccc} A_m & \xrightarrow{d_m} & A_{m-1} & \xrightarrow{d_{m-1}} & \cdots & \xrightarrow{d_2} & A_1 & \xrightarrow{d_1} & A_0 \\ \downarrow{\scriptstyle f_m} & & \downarrow{\scriptstyle f_{m-1}} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ B_m & \xrightarrow{e_m} & B_{m-1} & \xrightarrow{e_{m-1}} & \cdots & \xrightarrow{e_2} & B_1 & \xrightarrow{e_1} & B_0 \end{array}$$

be a morphism of complexes both satisfying (3.2) *and such that*

$$\mathrm{Hom}_{\mathbf{D}}(A_a, B_b[r]) = 0 \textit{ for any } a > b \textit{ and } r < 0.$$

Assume that the corresponding right (respectively left) convolutions are of the form $(d_0, 0) : A_0 \to A \oplus \bar{A}$ *and* $(e_0, 0) : B_0 \to B \oplus \bar{B}$ *(respectively* $(d_{m+1}, 0) : A' \oplus \bar{A}' \to A_m$ *and* $(e_{m+1}, 0) : B' \oplus \bar{B}' \to B_m$*) and that* $\mathrm{Hom}_{\mathbf{D}}(A_p, B[r]) = 0$ *(respectively* $\mathrm{Hom}_{\mathbf{D}}(A', B_p[r]) = 0$*) for* $r < 0$ *and any* p*. Then there exists a unique morphism* $f : A \to B$ *(respectively* $f' : A' \to B'$*) such that* $f \circ d_0 = e_0 \circ f_0$ *(respectively* $e_{m+1} \circ f' = f_m \circ d_{m+1}$*). If moreover each* f_i *is an isomorphism, then* f *(respectively* f'*) is an isomorphism as well.*

Proof. The first part is a particular case of Lemma 2.3 (respectively Lemma 2.6) of [11]. From this it is then straightforward to deduce that f (respectively f') is an isomorphism if each f_i is an isomorphism. □

Example 3.4. Let $\mathbf{D} := \mathrm{D}^{\mathrm{b}}(\mathbf{A})$ for some abelian category $\mathbf{A}$ and let Z be a complex as in (3.1) and such that every A_i is an object of $\mathbf{A}$. Then it is easy to see that a right (respectively left) convolution of Z (which is unique up to isomorphism by Lemma 3.2) is given by the natural morphism $A_0 \to Z^\bullet$ (respectively $Z^\bullet[-m] \to A_m$), where $Z^\bullet$ is the object of $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ naturally associated to Z (namely, $Z^i := A_{-i}$ for $-m \le i \le 0$ and otherwise $Z^i := 0$, with differential $d_{-i} : Z^i \to Z^{i+1}$ for $-m \le i < 0$).

3.2. **Extending isomorphisms of functors.** Let $\mathbf{A}$ be an abelian category with finite dimensional Hom's and assume that $\{P_i\}_{i\in\mathbb{Z}} \subset \mathrm{Ob}(\mathbf{A})$ is an ample sequence.

Lemma 3.5. *Any* $A \in \mathbf{A}$ *admits a resolution*

$$\cdots \to A_i^{\oplus k_i} \xrightarrow{d_i} A_{i-1}^{\oplus k_{i-1}} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} A_0^{\oplus k_0} \xrightarrow{d_0} A \to 0, \tag{3.3}$$

where $A_j \in \{P_i\}_{i\in\mathbb{Z}}$ *and* $k_j \in \mathbb{N}$*, for any* $j \in \mathbb{N}$*.*

Proof. To prove that such a(n infinite) resolution exists, it is clearly enough to show that for any $B \in \mathrm{Ob}(\mathbf{A})$ there exists $P \in \{P_i\}_{i\in\mathbb{Z}}$ and a surjective map $P^{\oplus k} \twoheadrightarrow B$, for some $k \in \mathbb{N}$. This follows from condition (1) in Definition 2.2. □

Remark 3.6. Consider a resolution of $A \in \mathrm{Ob}(\mathbf{A})$ as in Lemma 3.5 and assume that $\mathbf{A}$ has finite homological dimension N. Take $m > N$ and consider the bounded complex

$$S_m := \{A_m^{\oplus k_m} \xrightarrow{d_m} A_{m-1}^{\oplus k_{m-1}} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} A_0^{\oplus k_0}\}.$$

If $K_m := \ker(d_m)$, we have a distinguished triangle in $\mathrm{D^b}(\mathbf{A})$

$$K_m[m] \to S_m^\bullet \to A \to K_m[m+1].$$

Due to the choice of m, $\mathrm{Hom}_{\mathrm{D^b}(\mathbf{A})}(A, K_m[m+1]) = \mathrm{Hom}_{\mathrm{D^b}(\mathbf{A})}(A_0, K_m[m]) = 0$. Hence $S_m^\bullet \cong A \oplus K_m[m]$ and S_m has a (unique up to isomorphism) convolution $(d_0, 0) : A_0^{\oplus k_0} \to A \oplus K_m[m]$ (see Example 3.4).

The following result, whose proof relies on an extensive use of convolutions, improves [11, Lemma 6.5] and [14, Prop. 2.16].

Proposition 3.7. *Let $\mathbf{D}$ be a triangulated category and let $\mathbf{A}$ be an abelian category with finite dimensional* Hom*'s and of finite homological dimension. Assume that $\{P_i\}_{i\in\mathbb{Z}} \subseteq \mathrm{Ob}(\mathbf{A})$ is an ample sequence and denote by $\mathbf{C}$ the full subcategory of $\mathrm{D^b}(\mathbf{A})$ such that $\mathrm{Ob}(\mathbf{C}) = \{P_i\}_{i\in\mathbb{Z}}$. Let $F_1 : \mathrm{D^b}(\mathbf{A}) \to \mathbf{D}$ and $F_2 : \mathrm{D^b}(\mathbf{A}) \to \mathbf{D}$ be exact functors such that*

- (i) *there exists an isomorphism of functors $f : F_2|_{\mathbf{C}} \xrightarrow{\sim} F_1|_{\mathbf{C}}$;*
- (ii) $\mathrm{Hom}_{\mathbf{D}}(F_1(A), F_1(B)[j]) = 0$*, for any $A, B \in \mathrm{Ob}(\mathbf{A})$ and any $j < 0$;*
- (iii) *F_1 has a left adjoint F_1^*.*

Then there exists an isomorphism of functors $g : F_2 \xrightarrow{\sim} F_1$ extending f.

Proof. We denote by N the homological dimension of $\mathbf{A}$.

For any $i \in \mathbb{Z}$, let $f_i := f(P_i) : F_2(P_i) \xrightarrow{\sim} F_1(P_i)$. Given $A \in \mathrm{Ob}(\mathbf{A})$, we want to construct an isomorphism $f_A : F_2(A) \xrightarrow{\sim} F_1(A)$. According to Lemma 3.5, let

$$\cdots \to P_{i_j}^{\oplus k_j} \xrightarrow{d_j} P_{i_{j-1}}^{\oplus k_{j-1}} \xrightarrow{d_{j-1}} \cdots \xrightarrow{d_1} P_{i_0}^{\oplus k_0} \xrightarrow{d_0} A \to 0 \tag{3.4}$$

be a resolution of A. Fix $m > N$ and consider the bounded complex

$$R_m := \{P_{i_m}^{\oplus k_m} \xrightarrow{d_m} P_{i_{m-1}}^{\oplus k_{m-1}} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} P_{i_0}^{\oplus k_0}\}.$$

Due to Remark 3.6, a (unique up to isomorphism) convolution of R_m is $(d_0, 0) : P_{i_0}^{\oplus k_0} \to A \oplus K_m[m]$.

