UNIQUENESS OF DG ENHANCEMENTS FOR THE DERIVED CATEGORY OF A GROTHENDIECK CATEGORY

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Abstract. We prove that the derived category of a Grothendieck abelian category has a unique dg enhancement. Under some additional assumptions, we show that the same result holds true for its subcategory of compact objects. As a consequence, we deduce that the unbounded derived category of quasi-coherent sheaves on an algebraic stack and the category of perfect complexes on a noetherian concentrated algebraic stack with quasi-finite affine diagonal and enough perfect coherent sheaves have a unique dg enhancement. In particular, the category of perfect complexes on a noetherian semi-separated scheme with enough locally free sheaves has a unique dg enhancement.

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Introduction

The relation between triangulated categories and higher categorical structures is highly non-trivial, very rich in nature and with various appearances in the recent developments of derived algebraic geometry. The easiest thing we can do is to produce a triangulated category $\mathbf{T}$ out of a pretriangulated dg category $\mathbf{D}$ by taking the homotopy category of $\mathbf{D}$. Roughly speaking, a pretriangulated dg category $\mathbf{D}$ whose homotopy category is equivalent to a triangulated category $\mathbf{T}$ is called a *dg enhancement* (or *enhancement*, for short) of $\mathbf{T}$.

Now, there exist triangulated categories with no enhancements at all. For example, this happens to some triangulated categories naturally arising in topology (see [36] or [17, Section 3.6] for a

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discussion about this). Triangulated categories admitting an enhancement are called *algebraic* (as it is explained, for example, in [36, Section 3], algebraic triangulated categories are often defined in other equivalent ways). In practice, all triangulated categories one usually encounters in algebra or algebraic geometry are algebraic. For instance, the derived category of a Grothendieck category, as well as its full subcategory of compact objects are algebraic. Recall that a *Grothendieck category* is an abelian category $\mathbf{C}$ which is closed under small coproducts, has a set of generators $\mathbf{S}$ and the direct limits of short exact sequences are exact. The objects in $\mathbf{S}$ are generators in the sense that, for any C in $\mathbf{C}$, there exists an epimorphism $S \twoheadrightarrow C$ in $\mathbf{C}$, where S is a coproduct of objects in $\mathbf{S}$.

In particular, if X is a scheme or, more generally, an algebraic stack it is not difficult to construct explicit enhancements of the derived category $\mathrm{D}(\mathbf{Qcoh}(X))$ of quasi-coherent sheaves on X and, under mild assumptions, of the bounded derived categories of coherent sheaves $\mathrm{D}^b(X)$ and the category of perfect complexes $\mathbf{Perf}(X)$ on X. For example, this can be achieved by taking complexes of injective sheaves, Čech resolutions, chain complexes of sheaves in the corresponding categories or perfect complexes (see [3] and [25]).

Even when we know that an enhancement exists, one may wonder whether it is unique. Roughly, we say that a triangulated category $\mathbf{T}$ has a *unique enhancement* $\mathbf{D}$ if any other enhancement is related to $\mathbf{D}$ by a sequence of quasi-equivalences. These are the analogue, at the dg level, of the exact equivalences in the triangulated setting. Actually, at this level of generality, we may not expect a positive answer to the above question. Indeed, the result of Dugger and Shipley [11] easily yields an example of two $\mathbb{Z}$-linear pretriangulated dg categories which are not quasi-equivalent but whose homotopy categories are equivalent. The search of a similar example over a field rather than over a commutative ring is still a challenge.

Again, if we move to the geometric setting, then for a long while it was expected that any of the three triangulated categories $\mathrm{D}(\mathbf{Qcoh}(X))$, $\mathrm{D}^b(X)$ and $\mathbf{Perf}(X)$ should have unique enhancements, when X is a (quasi-)projective scheme. This was formally stated as a conjecture (even in a stronger form) by Bondal, Larsen and Lunts [3].

As we will explain later, this conjecture was positively solved by Lunts and Orlov in their seminal paper [24]. It should be noted that the quest for uniqueness of enhancements has a foundational relevance that cannot be overestimated by the 'working algebraic geometer'. Let us just mention an instance where the fact of having a unique enhancement has interesting consequences. The homological version of the so called *Mirror Symmetry Conjecture* by Kontsevich [18] predicts the existence of an A_∞-equivalence between a dg enhancement of $\mathrm{D}^b(X)$, for X a smooth projective scheme, and the Fukaya category of the mirror Y of X, which is actually an A_∞-category. The fact that the dg enhancements are unique allows us to conclude that finding an A_∞-equivalence (or rather a sequence of them) is the same as finding an exact equivalence between the corresponding homotopy categories. More generally, several geometric problems can be lifted to the dg level and treated there in a universal way (e.g. moduli problems or the characterization of exact functors). Having bridges between the different dg incarnations of the same triangulated or geometric problem is then crucial.