Due to Remark 3.1, for $i \in \{1, 2\}$, the complex

$$F_i(R_m) := \{F_i(P_{i_m}^{\oplus k_m}) \xrightarrow{F_i(d_m)} F_i(P_{i_{m-1}}^{\oplus k_{m-1}}) \xrightarrow{F_i(d_{m-1})} \cdots \xrightarrow{F_i(d_1)} F_i(P_{i_0}^{\oplus k_0})\}$$

admits a convolution $(F_i(d_0), 0) : F_i(P_{i_0}^{\oplus k_0}) \to F_i(A \oplus K_m[m])$. Lemma 3.2 and conditions (i) and (ii) ensure that such a convolution is unique up to isomorphism. Moreover, again by (i) and (ii), $\mathrm{Hom}_{\mathbf{D}}(F_2(P_{i_k}), F_1(A)[r]) \cong \mathrm{Hom}_{\mathbf{D}}(F_2(P_{i_l}), F_1(P_{i_j})[r]) = 0$, for any $i_j, i_l, i_k \in \{i_0, \ldots, i_m\}$ and $r < 0$. Hence we can apply Lemma 3.3 getting a unique isomorphism $f_A : F_2(A) \xrightarrow{\sim} F_1(A)$ making the following diagram commutative:

$$\begin{array}{ccccccccc}
F_2(P_{i_m}^{\oplus k_m}) & \xrightarrow{F_2(d_m)} & F_2(P_{i_{m-1}}^{\oplus k_{m-1}}) & \xrightarrow{F_2(d_{m-1})} \cdots \xrightarrow{F_2(d_1)} & F_2(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_2(d_0)} & F_2(A) \\
\downarrow f_{i_m}^{\oplus k_m} & & \downarrow f_{i_{m-1}}^{\oplus k_{m-1}} & & \downarrow f_{i_0}^{\oplus k_0} & & \downarrow f_A \\
F_1(P_{i_m}^{\oplus k_m}) & \xrightarrow{F_1(d_m)} & F_1(P_{i_{m-1}}^{\oplus k_{m-1}}) & \xrightarrow{F_1(d_{m-1})} \cdots \xrightarrow{F_1(d_1)} & F_1(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_1(d_0)} & F_1(A).
\end{array}$$

By Lemma 3.3, the definition of f_A does not depend on the choice of m. In other words, if we choose a different $m' > N$ and we truncate (3.4) in position m', the bounded complexes $F_i(R_{m'})$ give rise to the same isomorphism f_A.

To show that the definition of f_A does not depend on the choice of the resolution (3.4), consider another resolution

$$\cdots \to P_{i'_j}^{\oplus k'_j} \xrightarrow{d'_j} P_{i'_{j-1}}^{\oplus k'_{j-1}} \xrightarrow{d'_{j-1}} \cdots \xrightarrow{d'_1} P_{i'_0}^{\oplus k'_0} \xrightarrow{d'_0} A \to 0. \tag{3.5}$$

Suppose that there exists a third resolution

$$\cdots \to P_{i''_j}^{\oplus k''_j} \xrightarrow{d''_j} P_{i''_{j-1}}^{\oplus k''_{j-1}} \xrightarrow{d''_{j-1}} \cdots \xrightarrow{d''_1} P_{i''_0}^{\oplus k''_0} \xrightarrow{d''_0} A \to 0 \tag{3.6}$$

and morphisms $s_j : P_{i''_j}^{\oplus k''_j} \to P_{i_j}^{\oplus k_j}$ and $t_j : P_{i''_j}^{\oplus k''_j} \to P_{i'_j}^{\oplus k'_j}$, for any $j \geq 0$, fitting into the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{d'_{j+1}} & P_{i'_j}^{\oplus k'_j} & \xrightarrow{d'_j} & P_{i'_{j-1}}^{\oplus k'_{j-1}} & \xrightarrow{d'_{j-1}} \cdots \xrightarrow{d'_1} & P_{i'_0}^{\oplus k'_0} & & & \\
 & \uparrow {\scriptstyle t_j} & & \uparrow {\scriptstyle t_{j-1}} & & \uparrow {\scriptstyle t_0} & \searrow {\scriptstyle d'_0} & & \\
\cdots \xrightarrow{d''_{j+1}} & P_{i''_j}^{\oplus k''_j} & \xrightarrow{d''_j} & P_{i''_{j-1}}^{\oplus k''_{j-1}} & \xrightarrow{d''_{j-1}} \cdots \xrightarrow{d''_1} & P_{i''_0}^{\oplus k''_0} & \xrightarrow{d''_0} & A. & \\
 & \downarrow {\scriptstyle s_j} & & \downarrow {\scriptstyle s_{j-1}} & & \downarrow {\scriptstyle s_0} & \nearrow {\scriptstyle d_0} & & \\
\cdots \xrightarrow{d_{j+1}} & P_{i_j}^{\oplus k_j} & \xrightarrow{d_j} & P_{i_{j-1}}^{\oplus k_{j-1}} & \xrightarrow{d_{j-1}} \cdots \xrightarrow{d_1} & P_{i_0}^{\oplus k_0} & & &
\end{array}$$

Define the bounded complexes

$$\begin{aligned}
R''_m &:= \{P_{i''_m}^{\oplus k''_m} \xrightarrow{d''_m} P_{i''_{m-1}}^{\oplus k''_{m-1}} \xrightarrow{d''_{m-1}} \cdots \xrightarrow{d''_1} P_{i''_0}^{\oplus k''_0}\} \\
F_i(R''_m) &:= \{F_i(P_{i''_m}^{\oplus k''_m}) \xrightarrow{F_i(d''_m)} F_i(P_{i''_{m-1}}^{\oplus k''_{m-1}}) \xrightarrow{F_i(d''_{m-1})} \cdots \xrightarrow{F_i(d''_1)} F_i(P_{i''_0}^{\oplus k''_0})\}.
\end{aligned} \tag{3.7}$$

Let $f''_A : F_2(A) \xrightarrow{\sim} F_1(A)$ be the isomorphism constructed using (3.7). Due to Remark 3.6, these complexes and their convolutions give rise to the diagram

$$\begin{array}{ccccccc}
F_2(P_{i''_0}^{\oplus k''_0}) & & & \xrightarrow{F_2(d''_0)} & & & F_2(A) \\
 & \searrow {\scriptstyle F_2(s_0)} & & & & \swarrow {\scriptstyle \mathrm{id}} & \\
 & & F_2(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_2(d_0)} & F_2(A) & & \\
\downarrow {\scriptstyle f_{i''_0}^{\oplus k''_0}} & & \downarrow {\scriptstyle f_{i_0}^{\oplus k_0}} & & \downarrow {\scriptstyle f_A} & \bigstar & \downarrow {\scriptstyle f''_A} \\
 & & F_1(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_1(d_0)} & F_1(A) & & \\
 & \nearrow {\scriptstyle F_1(s_0)} & & & & \nwarrow {\scriptstyle \mathrm{id}} & \\
F_1(P_{i''_0}^{\oplus k''_0}) & & & \xrightarrow{F_1(d''_0)} & & & F_1(A)
\end{array}$$

where all squares but $\bigstar$ are commutative. Due to hypotheses (i), (ii) and Lemma 3.3 there exists a unique morphism $F_2(A) \to F_1(A)$ making the following diagram commutative:

$$\begin{array}{ccc}
F_2(P_{i''_0}^{\oplus k''_0}) & \xrightarrow{F_2(d''_0)} & F_2(A) \\
\downarrow {\scriptstyle F_1(s_0) \circ f_{i''_0}^{\oplus k''_0}} & & \downarrow \\
F_1(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_1(d_0)} & F_1(A).
\end{array}$$

Since $F_1(s_0) \circ f_{i_0''}^{\oplus k_0''} = f_{i_0}^{\oplus k_0} \circ F_2(s_0)$, both f_A and f_A'' have this property and then they coincide. Similarly one can prove that the morphism f_A'' is equal to the morphism f_A' constructed by means of (3.5).

To construct (3.6), we proceed as follows. First take $i_0'' \ll 0$ such that there exist a surjective morphism $d_0'' : P_{i_0''}^{\oplus k_0''} \twoheadrightarrow A$, for some $k_0'' \in \mathbb{N}$, and two morphisms s_0 and t_0 as required. Suppose now that $P_{i_j''}$, k_j'', d_j'', s_j and t_j are defined. Take $i_{j+1}'' \ll 0$, $k_{j+1}'' \in \mathbb{N}$ and $d_{j+1}'' : P_{i_{j+1}''}^{\oplus k_{j+1}''} \to P_{i_j''}^{\oplus k_j''}$ such that

- (a.1) $\ker(d_j'') = \operatorname{im}(d_{j+1}'')$;
- (b.1) the morphism $s_j \circ d_{j+1}''$ factorizes through d_{j+1};
- (c.1) the morphism $t_j \circ d_{j+1}''$ factorizes through d_{j+1}'.

Observe that this is always possible because $\operatorname{im}(s_j|_{\ker(d_j'')}) \subset \operatorname{im}(d_{j+1})$ and for $n \ll 0$ the natural map $\operatorname{Hom}_{\mathbf{A}}(P_n, P_{i_{j+1}}^{\oplus k_{j+1}}) \to \operatorname{Hom}_{\mathbf{A}}(P_n, \operatorname{im}(d_{j+1}))$ is surjective (the same holds true for d_{j+1}' and t_j).

To prove the functoriality, let $A, B \in \operatorname{Ob}(\mathbf{A})$ and let $\varphi : A \to B$ be a morphism. Consider a resolution

$$\cdots \to P_{l_j}^{\oplus h_j} \xrightarrow{e_j} P_{l_{j-1}}^{\oplus h_{j-1}} \xrightarrow{e_{j-1}} \cdots \xrightarrow{e_1} P_{l_0}^{\oplus h_0} \xrightarrow{e_0} B \to 0. \tag{3.8}$$

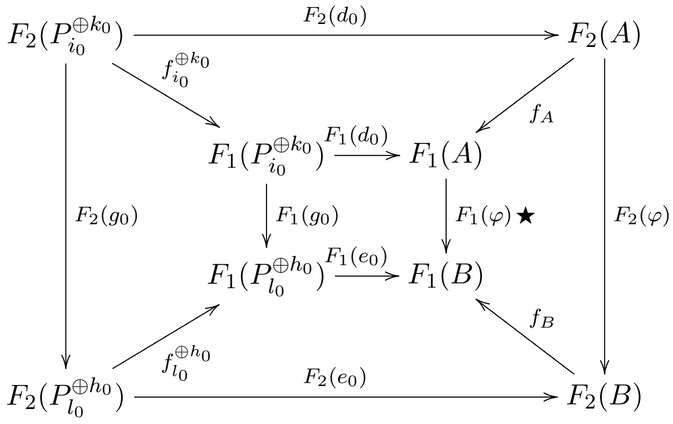
Reasoning as before, we can find a resolution

$$\cdots \to P_{i_j}^{\oplus k_j} \xrightarrow{d_j} P_{i_{j-1}}^{\oplus k_{j-1}} \xrightarrow{d_{j-1}} \cdots \xrightarrow{d_1} P_{i_0}^{\oplus k_0} \xrightarrow{d_0} A \to 0 \tag{3.9}$$

and morphisms $g_j : P_{i_j}^{\oplus k_j} \to P_{l_j}^{\oplus h_j}$ defining a morphism of complexes compatible with φ. Fix $m > N$ and take the bounded complexes

$$\begin{aligned} R_m &:= \{P_{i_m}^{\oplus k_m} \xrightarrow{d_m} P_{i_{m-1}}^{\oplus k_{m-1}} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} P_{i_0}^{\oplus k_0}\} \\ T_m &:= \{P_{l_m}^{\oplus h_m} \xrightarrow{e_m} P_{l_{m-1}}^{\oplus h_{m-1}} \xrightarrow{e_{m-1}} \cdots \xrightarrow{e_1} P_{l_0}^{\oplus h_0}\} \\ F_i(R_m) &:= \{F_i(P_{i_m}^{\oplus k_m}) \xrightarrow{F_i(d_m)} F_i(P_{i_{m-1}}^{\oplus k_{m-1}}) \xrightarrow{F_i(d_{m-1})} \cdots \xrightarrow{F_i(d_1)} F_i(P_{i_0}^{\oplus k_0})\} \\ F_i(T_m) &:= \{F_i(P_{l_m}^{\oplus h_m}) \xrightarrow{F_i(e_m)} F_i(P_{l_{m-1}}^{\oplus h_{m-1}}) \xrightarrow{F_i(e_{m-1})} \cdots \xrightarrow{F_i(e_1)} F_i(P_{l_0}^{\oplus h_0})\}. \end{aligned}$$

We can now consider the diagram



where all squares but ★ are commutative. Applying (i), (ii) and Lemma 3.3 we see that there is a unique morphism $F_2(A) \to F_1(B)$ completing the following diagram to a commutative square

$$\begin{array}{ccc} F_2(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_2(d_0)} & F_2(A) \\ {\scriptstyle F_1(g_0)\circ f_{i_0}^{\oplus k_0}}\big\downarrow & & \big\downarrow \\ F_1(P_{l_0}^{\oplus h_0}) & \xrightarrow{F_1(e_0)} & F_1(B). \end{array}$$

Since $F_1(g_0) \circ f_{i_0}^{\oplus k_0} = f_{l_0}^{\oplus h_0} \circ F_2(g_0)$, both $F_1(\varphi) \circ f_A$ and $f_B \circ F_2(\varphi)$ have this property. Thus $F_1(\varphi) \circ f_A = f_B \circ F_2(\varphi)$.

If $A \in \mathrm{Ob}(\mathbf{A})$, we can clearly put $f_{A[n]} := f_A[n]$ for every integer n. Moreover, for any $A, B \in \mathrm{Ob}(\mathbf{A})$, the morphisms f_A and f_B just constructed commute with any $g \in \mathrm{Hom}_{\mathrm{D^b}(\mathbf{A})}(A, B[j])$ (see [14, Sect. 2.16.4] for the proof).

The rest of the proof follows the strategy in [14, Sect. 2.16.5] and it proceeds by induction on the length of the segment in which the cohomologies of the objects are concentrated. In particular, let A be an object in $\mathrm{D^b}(\mathbf{A})$ and suppose, without loss of generality, that $H^p(A) = 0$ if $p \notin [a, 0]$ and $a < 0$. Consider a morphism $v : P_i^{\oplus k} \to A$ such that

(a.2) the natural morphism $u : P_i^{\oplus k} \to H^0(A)$ induced by v is surjective;
(b.2) $\mathrm{Hom}_{\mathbf{A}}(H^0(F_1^* \circ F_1(A)), P_i) = 0$.

Take a distinguished triangle

$$Z[-1] \to P_i^{\oplus k} \xrightarrow{v} A \to Z$$

and observe that $H^p(Z) = 0$ if $p \notin [a, -1]$. Hence, by induction hypothesis, we have an isomorphism $f_Z : F_2(Z) \xrightarrow{\sim} F_1(Z)$ and the following commutative diagram

$$\begin{array}{ccccccc} F_1(Z)[-1] & \longrightarrow & F_1(P_i^{\oplus k}) & \longrightarrow & F_1(A) & \longrightarrow & F_1(Z) \\ \big\downarrow {\scriptstyle f_Z^{-1}[-1]} & & \big\downarrow {\scriptstyle (f_i^{\oplus k})^{-1}} & & & & \big\downarrow {\scriptstyle f_Z^{-1}} \\ F_2(Z)[-1] & \longrightarrow & F_2(P_i^{\oplus k}) & \longrightarrow & F_2(A) & \longrightarrow & F_2(Z). \end{array}$$

By [14, Lemma 1.4], to complete the previous diagram with a unique isomorphism $f_A : F_2(A) \xrightarrow{\sim} F_1(A)$, we need to show that

$$\mathrm{Hom}_{\mathbf{D}}(F_1(A), F_2(P_i)) = 0.$$

To prove this we can suppose $A \in \mathrm{Ob}(\mathbf{A})$ because the cohomologies of A are concentrated in degrees less or equal to zero. Let $w = \max\{n \in \mathbb{Z} : H^n(F_1^* \circ F_1(A)) \neq 0\}$. Obviously, there exists a natural non-zero morphism $F_1^* \circ F_1(A) \to H^w(F_1^* \circ F_1(A))[-w]$. Hence

$$0 \neq \mathrm{Hom}_{\mathrm{D^b}(\mathbf{A})}(F_1^* \circ F_1(A), H^w(F_1^* \circ F_1(A))[-w]) \cong \mathrm{Hom}_{\mathbf{D}}(F_1(A), F_1(H^w(F_1^* \circ F_1(A)))[-w])$$

and $w \leq 0$ because of (ii). In particular $H^j(F_1^* \circ F_1(A)) = 0$ if $j \notin [-b, 0]$, for some positive integer b. Therefore, due to (b.2) and (ii),

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}}(F_1(A), F_2(P_i)) &\cong \mathrm{Hom}_{\mathbf{D}}(F_1(A), F_1(P_i)) \\ &\cong \mathrm{Hom}_{\mathrm{D^b}(\mathbf{A})}(F_1^* \circ F_1(A), P_i) \cong \mathrm{Hom}_{\mathbf{A}}(H^0(F_1^* \circ F_1(A)), P_i) = 0. \end{aligned}$$

To prove that f_A is well-defined and functorial, one has to repeat line by line the proof in Sections 2.16.6 and 2.16.7 of [14] using (ii) instead of the hypothesis that F_1 and F_2 are fully-faithful. We leave this to the reader. □

Corollary 3.8. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be abelian categories such that* $\mathbf{A}$ *has finite dimensional* Hom*'s, it is of finite homological dimension and it has an ample sequence. If* $F : \mathrm{D^b}(\mathbf{A}) \to \mathrm{D^b}(\mathbf{B})$ *is an* *exact functor with a left adjoint and such that* $F(\mathbf{A}) \subseteq \mathbf{B}$*, then* $G := F|_{\mathbf{A}} : \mathbf{A} \to \mathbf{B}$ *is exact and* $\mathrm{D^b}(G) \cong F$*.*

Proof. The exactness of G is trivial. Since $\mathrm{D^b}(G)|_{\mathbf{A}} \cong F|_{\mathbf{A}}$, we can apply Proposition 3.7 getting the desired conclusion. □

4. Proof of Theorem 1.1

We divide up our argument in several steps.