Let us now explain the contributions of this paper to the problem of showing the uniqueness of dg enhancements in geometric settings. The first point to make, which should be clear from now on, is that the analysis of these questions about $\mathrm{D}(\mathbf{Qcoh}(X))$ or $\mathbf{Perf}(X)$ (or $\mathrm{D}^b(X)$) follows slightly different paths. In particular, they can be deduced from two different general criteria whose statements are similar but whose proofs are rather different in nature.

We first consider the case of $\mathrm{D}(\mathbf{Qcoh}(X))$ and, setting the problem at a more abstract level, we first prove the following general result.

Theorem A. *If $\mathbf{C}$ is a Grothendieck category, then $\mathrm{D}(\mathbf{C})$ has a unique enhancement.*

We will explain later some key features in the proof. For the moment, we just recall that the main geometric applications are the following:

- If X is an algebraic stack, $\mathrm{D}(\mathbf{Qcoh}(X))$ has a unique enhancement (see Corollary 5.4);
- If X is a quasi-compact, quasi-separated scheme and α is an element in the Brauer group $\mathrm{Br}(X)$ of X, then the twisted derived category $\mathrm{D}(\mathbf{Qcoh}(X, \alpha))$ has a unique enhancement (see Corollary 5.6).

Now, if we want to study the enhancements of $\mathbf{Perf}(X)$ (and, consequently, of $\mathrm{D}^b(X)$), we should keep in mind that if X is a quasi-compact and semi-separated scheme, then $\mathbf{Perf}(X)$ is the triangulated subcategory of $\mathrm{D}(\mathbf{Qcoh}(X))$ consisting of compact objects. Our general result in this direction is then the following.

Theorem B. *Let $\mathbf{C}$ be a Grothendieck category with a set $\mathbf{A}$ of generators such that*

(1) *$\mathbf{A}$ is closed under finite direct sums;*
(2) *Every object of $\mathbf{A}$ is a noetherian object in $\mathbf{C}$;*
(3) *If $f\colon A' \twoheadrightarrow A$ is an epimorphism of $\mathbf{C}$ with $A, A' \in \mathbf{A}$, then $\ker f \in \mathbf{A}$;*
(4) *For every $A \in \mathbf{A}$ there exists $N(A) > 0$ such that $\mathrm{D}(\mathbf{C})\,(A, A'[N(A)]) = 0$ for every $A' \in \mathbf{A}$.*

Then $\mathrm{D}(\mathbf{C})^c$ has a unique enhancement.

Here $\mathrm{D}(\mathbf{C})^c$ denotes the subcategory of compact objects in $\mathrm{D}(\mathbf{C})$. One may wonder why the result above is conditional while Theorem A does not include any specific assumption on $\mathbf{C}$. We will try to explain later that this is, in a sense, unavoidable but to reassure the reader about the mildness of (1)–(4), let us now discuss some geometric cases where Theorem B applies:

- If X is a noetherian concentrated algebraic stack with quasi-finite affine diagonal and with enough perfect coherent sheaves, then $\mathbf{Perf}(X)$ has a unique enhancement (see Proposition 6.10);
- As a special (but maybe easier to understand) instance of the above case, we have that if X is a noetherian semi-separated scheme with enough locally free sheaves, then $\mathbf{Perf}(X)$ has a unique enhancement (see Corollary 6.11);
- Under the same assumptions on the scheme X, the category $\mathrm{D}^b(X)$ has a unique enhancement (see Corollary 7.2).

The (more or less) standard terminology involved in the above statements will be briefly recalled in Section 6.3.

It is very likely that Theorem A and Theorem B may be used in other geometric contexts. One direct application of the circle of ideas appearing in the proofs of these two results concerns the existence of exact equivalences. In particular, if X_1 and X_2 are noetherian, semi-separated schemes with enough locally free sheaves then the set of equivalences between $\mathbf{Perf}(X_1)$ and $\mathbf{Perf}(X_2)$ is not empty if and only if the same is true for the set of equivalences between $\mathrm{D}(\mathbf{Qcoh}(X_1))$ and $\mathrm{D}(\mathbf{Qcoh}(X_2))$. This is Proposition 7.4.

The strategy of the proof. Before entering into some details of the proof it is worth pointing out the general approach to Theorem A and Theorem B. Even if these results have a dg flavour, the idea is to reduce them to questions about Verdier quotients of triangulated categories. Unfortunately, some of these latter questions are highly non-trivial and involve deep problems concerning the description of the subcategory of compact objects of a quotient. This is the reason why our proofs, which are conceptually quite simple, become technically rather involved.