4.1. **Resolution of the diagonal.** Denoting by $d : X \hookrightarrow X \times X$ the diagonal morphism, $\mathcal{O}_\Delta := d_*\mathcal{O}_X$ can be regarded as an $(\alpha^{-1} \boxtimes \alpha)$-twisted coherent sheaf on $X \times X$ in a natural way, since $d^*(\alpha^{-1} \boxtimes \alpha) = 1$. It is easy to see that $\mathcal{O}_\Delta \in \mathbf{Coh}(X \times X, \alpha^{-1} \boxtimes \alpha)$ admits a resolution

$$\cdots \to A_i \boxtimes B_i \xrightarrow{\delta_i} A_{i-1} \boxtimes B_{i-1} \xrightarrow{\delta_{i-1}} \cdots \xrightarrow{\delta_1} A_0 \boxtimes B_0 \xrightarrow{\delta_0} \mathcal{O}_\Delta \to 0, \tag{4.1}$$

where $A_j \in \mathbf{Coh}(X, \alpha^{-1})$ and $B_j \in \mathbf{Coh}(X, \alpha)$ are locally free for any $j \in \mathbb{N}$. Indeed, if L is an ample line bundle on X, $L \boxtimes L$ is ample on $X \times X$. Hence, given a locally free sheaf $E \in \mathbf{Coh}(X, \alpha)$, Lemma 2.3 proves that $\{(E^\vee \boxtimes E) \otimes (L \boxtimes L)^{\otimes k}\}_{k\in\mathbb{Z}}$ is an ample sequence in $\mathbf{Coh}(X \times X, \alpha^{-1} \boxtimes \alpha)$. As $(E^\vee \boxtimes E) \otimes (L \boxtimes L)^{\otimes k} \cong (E^\vee \otimes L^{\otimes k}) \boxtimes (E \otimes L^{\otimes k})$, we conclude by Lemma 3.5.

4.2. **Some bounded complexes.** Since F is a bounded functor by Proposition 2.4, we can assume without loss of generality that $H^i(F(\mathcal{F})) = 0$ for any $\mathcal{F} \in \mathbf{Coh}(X, \alpha)$ and any $i \notin [-M, 0]$ for some $M \in \mathbb{N}$. Then we fix once and for all a resolution of $\mathcal{O}_\Delta$ as in (4.1), and for every integer $m > \dim(X) + \dim(Y) + M$ we define the following complexes

$$C_m := \{A_m \boxtimes B_m \xrightarrow{\delta_m} \cdots \xrightarrow{\delta_1} A_0 \boxtimes B_0\}$$
$$\tilde{C}_m := \{A_m \boxtimes F(B_m) \xrightarrow{\tilde{\delta}_m} \cdots \xrightarrow{\tilde{\delta}_1} A_0 \boxtimes F(B_0)\}$$

in $\mathrm{D^b}(X \times X, \alpha^{-1} \boxtimes \alpha)$ and $\mathrm{D^b}(X \times Y, \alpha^{-1} \boxtimes \beta)$ respectively, where $\tilde{\delta}_i$ denotes the image of δ_i through the map

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D^b}(X\times X,\alpha^{-1}\boxtimes\alpha)}(A_i \boxtimes B_i, A_{i-1} \boxtimes B_{i-1}) &\cong \mathrm{Hom}_{\mathrm{D^b}(X,\alpha^{-1})}(A_i, A_{i-1}) \otimes \mathrm{Hom}_{\mathrm{D^b}(X,\alpha)}(B_i, B_{i-1})\\ &\xrightarrow{\mathrm{id}\otimes F} \mathrm{Hom}_{\mathrm{D^b}(X,\alpha^{-1})}(A_i, A_{i-1}) \otimes \mathrm{Hom}_{\mathrm{D^b}(Y,\beta)}(F(B_i), F(B_{i-1}))\\ &\cong \mathrm{Hom}_{\mathrm{D^b}(X\times Y,\alpha^{-1}\boxtimes\beta)}(A_i \boxtimes F(B_i), A_{i-1} \boxtimes F(B_{i-1})).\end{aligned}$$

Setting $\mathcal{K}_m := \ker(\delta_m) \in \mathbf{Coh}(X \times X, \alpha^{-1} \boxtimes \alpha)$ and proceeding as in Remark 3.6, we see that, if $m > 2\dim(X)$, $C_m^\bullet \cong \mathcal{O}_\Delta \oplus \mathcal{K}_m[m]$ and C_m has a (unique up to isomorphism by Lemma 3.2) right convolution $(\delta_0, 0) : A_0 \boxtimes B_0 \to \mathcal{O}_\Delta \oplus \mathcal{K}_m[m]$. Observe that the assumption on F implies that also $\tilde{C}_m$ satisfies the hypothesis of Lemma 3.2, hence it has a unique up to isomorphism right convolution $\tilde{\delta}'_{0,m} : A_0 \boxtimes F(B_0) \to \mathcal{G}_m$.

We proceed now as in [11, Lemma 6.1]. We denote by $\mathbf{K}_m$ the full subcategory of $\mathbf{Coh}(X, \alpha)$ with objects the locally free sheaves E such that $H^i(X, E \otimes A_j) = 0$ for $i > 0$ and $0 \le j \le m + \dim(X)$. Observe that, for any locally free $E' \in \mathbf{Coh}(X, \alpha)$ and any ample line bundle $L \in \mathbf{Coh}(X)$, $E' \otimes L^{\otimes k} \in \mathbf{K}_m$, when $k \gg 0$.

As $\mathbf{R}^i p_{2*}(A_j \boxtimes B_j \otimes p_1^*\mathcal{F}) \cong H^i(X, A_j \otimes \mathcal{F}) \otimes B_j$ for $\mathcal{F} \in \mathbf{Coh}(X, \alpha)$ and $i, j \in \mathbb{N}$ (where $p_l : X \times X \to X$ is the projection onto the l^{th} factor),

$$\mathbf{R}p_{2*}(A_j \boxtimes B_j \otimes p_1^*E) \cong p_{2*}(A_j \boxtimes B_j \otimes p_1^*E) \cong H^0(X, A_j \otimes E) \otimes B_j$$

if $E \in \mathbf{K}_m$ and $0 \le j \le m + \dim(X)$. It follows that the exact functor $\mathbf{R}p_{2*}(- \otimes p_1^*E)$ maps C_m to a complex

$$C_{m,E} = \{H^0(X, A_m \otimes E) \otimes B_m \xrightarrow{\delta_{m,E}} \cdots \xrightarrow{\delta_{1,E}} H^0(X, A_0 \otimes E) \otimes B_0\}$$

in $\mathrm{D^b}(X,\alpha)$, which has a (unique up to isomorphism) right convolution

$$(\delta_{0,E},0): H^0(X,A_0\otimes E)\otimes B_0 \to E\oplus \mathcal{K}_{m,E}[m],$$

where $\mathcal{K}_{m,E} := \ker(\delta_{m,E})$. If $p: X\times Y\to Y$ and $q: X\times Y\to X$ are the natural projections, a similar argument shows that the exact functor $\mathbf{R}p_*(-\otimes q^*E)$ maps $\tilde{C}_m$ to a complex

$$\tilde{C}_{m,E} = \{H^0(X,A_m\otimes E)\otimes F(B_m) \xrightarrow{\tilde{\delta}_{m,E}} \cdots \xrightarrow{\tilde{\delta}_{1,E}} H^0(X,A_0\otimes E)\otimes F(B_0)\}$$

in $\mathrm{D^b}(Y,\beta)$, which has a unique up to isomorphism right convolution

$$\tilde{\delta}'_{0,m,E}: H^0(X,A_0\otimes E)\otimes F(B_0)\to \mathbf{R}p_*(\mathcal{G}_m\otimes q^*E)=\Phi_{\mathcal{G}_m}(E). \tag{4.2}$$