Let us try to make this more precise and consider first Theorem A. The key observation is that the derived category $\mathrm{D}(\mathbf{C})$ of a Grothendieck category $\mathbf{C}$ is *well generated* in the sense of Neeman [30]. Thus one can choose a set $\mathbf{A}$ of generators for $\mathbf{C}$ such that $\mathrm{D}(\mathbf{C})$ is naturally equivalent to the quotient $\mathcal{D}(\mathbf{A})/\mathbf{L}$, where $\mathcal{D}(\mathbf{A})$ is the derived category of $\mathbf{A}$, seen as a dg category, and $\mathbf{L}$ is an appropriate localizing subcategory of $\mathcal{D}(\mathbf{A})$. This is carried out in Section 5.1.

In Section 5.1 we explain that, after this, Theorem A follows easily once we prove the following general criterion.

Theorem C. *Let $\mathbf{A}$ be a small category and let $\mathbf{L}$ be a localizing subcategory of $\mathcal{D}(\mathbf{A})$ such that:*

(a) *The quotient $\mathcal{D}(\mathbf{A})/\mathbf{L}$ is a well generated triangulated category;*

(b) $\mathcal{D}(\mathbf{A})/\mathbf{L}\left(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_1)),\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_2))[i]\right)\cong 0$*, for all $A_1, A_2 \in \mathbf{A}$ and all integers $i < 0$.*

Then $\mathcal{D}(\mathbf{A})/\mathbf{L}$ has a unique enhancement.

Here $\mathsf{Y}^{\mathbf{A}}\colon \mathbf{A} \to \mathcal{D}(\mathbf{A})$ denotes the Yoneda functor while $\mathsf{Q}\colon \mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{A})/\mathbf{L}$ is the natural quotient functor. We will give the precise definition of well generated triangulated category in Section 1.1. For the moment, it is enough to keep in mind that it is a natural generalization of the usual notion of compactly generated triangulated category. The idea of the proof, which is carried out all along Section 4, is very much inspired by the proof of [24, Theorem 2.7] but it differs at several important technical steps. We will try to clarify them in a while when comparing our results to those in [24]. The geometric applications mentioned above and discussed in Section 5.2, can be deduced easily from the fact that, in all these cases, the category of quasi-coherent sheaves is a Grothendieck category, under our assumptions on X.

Once we have the equivalence $\mathrm{D}(\mathbf{C}) \cong \mathcal{D}(\mathbf{A})/\mathbf{L}$ as above, it is clear that to prove Theorem B we have to show that the triangulated subcategory $(\mathcal{D}(\mathbf{A})/\mathbf{L})^c$ of compact objects in $\mathcal{D}(\mathbf{A})/\mathbf{L}$ has a unique enhancement, for a smart choice of $\mathbf{A}$ (see Section 6.1). For this, one may hope to use Theorem 6.2 which was proved by Lunts and Orlov in [24]. Indeed, this criterion for uniqueness asserts that if we change (a) in Theorem C to

(a') $\mathbf{L}^c = \mathbf{L} \cap \mathcal{D}(\mathbf{A})^c$ and $\mathbf{L}$ is generated by $\mathbf{L}^c$,

and we keep (b), then we can deduce that $(\mathcal{D}(\mathbf{A})/\mathbf{L})^c$ has a unique enhancement as well. In this case, the proof is not too difficult, as we can use the fact that $(\mathcal{D}(\mathbf{A})/\mathbf{L})^c$ and $\mathcal{D}(\mathbf{A})^c/\mathbf{L}^c$ are nicely related, as explained in [28] (see Theorem 1.3).

The issue here is that (a') is not easily verified. Indeed, if $\mathbf{L}$ satisfies (a'), then the inclusion functor $\mathbf{L} \hookrightarrow \mathbf{T}$ has a right adjoint which preserves coproducts. In general, given a compactly generated triangulated category $\mathbf{T}$ closed under small coproducts, a localizing subcategory $\mathbf{L}$ of $\mathbf{T}$ such that the inclusion $\mathbf{L} \hookrightarrow \mathbf{T}$ has the above property is called *smashing subcategory.*

For a while, it was conjectured that all smashing subcategories $\mathbf{L}$ of a triangulated category $\mathbf{T}$ as above should verify (a'). This goes under the name of *Telescope Conjecture* (see [33, 1.33] and [5, 3.4]). Unfortunately, the Telescope Conjecture is known to be false in this generality [15] and to be true in very few examples (see, for example, [29]). This shows that we cannot expect that (a') holds true in general or easily.

In view of this discussion, the main task which is carried out in Section 6.2 is to identify the correct choice for $\mathbf{A}$ such that (a') holds for the corresponding localizing subcategory $\mathbf{L}$. To get this, one has to impose some additional assumptions on $\mathbf{C}$ and $\mathbf{A}$. This is the reason why the hypotheses (1)–(4) appear in Theorem B. Assuming this, the proof of Theorem B is contained in Section 6.2 and the core of the argument is then Theorem 6.6.

The applications concerning the uniqueness of enhancements for $\mathbf{Perf}(X)$ (see Proposition 6.10 and Corollary 6.11) and $\mathrm{D}^b(X)$ (see Corollary 7.2) are rather easy consequences once Theorem B and, respectively, Theorem 6.6 are established.

Related work. As we recalled before, in [3], Bondal, Larsen and Lunts first conjectured that all enhancements of $\mathrm{D}^b(X)$, for X a smooth projective scheme, should be unique. In the same paper, they show that all 'standard' enhancements are related by quasi-equivalences, giving the first evidence to their conjecture.

After that, the main reference is [24] which is certainly the principal source of inspiration for this paper as well. Let us briefly summarize the results contained in that paper and compare them to ours. For $\mathbf{A}$ a small category as in Theorem C, Lunts and Orlov show that $\mathcal{D}(\mathbf{A})/\mathbf{L}$ has a unique enhancement if (b) holds and (a) is replaced by:

(a.1) Q sends $\mathcal{D}(\mathbf{A})^c$ to $(\mathcal{D}(\mathbf{A})/\mathbf{L})^c$, where $(-)^c$ denotes the full subcategory of compact objects;
(a.2) Q has a right adjoint.

This is [24, Theorem 2.7]. It should be noted that (a.1) and (a.2) together imply that $\mathcal{D}(\mathbf{A})/\mathbf{L}$ is compactly generated. This is a special instance of our assumption (a) in Theorem C. Moreover, by [27], there are examples of Grothendieck categories whose derived category is not compactly generated but it is well generated. Hence Theorem C is certainly a generalization of [24, Theorem 2.7]. The geometric consequences of [24, Theorem 2.7], which are discussed in the same paper, are then:

- For a Grothendieck category $\mathbf{C}$, the derived category $\mathrm{D}(\mathbf{C})$ has a unique enhancement, if $\mathbf{C}$ has a small set of generators which are compact in $\mathrm{D}(\mathbf{C})$ (see [24, Theorem 2.9]);
- This implies that if X is a quasi-compact and separated scheme that has enough locally free sheaves, then $\mathrm{D}(\mathbf{Qcoh}(X))$ has a unique enhancement (see [24, Theorem 2.10]).

As a second step, Lunts and Orlov deduce from [24, Theorem 2.8] that if X is a quasi-projective scheme, then both $\mathbf{Perf}(X)$ and $\mathrm{D}^b(X)$ have unique enhancements. A strong version of uniqueness is then discussed. Namely, they prove that these two categories have strongly unique enhancements when X is projective and another technical assumption is satisfied. This is out of the scope of this paper but we believe that the techniques discussed here might have applications to show the strongly uniqueness of dg enhancements in new cases. Indeed, Theorem 6.6 has already been applied to prove the strongly uniqueness of the category of perfect supported complexes (see [8, Theorem 1.2]).

New interesting enhancements of geometric nature have been recently introduced by Lunts and Schnürer in [25]. Roughly speaking, they were used to show that the dg notion of Fourier–Mukai functor and the triangulated one agree, under some assumptions on the schemes. This important result was previously stated in [38] but without a rigorous proof.

Plan of the paper. This paper starts with a quick recollection of results about localizations of triangulated categories and of some properties of well generated triangulated categories (see Section 1.1). After this, we explore further in Section 1.2 the relation between localizations and well generation. This is a crucial step in the proof of Theorem A.

Section 2 and Section 3 have a rather abstract nature. They cover some basic material about dg categories and dg enhancements with an emphasis on the case of enhancements of well generated triangulated categories. Section 3 provides some properties of special functors which are used in the proof of Theorem C.

In Section 4 we prove Theorem C while Theorem A, together with its geometric applications, is proved in Section 5. The proof of Theorem B, of Proposition 6.10 and of Corollary 6.11 are the contents of Section 6.

Section 7 contain two further applications. The first one, concerning the uniqueness of enhancements for $\mathrm{D}^b(X)$, is proved in Section 7.1. The second one, about Fourier–Mukai functors is explained in Section 7.2.