On the other hand, $\tilde{C}_{m,E}$ can be identified with the image of $C_{m,E}$ through F, so that a right convolution of $\tilde{C}_{m,E}$ is given also by

$$(F(\delta_{0,E}),0): H^0(X,A_0\otimes E)\otimes F(B_0)\to F(E)\oplus F(\mathcal{K}_{m,E})[m]. \tag{4.3}$$

Therefore $\Phi_{\mathcal{G}_m}(E)\cong F(E)\oplus F(\mathcal{K}_{m,E})[m]$, and so, in particular, $H^i(\Phi_{\mathcal{G}_m}(E))=0$ unless $i\in[-m-M,-m]\cup[-M,0]$. Since this holds for every $E\in\mathbf{K}_m$, applying Lemma 2.5 we deduce that also $H^i(\mathcal{G}_m)=0$ unless $i\in[-m-M,-m]\cup[-M,0]$. This implies that $\mathcal{G}_m\cong\mathcal{E}_m\oplus\mathcal{F}_m$ with $H^i(\mathcal{E}_m)=0$ unless $i\in[-M,0]$ and $H^i(\mathcal{F}_m)=0$ unless $i\in[-m-M,-m]$.

4.3. **Uniqueness of the kernel.** We are going to show that a kernel of F (if it exists) is necessarily isomorphic to $\mathcal{E}_m$ for $m\gg 0$. Indeed, assume that $\mathcal{E}\in\mathrm{D^b}(X\times Y,\alpha^{-1}\boxtimes\beta)$ is such that $F\cong\Phi_{\mathcal{E}}$. A standard computation shows that

$$\mathcal{E}' := p_{13}^*\mathcal{O}_\Delta \overset{\mathbf{L}}{\otimes} p_{24}^*\mathcal{E}\in\mathrm{D^b}(X\times X\times X\times Y,\alpha\boxtimes\alpha^{-1}\boxtimes\alpha^{-1}\boxtimes\beta)$$

(where p_{ij} is the obvious projection from $X\times X\times X\times Y$ and $\mathcal{O}_\Delta$ is now considered to be $(\alpha\boxtimes\alpha^{-1})$-twisted) defines a functor of Fourier-Mukai type

$$\Phi_{\mathcal{E}'}:\mathrm{D^b}(X\times X,\alpha^{-1}\boxtimes\alpha)\to\mathrm{D^b}(X\times Y,\alpha^{-1}\boxtimes\beta)$$

such that $\Phi_{\mathcal{E}'}(\mathcal{F})\cong\mathbf{R}(q_{13})_*(q_{12}^*\mathcal{F}\overset{\mathbf{L}}{\otimes} q_{23}^*\mathcal{E})$ for every $\mathcal{F}\in\mathrm{D^b}(X\times X,\alpha^{-1}\boxtimes\alpha)$ (here, again, q_{ij} denotes the obvious projection from $X\times X\times Y$). It follows easily that $\Phi_{\mathcal{E}'}(\mathcal{O}_\Delta)\cong\mathcal{E}$ and

$$\Phi_{\mathcal{E}'}(\mathcal{A}\overset{\mathbf{L}}{\boxtimes}\mathcal{B})\cong\mathcal{A}\overset{\mathbf{L}}{\boxtimes}\Phi_{\mathcal{E}}(\mathcal{B})\cong\mathcal{A}\overset{\mathbf{L}}{\boxtimes}F(\mathcal{B})$$

for $\mathcal{A}\in\mathrm{D^b}(X,\alpha^{-1})$ and $\mathcal{B}\in\mathrm{D^b}(X,\alpha)$. In particular, we see that $\Phi_{\mathcal{E}'}$ maps the complex C_m to $\tilde{C}_m$, hence if $m>2\dim(X)$ a convolution of the latter complex is given by $\Phi_{\mathcal{E}'}(\mathcal{O}_\Delta\oplus\mathcal{K}_m[m])\cong\mathcal{E}\oplus\Phi_{\mathcal{E}'}(\mathcal{K}_m)[m]$. Therefore $\mathcal{E}\oplus\Phi_{\mathcal{E}'}(\mathcal{K}_m)[m]\cong\mathcal{G}_m\cong\mathcal{E}_m\oplus\mathcal{F}_m$, and we can conclude that $\mathcal{E}\cong\mathcal{E}_m$ (and $\Phi_{\mathcal{E}'}(\mathcal{K}_m)[m]\cong\mathcal{F}_m$) provided $m\gg 0$ (more precisely, it is enough that $\mathrm{Hom}_{\mathrm{D^b}(X\times Y,\alpha^{-1}\boxtimes\beta)}(\mathcal{E},\mathcal{F}_m)=0$ and $\mathrm{Hom}_{\mathrm{D^b}(X\times Y,\alpha^{-1}\boxtimes\beta)}(\mathcal{E}_m,\Phi_{\mathcal{E}'}(\mathcal{K}_m)[m])=0$, which is certainly true for large m by definition of $\mathcal{E}_m$ and $\mathcal{F}_m$ and because $\Phi_{\mathcal{E}'}$ is bounded).

4.4. **Isomorphism of functors on a subcategory.** Now we fix an integer $m>\dim(X)+\dim(Y)+M$ and we will prove that $\mathcal{E}:=\mathcal{E}_m$ is really a kernel of F. To simplify the notation we will suppress the subscript m also from $\mathcal{G}_m$, $\mathcal{F}_m$, $\tilde{C}_{m,E}$, $\tilde{\delta}'_{0,m,E}$ and $\mathbf{K}_m$. As a first step, we will show that $\Phi_{\mathcal{E}}|_{\mathbf{K}}$ and $F|_{\mathbf{K}}$ are isomorphic as functors from $\mathbf{K}$ to $\mathrm{D^b}(Y,\beta)$. To see this we use the argument in [11, Lemma 6.2]. In fact for every $E\in\mathbf{K}$ by (4.2) and (4.3) the complex $\tilde{C}_E$ has two right convolutions, namely

$$\tilde{\delta}'_{0,E}=(\tilde{\delta}_{0,E},0): H^0(X,A_0\otimes E)\otimes F(B_0)\to\Phi_{\mathcal{G}}(E)\cong\Phi_{\mathcal{E}}(E)\oplus\Phi_{\mathcal{F}}(E)$$

and $(F(\delta_{0,E}),0)$. Due to Lemma 3.3 this implies that there exists a unique isomorphism $\varphi(E):\Phi_{\mathcal{E}}(E)\xrightarrow{\sim}F(E)$ such that $F(\delta_{0,E})=\varphi(E)\circ\tilde{\delta}_{0,E}$. In order to see that this isomorphism is functorial, just notice that for every morphism $\gamma : E \to E'$ of $\mathbf{K}$, again by Lemma 3.3, there is a unique morphism $\Phi_{\mathcal{E}}(E) \to F(E')$ such that the diagram

$$\begin{array}{ccc} H^0(X, A_0 \otimes E) \otimes F(B_0) & \xrightarrow{\tilde{\delta}_{0,E}} & \Phi_{\mathcal{E}}(E) \\ \Big\downarrow{\scriptstyle H^0(\mathrm{id}\otimes\gamma)\otimes\mathrm{id}} & & \Big\downarrow \\ H^0(X, A_0 \otimes E') \otimes F(B_0) & \xrightarrow{F(\delta_{0,E'})} & F(E') \end{array}$$

commutes. Since both $F(\gamma) \circ \varphi(E)$ and $\varphi(E') \circ \Phi_{\mathcal{E}}(\gamma)$ satisfy this property, they must be equal.