Notation. All categories and functors are assumed to be $\Bbbk$-linear, for a fixed commutative ring $\Bbbk$. By a $\Bbbk$-linear category we mean a category whose Hom-spaces are $\Bbbk$-modules and such that the compositions are $\Bbbk$-bilinear, not assuming that finite direct sums exist.

Throughout the paper, we assume that a universe containing an infinite set is fixed. Several definitions concerning dg categories need special care because they may, in principle, require a change of universe. All possible subtle logical issues in this sense can be overcome in view of [24, Appendix A]. The careful reader should have a look at it. After these warnings and to simplify the notation, we will not mention explicitly the universe where we are working any longer in the paper, as it should be clear from the context. The members of this universe will be called small sets. For example, when we speak about small coproducts in a category, we mean coproducts indexed by a set in this universe. If not stated otherwise, we always assume that the Hom-spaces in a category form a small set. A category is called *small* if the isomorphism classes of its objects form a small set.

If $\mathbf{T}$ is a triangulated category and $\mathbf{S}$ a full triangulated subcategory of $\mathbf{T}$, we denote by $\mathbf{T}/\mathbf{S}$ the Verdier quotient of $\mathbf{T}$ by $\mathbf{S}$. In general, $\mathbf{T}/\mathbf{S}$ is not a category according to our convention (namely, the Hom-spaces in $\mathbf{T}/\mathbf{S}$ need not be small sets), but it is in many common situations, for instance when $\mathbf{T}$ is small.

Given a category $\mathbf{C}$ and two objects C_1 and C_2 in $\mathbf{C}$, we denote by $\mathbf{C}(C_1, C_2)$ the Hom-space between C_1 and C_2. If $\mathsf{F}\colon \mathbf{C} \to \mathbf{D}$ is a functor and C_1 and C_2 are objects of $\mathbf{C}$, then we denote by F_{C_1,C_2} the induced morphism $\mathbf{C}(C_1, C_2) \to \mathbf{D}(\mathsf{F}(C_1), \mathsf{F}(C_2))$.

If I is a set, $|I|$ denotes the cardinality of I.

1. Well generated triangulated categories and localizations

This section discusses the notion of well generated triangulated categories and, in particular, its behaviour under taking Verdier quotients.

1.1. Well generated triangulated categories.

In this section we use Krause's equivalent treatment (see [20]) of Neeman's notion of well generated triangulated category (see [30]). For a very clear survey about this subject, the reader can have a look at [21].

From now on, we assume $\mathbf{T}$ to be a triangulated category with small coproducts. Given a cardinal α, an object S of $\mathbf{T}$ is *α-small* if every map $S \to \coprod_{i\in I} X_i$ in $\mathbf{T}$ factors through $\coprod_{i\in J} X_i$, for some $J \subseteq I$ with $|J| < \alpha$. Recall that a cardinal α is called *regular* if it is not the sum of fewer than α cardinals, all of them smaller than α.

Definition 1.1. The category $\mathbf{T}$ is *well generated* if there exists a set $\mathbf{S}$ of objects in $\mathbf{T}$ satisfying the following properties:

(G1) An object $X \in \mathbf{T}$ is isomorphic to 0, if and only if $\mathbf{T}(S, X[j]) = 0$, for all $S \in \mathbf{S}$ and all $j \in \mathbb{Z}$;

(G2) For every set of maps $\{X_i \to Y_i\}_{i\in I}$ in $\mathbf{T}$, the induced map $\mathbf{T}(S, \coprod_i X_i) \to \mathbf{T}(S, \coprod_i Y_i)$ is surjective for all $S \in \mathbf{S}$, if $\mathbf{T}(S, X_i) \to \mathbf{T}(S, Y_i)$ is surjective, for all $i \in I$ and all $S \in \mathbf{S}$;

(G3) Every object of $\mathbf{S}$ is α-small, for some regular cardinal α.

When the category $\mathbf{T}$ is well generated and we want to put emphasis on the cardinal α in (G3), we say that $\mathbf{T}$ is *α-compactly generated* by the set of *α-compact generators* $\mathbf{S}$.

If $\mathbf{T}$ is α-compactly generated by $\mathbf{S}$, following [20], we denote by $\mathbf{T}^\alpha$ the smallest α-localizing subcategory of $\mathbf{T}$ containing $\mathbf{S}$. Recall that a full triangulated subcategory $\mathbf{L}$ of $\mathbf{T}$ is *α-localizing* if it is closed under α-coproducts and under direct summands (the latter condition is actually redundant if $\alpha > \aleph_0$). By definition, an α-coproduct is a coproduct of strictly less than α summands. On the other hand, $\mathbf{L}$ is *localizing* if it is closed under small coproducts in $\mathbf{T}$.

Remark 1.2. (i) As alluded by the notation and explained in [20, 30], the subcategory $\mathbf{T}^\alpha$ does not depend on the choice of the set $\mathbf{S}$ of α-compact generators.

(ii) When $\alpha = \aleph_0$, then $\mathbf{T}^\alpha = \mathbf{T}^c$, the full triangulated subcategory of compact objects in $\mathbf{T}$. Recall that, in this case, $\mathbf{T}$ is $\aleph_0$-compactly generated by $\mathbf{S} \subseteq \mathbf{T}^c$ if (G1) holds (notice that (G3) holds by definition of compact object, whereas (G2) is automatically satisfied). Following the usual convention, we simply say that $\mathbf{T}$ is *compactly generated* by $\mathbf{S}$.

Later we will need the following result.

Theorem 1.3 ([28], Theorem 2.1)**.** *Let $\mathbf{T}$ be a compactly generated triangulated category and let $\mathbf{L}$ be a localizing subcategory which is generated by a set of compact objects. Then*

(i) *$\mathbf{T}/\mathbf{L}$ has small Hom-sets and it is compactly generated;*

(ii) $\mathbf{L}^c = \mathbf{L} \cap \mathbf{T}^c$;

(iii) *The quotient functor* $\mathsf{Q}\colon \mathbf{T} \to \mathbf{T}/\mathbf{L}$ *sends* $\mathbf{T}^c$ *to* $(\mathbf{T}/\mathbf{L})^c$;

(iv) *The induced functor* $\mathbf{T}^c/\mathbf{L}^c \to (\mathbf{T}/\mathbf{L})^c$ *if fully faithful and identifies* $(\mathbf{T}/\mathbf{L})^c$ *with the idempotent completion of* $\mathbf{T}^c/\mathbf{L}^c$.

Recall that the the fact that $(\mathbf{T}/\mathbf{L})^c$ is the idempotent completion of $\mathbf{T}^c/\mathbf{L}^c$ simply means that any object in $(\mathbf{T}/\mathbf{L})^c$ is isomorphic to a summand of an object in $\mathbf{T}^c/\mathbf{L}^c$. A similar result holds for well generated triangulated categories (see, for example, [21, Theorem 7.2.1]).

Example 1.4. Let $\mathbf{C}$ be a Grothendieck category. Then the derived category $\mathrm{D}(\mathbf{C})$ is α-compactly generated, for some regular cardinal α (see [27, Theorem 0.2] and [21, Example 7.7]).

1.2. **Well generation under Verdier localizations.** Assume now that $\mathbf{T}$ is α-compactly generated by a set $\mathbf{S}$ of α-compact generators. Let $\mathbf{L}$ be a localizing subcategory of $\mathbf{T}$ such that the quotient $\mathbf{T}/\mathbf{L}$ is α-compactly generated and the quotient functor

$$\mathsf{Q}\colon \mathbf{T} \longrightarrow \mathbf{T}/\mathbf{L}$$

maps $\mathbf{T}^\alpha$ to $(\mathbf{T}/\mathbf{L})^\alpha$.

Remark 1.5. As we assume that $\mathbf{T}/\mathbf{L}$ is well generated, in particular, it has small Hom-sets. Moreover, $\mathbf{T}/\mathbf{L}$ has small coproducts and the quotient functor Q commutes with them by [30, Corollary 3.2.11]. Then it follows from Theorem 5.1.1 and Proposition 2.3.1 in [21] that the functor Q has a fully faithful right adjoint Q^R (hence $\mathsf{Q} \circ \mathsf{Q}^R \cong \mathrm{id}$).

The following will be used later.

Proposition 1.6. *For* $\mathbf{T}$, $\mathbf{S}$ *and* $\mathbf{L}$ *as above, the set* $\mathbf{S}' := \{\mathsf{Q}(S) : S \in \mathbf{S}\}$ *is a set of* α*-compact generators for* $\mathbf{T}/\mathbf{L}$.

Proof. Let Q^R be the fully faithful right adjoint of Q, which exists by Remark 1.5. Thus we have

$$\mathbf{T}/\mathbf{L}(\mathsf{Q}(S), X[j]) \cong \mathbf{T}(S, \mathsf{Q}^R(X)[j]),$$

for all $S \in \mathbf{S}$, all $X \in \mathbf{T}/\mathbf{L}$ and all $j \in \mathbb{Z}$. Hence $\mathbf{S}'$ satisfies (G1) as $\mathbf{S}$ does.