4.5. **Extending the isomorphism.** Now we choose an α-twisted locally free sheaf E and a very ample line bundle L on X (defining an embedding $X \hookrightarrow \mathbb{P}^N$) and we denote by $\mathbf{C}$ the full subcategory of $\mathbf{Coh}(X, \alpha)$ with objects $\{E \otimes L^{\otimes k}\}_{k \in \mathbb{Z}}$. Now, by Lemma 2.3 this set of objects is an ample sequence in $\mathbf{Coh}(X, \alpha)$, hence by Proposition 3.7 in order to prove that $F \cong \Phi_{\mathcal{E}}$ it is enough to show that $F|_{\mathbf{C}} \cong \Phi_{\mathcal{E}}|_{\mathbf{C}}$. To this purpose, we proceed as in [11, Lemma 6.4] and we define isomorphisms $\varphi_k : F(E \otimes L^{\otimes k}) \xrightarrow{\sim} \Phi_{\mathcal{E}}(E \otimes L^{\otimes k})$ (for $k \in \mathbb{Z}$) such that

$$\Phi_{\mathcal{E}}(\gamma) \circ \varphi_{k_1} = \varphi_{k_2} \circ F(\gamma) \tag{4.4}$$

for every morphism $\gamma : E \otimes L^{\otimes k_1} \to E \otimes L^{\otimes k_2}$ of $\mathbf{C}$ and for every $k_1, k_2 \in \mathbb{Z}$. By definition of ample sequence there exists $k_0 \in \mathbb{Z}$ such that $E \otimes L^{\otimes k} \in \mathbf{K}$ for $k \geq k_0$. Then, setting $\varphi_k := \varphi(E \otimes L^{\otimes k})^{-1}$ for $k \geq k_0$, the equation (4.4) is satisfied for $k_1, k_2 \geq k_0$. Now we proceed by descending induction: assuming φ_k is defined for $k > n$ and (4.4) is satisfied for $k_1, k_2 > n$, we define φ_n as follows. As in (2.1), Beilinson's resolution gives an exact sequence in $\mathbf{Coh}(X)$

$$0 \to \mathcal{O}_X \xrightarrow{\rho_{N+1}} L \otimes V_N \xrightarrow{\rho_N} \cdots \xrightarrow{\rho_2} L^{\otimes N} \otimes V_1 \xrightarrow{\rho_1} L^{\otimes N+1} \otimes V_0 \to 0$$

(where each V_i is a finite dimensional vector space), hence, setting $\rho_i^{(n)} := \mathrm{id}_{E \otimes L^{\otimes n}} \otimes \rho_i$, the complex

$$E \otimes L^{\otimes n+1} \otimes V_N \xrightarrow{\rho_N^{(n)}} \cdots \xrightarrow{\rho_1^{(n)}} E \otimes L^{\otimes n+N+1} \otimes V_0$$

in $\mathrm{D^b}(X, \alpha)$ has a unique up to isomorphism left convolution $\rho_{N+1}^{(n)} : E \otimes L^{\otimes n} \to E \otimes L^{\otimes n+1} \otimes V_N$. The inductive hypothesis implies that

$$\begin{array}{ccccc} F(E \otimes L^{\otimes n+1}) \otimes V_N & \xrightarrow{F(\rho_N^{(n)})} & \cdots & \xrightarrow{F(\rho_1^{(n)})} & F(E \otimes L^{\otimes n+N+1}) \otimes V_0 \\ \Big\downarrow{\scriptstyle \varphi_{n+1}\otimes\mathrm{id}} & & & & \Big\downarrow{\scriptstyle \varphi_{n+N+1}\otimes\mathrm{id}} \\ \Phi_{\mathcal{E}}(E \otimes L^{\otimes n+1}) \otimes V_N & \xrightarrow{\Phi_{\mathcal{E}}(\rho_N^{(n)})} & \cdots & \xrightarrow{\Phi_{\mathcal{E}}(\rho_1^{(n)})} & \Phi_{\mathcal{E}}(E \otimes L^{\otimes n+N+1}) \otimes V_0 \end{array}$$

is an isomorphism of complexes in $\mathrm{D^b}(Y, \beta)$ which satisfies the assumptions of Lemma 3.3, hence there is a unique isomorphism φ_n such that the diagram

$$\begin{array}{ccc} F(E \otimes L^{\otimes n}) & \xrightarrow{F(\rho_{N+1}^{(n)})} & F(E \otimes L^{\otimes n+1}) \otimes V_N \\ \Big\downarrow{\scriptstyle \varphi_n} & & \Big\downarrow{\scriptstyle \varphi_{n+1}\otimes\mathrm{id}} \\ \Phi_{\mathcal{E}}(E \otimes L^{\otimes n}) & \xrightarrow{\Phi_{\mathcal{E}}(\rho_{N+1}^{(n)})} & \Phi_{\mathcal{E}}(E \otimes L^{\otimes n+1}) \otimes V_N \end{array}$$

commutes. Moreover, for every morphism $\gamma : E \otimes L^{\otimes k_1} \to E \otimes L^{\otimes k_2}$ of $\mathbf{C}$ and for every $k_1, k_2 \geq n$

$$\begin{array}{ccccc}
F(E \otimes L^{\otimes k_1+1}) \otimes V_N & \xrightarrow{F(\rho_N^{(k_1)})} & \cdots & \xrightarrow{F(\rho_1^{(k_1)})} & F(E \otimes L^{\otimes k_1+N+1}) \otimes V_0 \\
\downarrow{\scriptstyle \tilde{\gamma}_N \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \tilde{\gamma}_0 \otimes \mathrm{id}} \\
\Phi_{\mathcal{E}}(E \otimes L^{\otimes k_2+1}) \otimes V_N & \xrightarrow{\Phi_{\mathcal{E}}(\rho_N^{(k_2)})} & \cdots & \xrightarrow{\Phi_{\mathcal{E}}(\rho_1^{(k_2)})} & \Phi_{\mathcal{E}}(E \otimes L^{\otimes k_2+N+1}) \otimes V_0
\end{array}$$

(where $\tilde{\gamma}_i := \varphi_{k_2+N+1-i} \circ F(\gamma \otimes \mathrm{id}_{L^{\otimes N+1-i}}) = \Phi_{\mathcal{E}}(\gamma \otimes \mathrm{id}_{L^{\otimes N+1-i}}) \circ \varphi_{k_1+N+1-i}$) is a morphism of complexes in $\mathrm{D^b}(Y, \beta)$ which again satisfies the assumptions of Lemma 3.3. Therefore, there is a unique morphism $F(E \otimes L^{\otimes k_1}) \to \Phi_{\mathcal{E}}(E \otimes L^{\otimes k_2})$ such that the diagram

$$\begin{array}{ccc}
F(E \otimes L^{\otimes k_1}) & \xrightarrow{F(\rho_{N+1}^{(k_1)})} & F(E \otimes L^{\otimes k_1+1}) \otimes V_N \\
\downarrow & & \downarrow{\scriptstyle \tilde{\gamma}_N \otimes \mathrm{id}} \\
\Phi_{\mathcal{E}}(E \otimes L^{\otimes k_2}) & \xrightarrow{\Phi_{\mathcal{E}}(\rho_{N+1}^{(k_2)})} & \Phi_{\mathcal{E}}(E \otimes L^{\otimes k_2+1}) \otimes V_N
\end{array}$$

commutes, and, since both $\Phi_{\mathcal{E}}(\gamma) \circ \varphi_{k_1}$ and $\varphi_{k_2} \circ F(\gamma)$ satisfy this property, we conclude that (4.4) holds.

Remark 4.1. Theorem 1.1 in [11] concerns fully faithful functors. This requirement is essential in Kawamata's proof only in [11, Lemma 6.5] (which depends on [14, Prop. 2.16]). Kawamata's argument can now be reconsidered using Proposition 3.7 instead of [11, Lemma 6.5]. Hence we immediately get the following generalization of Kawamata's result. Let X and Y be normal projective varieties with only quotient singularities and let $\mathcal{X}$ and $\mathcal{Y}$ be the smooth stacks naturally associated to them. Let $F : \mathrm{D^b}(\mathbf{Coh}(\mathcal{X})) \to \mathrm{D^b}(\mathbf{Coh}(\mathcal{Y}))$ be an exact functor with a left adjoint and such that, for any $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}(\mathcal{X})$,

$$\mathrm{Hom}_{\mathrm{D^b}(\mathbf{Coh}(\mathcal{Y}))}(F(\mathcal{F}), F(\mathcal{G})[j]) = 0$$

if $j < 0$. Then there exists a unique up to isomorphism $\mathcal{E} \in \mathrm{D^b}(\mathbf{Coh}(\mathcal{X} \times \mathcal{Y}))$ and an isomorphism of functors $F \cong \Phi_{\mathcal{E}}$.

Observe moreover that, in Kawamata's proof, the results in [11, Sect. 3] can be replaced by our shorter argument in Section 4.1.