To verify that $\mathbf{S}'$ satisfies (G2), consider the closure $\overline{\mathbf{S}}$ and $\overline{\mathbf{S}}'$ of $\mathbf{S}$ and $\mathbf{S}'$ respectively under α-coproducts. We consider them as full subcategories of $\mathbf{T}$ and $\mathbf{T}/\mathbf{L}$ respectively. By [21, Appendix B], we have the functors

$$\mathsf{H}_1\colon \mathbf{T} \to \mathbf{Add}_\alpha(\overline{\mathbf{S}}^\circ, \mathbf{Ab}) \qquad X \mapsto \mathbf{T}(-, X)|_{\overline{\mathbf{S}}}$$

and

$$\mathsf{H}_2\colon \mathbf{T}/\mathbf{L} \to \mathbf{Add}_\alpha((\overline{\mathbf{S}}')^\circ, \mathbf{Ab}) \qquad X \mapsto \mathbf{T}/\mathbf{L}(-, X)|_{\overline{\mathbf{S}}'}.$$

Given an additive category $\mathbf{D}$ closed under α-coproducts, we denote by $\mathbf{Add}(\mathbf{D}^\circ, \mathbf{Ab})$ the category whose objects are all the additive functors $\mathsf{F}\colon \mathbf{D}^\circ \to \mathbf{Ab}$ and by $\mathbf{Add}_\alpha(\mathbf{D}^\circ, \mathbf{Ab})$ the full subcategory of $\mathbf{Add}(\mathbf{D}^\circ, \mathbf{Ab})$ consisting of those functors preserving α-products.

If we set $\mathsf{Q}_1 := \mathsf{Q}|_{\overline{\mathbf{S}}}$, then we can apply again [21, Appendix B], getting a commutative diagram

$$\begin{array}{ccc} \mathbf{T} & \xrightarrow{\quad \mathsf{Q} \quad} & \mathbf{T}/\mathbf{L} \\ {\scriptstyle \mathsf{H}_1}\downarrow & & \downarrow{\scriptstyle \mathsf{H}_2} \\ \mathbf{Add}_\alpha\left(\overline{\mathbf{S}}^\circ, \mathbf{Ab}\right) & \xrightarrow{\quad \mathsf{Q}_1^* \quad} & \mathbf{Add}_\alpha\left((\overline{\mathbf{S}}')^\circ, \mathbf{Ab}\right). \end{array}$$

Given this, we can observe now that Q commutes with coproducts and H_1 does the same in view of [20, Theorem C].

Now, the key point is that Q_1^* has a right adjoint. To prove this one uses the same argument as in the proof of the first part of [21, Lemma B.8], where it is shown that the functor f_* has a left adjoint f^*. Indeed, a careful check shows that the same proof works without assuming that the categories $\mathbf{C}$ and $\mathbf{D}$ there are triangulated[1].

The important consequence of the existence of the right adjoint to Q_1^* is that Q_1^* preserves coproducts. Putting all together, the functor H_2 must preserve coproducts. But then [20, Theorem C] implies that $\mathbf{S}'$ satisfies (G2).

As for (G3), observe that since Q maps $\mathbf{T}^\alpha$ to $(\mathbf{T}/\mathbf{L})^\alpha$, the set $\mathbf{S}'$ is contained in $(\mathbf{T}/\mathbf{L})^\alpha$. Then the objects in $\mathbf{S}'$ satisfy (G3) by [20, Lemma 5]. □

Example 1.7. Let us reconsider the situation in Example 1.4 concerning the triangulated category $\mathrm{D}(\mathbf{C})$, for $\mathbf{C}$ a Grothendieck category. By the construction in [27, Theorem 0.2] and [21, Example 7.7], there is an exact equivalence $\mathrm{D}(\mathbf{C}) \cong \mathcal{D}(R)/\mathbf{L}$, where R is the endomorphism ring of a generator G of $\mathbf{C}$, as a Grothendieck category, and $\mathbf{L}$ is a localizing subcategory of $\mathcal{D}(R)$. Moreover, the ring R is mapped to the generator G under the composition $\mathcal{D}(R) \to \mathcal{D}(R)/\mathbf{L} \cong \mathrm{D}(\mathbf{C})$ and it is a compact generator of $\mathcal{D}(R)$ (see, for example, [24, Example 1.9]). Hence, as an easy application of Proposition 1.6, any generator G of $\mathbf{C}$, as a Grothendieck category, yields a set of α-compact generators for $\mathrm{D}(\mathbf{C})$.

We will study a similar situation with a more general approach in Section 5.1. See also [19] for general results about the well generation of the derived categories of Grothendieck categories.

The following easy result will be used later on.