5. Exact functors between the abelian categories of twisted sheaves

Theorem 1.1 can be used to classify exact functors from $\mathbf{Coh}(X, \alpha)$ to $\mathbf{Coh}(Y, \beta)$.

Proposition 5.1. *Let (X, α) and (Y, β) be twisted varieties. If $\mathcal{E}$ is in $\mathbf{Coh}(X \times Y, \alpha^{-1} \boxtimes \beta)$, then the additive functor*

$$\Psi_{\mathcal{E}} := p_*(\mathcal{E} \otimes q^*(-)) : \mathbf{Coh}(X, \alpha) \to \mathbf{Coh}(Y, \beta)$$

is exact if and only if $\mathcal{E}$ is flat over X and $p|_{\mathrm{Supp}(\mathcal{E})} : \mathrm{Supp}(\mathcal{E}) \to Y$ is a finite morphism.

Moreover, for every exact functor $G : \mathbf{Coh}(X, \alpha) \to \mathbf{Coh}(Y, \beta)$ there exists unique up to isomorphism $\mathcal{E} \in \mathbf{Coh}(X \times Y, \alpha^{-1} \boxtimes \beta)$ (flat over X and with $p|_{\mathrm{Supp}(\mathcal{E})}$ finite) such that $G \cong \Psi_{\mathcal{E}}$.

Proof. Clearly $\mathcal{E}$ is flat over X if and only if the functor $\mathcal{E} \otimes q^*(-)$ is exact, and in this case $\Psi_{\mathcal{E}}$ is left exact and $\mathbf{R}\Psi_{\mathcal{E}} \cong \mathbf{R}p_*(\mathcal{E} \otimes q^*(-))$. Notice also that $\mathcal{E} \otimes q^*(-)$ is exact if $\Psi_{\mathcal{E}}$ is exact. Indeed, given an injective morphism $\mathcal{F} \hookrightarrow \mathcal{G}$ in $\mathbf{Coh}(X, \alpha)$ and setting

$$\mathcal{K} := \ker(\mathcal{E} \otimes q^*\mathcal{F} \to \mathcal{E} \otimes q^*\mathcal{G}),$$

for every $E \in \mathbf{Coh}(X)$ locally free there is an exact sequence in $\mathbf{Coh}(Y, \beta)$

$$0 \to p_*(\mathcal{K} \otimes q^*E) \to p_*(\mathcal{E} \otimes q^*(\mathcal{F} \otimes E)) = \Psi_{\mathcal{E}}(\mathcal{F} \otimes E) \to p_*(\mathcal{E} \otimes q^*(\mathcal{G} \otimes E)) = \Psi_{\mathcal{E}}(\mathcal{G} \otimes E),$$

from which we see that, if $\Psi_{\mathcal{E}}$ is exact, $p_*(\mathcal{K}\otimes q^*E)=0$; therefore $\mathcal{K}=0$, and this proves that $\mathcal{E}\otimes q^*(-)$ is exact. It follows that, in order to conclude the proof of the first statement, it is enough to show that $p|_{\mathrm{Supp}(\mathcal{E})}$ is finite if and only if $\mathbf{R}^jp_*(\mathcal{E}\otimes q^*\mathcal{F})=0$ for $j>0$ and for every $\mathcal{F}\in\mathbf{Coh}(X,\alpha)$. To this purpose, up to replacing $\mathcal{E}$ with $\mathcal{E}\otimes p^*F$ for some locally free sheaf $F\in\mathbf{Coh}(Y,\beta^{-1})$, we can assume that β is trivial (because $\mathrm{Supp}(\mathcal{E})=\mathrm{Supp}(\mathcal{E}\otimes p^*F)$ and $\mathbf{R}^jp_*(\mathcal{E}\otimes p^*F\otimes q^*\mathcal{F})\cong F\otimes\mathbf{R}^jp_*(\mathcal{E}\otimes q^*\mathcal{F})$).

If $p|_{\mathrm{Supp}(\mathcal{E})}$ is finite, we can find a cover by open affine subsets $\{V_i\}_{i\in I}$ of Y and $U_i\subset X$ open affine such that $\mathrm{Supp}(\mathcal{E}|_{X\times V_i})\subset U_i\times V_i$ for every $i\in I$. Then, denoting by $p_i:X\times V_i\to V_i$ and $p'_i:U_i\times V_i\to V_i$ the projections, for $j>0$ and for every $\mathcal{F}\in\mathbf{Coh}(X,\alpha)$ we have

$$\mathbf{R}^jp_*(\mathcal{E}\otimes q^*\mathcal{F})|_{V_i}\cong\mathbf{R}^jp_{i*}((\mathcal{E}\otimes q^*\mathcal{F})|_{X\times V_i})\cong\mathbf{R}^jp'_{i*}((\mathcal{E}\otimes q^*\mathcal{F})|_{U_i\times V_i})=0$$

because p'_i is an affine morphism, hence $\mathbf{R}^jp_*(\mathcal{E}\otimes q^*\mathcal{F})=0$. On the other hand, if $p|_{\mathrm{Supp}(\mathcal{E})}$ is not finite, there exist a closed point $y\in Y$ and a closed irreducible subset $X'\subseteq X$ such that $d:=\dim(X')>0$ and $X'\subseteq\mathrm{Supp}(\mathcal{E}_y)$, where $\mathcal{E}_y\in\mathbf{Coh}(X,\alpha^{-1})$ corresponds to $\mathcal{E}|_{X\times\{y\}}$ under the natural isomorphism $X\cong X\times\{y\}$. We claim that there exists $\mathcal{F}_0\in\mathbf{Coh}(X,\alpha)$ such that $\mathrm{Supp}(\mathcal{F}_0)=X'$ and $H^d(X,\mathcal{E}_y\otimes\mathcal{F}_0)\neq0$. For instance, denoting by E a locally free α-twisted sheaf on X and by $\mathcal{O}_{X'}(1)$ a very ample line bundle on X' (regarded as a subscheme of X with the reduced induced structure), we can take $\mathcal{F}_0=E\otimes\mathcal{O}_{X'}(-n)$ for $n\gg0$. Indeed, by definition of the dualizing sheaf $\omega^\circ_{X'}$ (see [7, p. 241]), we have

$$\begin{aligned}H^d(X,\mathcal{E}_y\otimes E\otimes\mathcal{O}_{X'}(-n))^\vee&\cong H^d(X',(\mathcal{E}_y\otimes E)|_{X'}(-n))^\vee\\&\cong\mathrm{Hom}_{X'}((\mathcal{E}_y\otimes E)|_{X'},\omega^\circ_{X'}(n))\cong H^0(\mathcal{H}om_{X'}((\mathcal{E}_y\otimes E)|_{X'},\omega^\circ_{X'})(n)),\end{aligned}$$

and the last term is not 0 for $n\gg0$, since $\mathcal{H}om_{X'}((\mathcal{E}_y\otimes E)|_{X'},\omega^\circ_{X'})\neq0$ due to the fact that $\mathrm{Supp}((\mathcal{E}_y\otimes E)|_{X'})=X'$ and $\omega^\circ_{X'}\cong\omega_{X'}$ on the non-empty open subset where X' is smooth.

Then let $V\subset Y$ be an open affine subset containing y and denote by $q':X\times V\to X$ the projection. Applying the right exact functor $q'_*(-)\otimes\mathcal{F}_0$ to the natural surjective morphism $\mathcal{E}|_{X\times V}\twoheadrightarrow\mathcal{E}|_{X\times}\{y\}$, we get a surjective morphism in $\mathbf{QCoh}(X)$

$$\varphi:q'_*(\mathcal{E}|_{X\times V})\otimes\mathcal{F}_0\twoheadrightarrow q'_*(\mathcal{E}|_{X\times\{y\}})\otimes\mathcal{F}_0\cong\mathcal{E}_y\otimes\mathcal{F}_0.$$

As $\mathrm{Supp}(\ker(\varphi))\subseteq X'$, we have $H^{d+1}(X,\ker(\varphi))=0$, hence the assumption $H^d(X,\mathcal{E}_y\otimes\mathcal{F}_0)\neq0$ implies that

$$0\neq H^d(X,q'_*(\mathcal{E}|_{X\times V})\otimes\mathcal{F}_0)\cong H^d(X\times V,(\mathcal{E}\otimes q^*\mathcal{F}_0)|_{X\times V}),$$

and this proves that $\mathbf{R}^dp_*(\mathcal{E}\otimes q^*\mathcal{F}_0)\neq0$.

Assume now that $G:\mathbf{Coh}(X,\alpha)\to\mathbf{Coh}(Y,\beta)$ is an exact functor. By Theorem 1.1 there exists (unique up to isomorphism) $\mathcal{E}\in\mathrm{D^b}(X\times Y,\alpha^{-1}\boxtimes\beta)$ such that $\mathrm{D^b}(G)\cong\Phi_{\mathcal{E}}$, and $\mathcal{E}\in\mathbf{Coh}(X\times Y,\alpha^{-1}\boxtimes\beta)$ by Lemma 2.5. From the fact that $\Phi_{\mathcal{E}}(\mathbf{Coh}(X,\alpha))\subseteq\mathbf{Coh}(Y,\beta)$ it is easy to deduce that

$$G\cong\Phi_{\mathcal{E}}|_{\mathbf{Coh}(X,\alpha)}\cong\Psi_{\mathcal{E}}.$$

The uniqueness of $\mathcal{E}$ follows from Corollary 3.8. □

Remark 5.2. The above result implies that there are no non-zero exact functors from $\mathbf{Coh}(X,\alpha)$ to $\mathbf{Coh}(Y,\beta)$ if $\dim(X)>\dim(Y)$: to prove this, just note that if $0\neq\mathcal{E}\in\mathbf{Coh}(X\times Y,\alpha^{-1}\boxtimes\beta)$ is flat over X then $\dim(\mathrm{Supp}(\mathcal{E}))\geq\dim(X)$, and that $\dim(\mathrm{Supp}(\mathcal{E}))\leq\dim(Y)$ if $p|_{\mathrm{Supp}(\mathcal{E})}$ is finite.

It was proved by Gabriel in [6] that if X and Y are noetherian schemes then there exists an exact equivalence $\mathbf{QCoh}(X)\cong\mathbf{QCoh}(Y)$ if and only if X and Y are isomorphic. For smooth projective varieties, a short proof (relying on Orlov's result) of an analogous statement involving coherent sheaves was given in [8]. Following this last approach and using Proposition 5.1 we prove a Gabriel-type result for twisted varieties.

Corollary 5.3. *Let (X, α) and (Y, β) be twisted varieties. Then the following three conditions are equivalent:*

(i) *there is an exact equivalence between $\mathbf{QCoh}(X, \alpha)$ and $\mathbf{QCoh}(Y, \beta)$;*
(ii) *there is an exact equivalence between $\mathbf{Coh}(X, \alpha)$ and $\mathbf{Coh}(Y, \beta)$;*
(iii) *there exists an isomorphism $f : X \xrightarrow{\sim} Y$ such that $f^*(\beta) = \alpha$.*

Proof. The implications (iii) ⇒ (i) and (iii) ⇒ (ii) are trivial. Suppose that an exact equivalence $G : \mathbf{QCoh}(X, \alpha) \xrightarrow{\sim} \mathbf{QCoh}(Y, \beta)$ is assigned and consider the equivalence $\mathrm{D}(G) : \mathrm{D}(\mathbf{QCoh}(X, \alpha)) \xrightarrow{\sim} \mathrm{D}(\mathbf{QCoh}(Y, \beta))$ induced by G. Due to [15, Thm. 18] and [3, Lemma 2.1.4, Prop. 2.1.8], $\mathrm{D}(G)$ restricts to an equivalence $F : \mathrm{D}^{\mathrm{b}}(X, \alpha) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y, \beta)$ which yields an exact equivalence $G' : \mathbf{Coh}(X, \alpha) \xrightarrow{\sim} \mathbf{Coh}(Y, \beta)$. This proves that (i) implies (ii).

The proof of the implication (ii) ⇒ (iii) proceeds now as in [8, Cor. 5.22, Cor. 5.23]. First of all, recall that given an abelian category $\mathbf{A}$, $A \in \mathrm{Ob}(\mathbf{A})$ is *minimal* if any non-trivial surjective morphism $A \to B$ in $\mathbf{A}$ is an isomorphism. Notice that an equivalence $F : \mathbf{A} \xrightarrow{\sim} \mathbf{B}$ sends minimal objects to minimal objects.

It is easy to see that the set of minimal objects of $\mathbf{Coh}(X, \alpha)$ consists of all skyscraper sheaves $\mathcal{O}_x$, where x is a closed point of X. Let $G : \mathbf{Coh}(X, \alpha) \xrightarrow{\sim} \mathbf{Coh}(Y, \beta)$ be an exact equivalence. By Proposition 5.1, $G \cong \Psi_{\mathcal{E}}$, for some $\mathcal{E} \in \mathbf{Coh}(X \times Y, \alpha^{-1} \boxtimes \beta)$.

Since G maps skyscraper sheaves to skyscraper sheaves, $\mathcal{E}|_{\{x\} \times Y}$ is isomorphic to a skyscraper sheaf and we naturally get a morphism $f : X \to Y$ and $L \in \mathrm{Pic}(X)$ such that

$$G \cong L \otimes f_*(-).$$

The morphism f is actually an isomorphism since G is an equivalence ([8, Cor. 5.23]) and, by definition, $f^*(\beta) = \alpha$. □

This proves Conjecture 1.3.17 in [3] for quasi-coherent sheaves on smooth projective varieties.

Remark 5.4. Suppose that X and Y are smooth separated schemes of finite type over a field K and that $\alpha \in \mathrm{Br}(X)$ while $\beta \in \mathrm{Br}(Y)$. In [15] it was proved that if there exists an exact equivalence $\mathbf{Coh}(X, \alpha) \cong \mathbf{Coh}(Y, \beta)$, then there is an isomorphism $f : X \xrightarrow{\sim} Y$. On the other hand the approach in [15] does not allow to conclude that $f^*(\beta) = \alpha$.

Acknowledgements. This paper was partially written while the second named author was visiting the Max-Planck-Institut für Mathematik (Bonn). It is a great pleasure to thank the staff for the exciting mathematical atmosphere at the Institute.

References

[1] A. Beilinson, *Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra*, Funct. Anal. Appl. **12** (1979), 214–216.
[2] A. Bondal, M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Moscow Math. J. **3** (2003), 1–36.
[3] A. Căldăraru, *Derived categories of twisted sheaves on Calabi-Yau manifolds*, PhD-Thesis, Cornell University (2000).
[4] A. Căldăraru, *Non-fine moduli spaces of sheaves on K3 surfaces*, Int. Math. Res. Not. **20** (2002), 1027–1056.
[5] A.J. de Jong, *A result of Gabber*, preprint (2003).
[6] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–348.
[7] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics Vol. 52, Springer-Verlag (1977).
[8] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, Oxford Science Publications (2006).
[9] D. Huybrechts, P. Stellari, *Equivalences of twisted K3 surfaces*, Math. Ann. **332** (2005), 901–936.
[10] D. Huybrechts, P. Stellari, *Proof of Căldăraru's conjecture*, to appear in: The 13th MSJ Inter. Research Inst. Moduli Spaces and Arithmetic Geometry, Adv. Stud. Pure Math., AMS, math.AG/0411541.
[11] Y. Kawamata, *Equivalences of derived categories of sheaves on smooth stacks*, Am. J. Math. **126** (2004), 1057–1083.
[12] M. Lieblich, *Moduli of twisted sheaves and generalized Azumaya algebras*, PhD-thesis, MIT (2004).

[13] M. Lieblich, *Moduli of twisted sheaves*, to appear in: Duke Math. J., math.AG/0411337.
[14] D. Orlov, *Equivalences of derived categories and K3 surfaces*, J. Math. Sci. **84** (1997), 1361–1381.
[15] A. Perego, *A Gabriel theorem for coherent twisted sheaves*, math.AG/0607025.
[16] R. Rouquier, *Dimensions of triangulated categories*, math.CT/0310134.
[17] K. Yoshioka, *Moduli spaces of twisted sheaves on a projective variety*, to appear in: The 13th MSJ Inter. Research Inst. Moduli Spaces and Arithmetic Geometry, Adv. Stud. Pure Math., AMS, math.AG/0411538.

A.C.: Dipartimento di Matematica "F. Casorati", Università degli Studi di Pavia, Via Ferrata 1, 27100 Pavia, Italy
E-mail address: alberto.canonaco@unipv.it

P.S.: Dipartimento di Matematica "F. Enriques", Università degli Studi di Milano, Via Cesare Saldini 50, 20133 Milano, Italy
E-mail address: paolo.stellari@unimi.it
URL: http://users.unimi.it/stellari