Lemma 1.8. *Let $\mathbf{T}$ be a well generated triangulated category and let $\mathbf{S}$ be a set of α-compact generators for $\mathbf{T}$. Assume that $\{X_i\}_{i\in I}$ is a family of objects in $\mathbf{T}$ with the property that $\mathbf{T}(S, X_i) = 0$, for all $S \in \mathbf{S}$ and all $i \in I$. Then $\mathbf{T}(S, \coprod_i X_i) = 0$, for all $S \in \mathbf{S}$.*

Proof. It is obvious that the family of maps $\{0 \to X_i\}_{i\in I}$ is such that the induced maps

$$\mathbf{T}(S, 0) \longrightarrow \mathbf{T}(S, X_i) \cong 0$$

are surjective, for all $S \in \mathbf{S}$. Since the set $\mathbf{S}$ satisfies (G2), the induced map

$$\mathbf{T}(S, 0) \longrightarrow \mathbf{T}\left(S, \coprod_i X_i\right)$$

is surjective for all $S \in \mathbf{S}$ and so $\mathbf{T}(S, \coprod_i X_i) = 0$, for all $S \in \mathbf{S}$. □

[1] We thank Henning Krause for several explanations about his results in Appendix B of [21].

2. Dg categories and dg enhancements

In this section, we recall some general facts about dg categories and stick to the description of dg enhancements for well generated triangulated categories.

2.1. A quick tour about dg categories. An excellent survey about dg categories is [17]. Nevertheless, we briefly summarize here what we need in the rest of the paper.

First of all, recall that a *dg category* is a $\Bbbk$-linear category $\mathbf{C}$ such that, for all A, B in $\mathbf{C}$, the morphism spaces $\mathbf{C}(A, B)$ are $\mathbb{Z}$-graded $\Bbbk$-modules with a differential $d\colon \mathbf{C}(A, B) \to \mathbf{C}(A, B)$ of degree 1 and the composition maps are morphisms of complexes. By definition, the identity of each object is a closed morphism of degree 0.

Example 2.1. (i) Any $\Bbbk$-linear category has a (trivial) structure of dg category, with morphism spaces concentrated in degree 0.

(ii) For a dg category $\mathbf{C}$, one defines the opposite dg category $\mathbf{C}^\circ$ with the same objects as $\mathbf{C}$ while $\mathbf{C}^\circ(A, B) := \mathbf{C}(B, A)$.

(iii) Following [10], given a dg category $\mathbf{C}$ and a full dg subcategory $\mathbf{D}$ of $\mathbf{C}$, one can take the quotient $\mathbf{C}/\mathbf{D}$ which is again a dg category.

Given a dg category $\mathbf{C}$ we denote by $\mathrm{H}^0(\mathbf{C})$ its *homotopy category*. To be precise, the objects of $\mathrm{H}^0(\mathbf{C})$ are the same as those of $\mathbf{C}$ while the morphisms from A to B are obtained by taking the 0-th cohomology $H^0(\mathbf{C}(A, B))$ of the complex $\mathbf{C}(A, B)$.

A *dg functor* $\mathsf{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$ between two dg categories is the datum of a map $\mathrm{Ob}(\mathbf{C}_1) \to \mathrm{Ob}(\mathbf{C}_2)$ and of morphisms of complexes of $\Bbbk$-modules $\mathbf{C}_1(A, B) \to \mathbf{C}_2(\mathsf{F}(A), \mathsf{F}(B))$, for $A, B \in \mathbf{C}_1$, which are compatible with the compositions and the units. Clearly, a dg functor $\mathsf{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$ induces a functor $\mathrm{H}^0(\mathsf{F})\colon \mathrm{H}^0(\mathbf{C}_1) \to \mathrm{H}^0(\mathbf{C}_2)$.

A dg functor $\mathsf{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$ is a *quasi-equivalence*, if the maps $\mathbf{C}_1(A, B) \to \mathbf{C}_2(\mathsf{F}(A), \mathsf{F}(B))$ are quasi-isomorphisms, for every $A, B \in \mathbf{C}_1$, and $\mathrm{H}^0(\mathsf{F})$ is an equivalence. If only the first condition holds true, we say that F is *quasi-fully faithful*.

One can consider the localization $\mathbf{Hqe}$ of the category of (small) dg categories with respect to quasi-equivalences. Given a dg functor F, we will denote with the same symbol its image in $\mathbf{Hqe}$. In particular, if F is a quasi-equivalence, we denote by F^{-1} the morphism in $\mathbf{Hqe}$ which is the inverse of F. A morphism in $\mathbf{Hqe}$ is called a *quasi-functor*. By the general theory of localizations and model categories (see, for example, [17, 38]), a quasi-functor between two dg categories $\mathbf{C}_1$ and $\mathbf{C}_2$ can be represented by a roof

$$\mathbf{C}_1 \xleftarrow{\ \mathsf{I}\ } \mathbf{C} \xrightarrow{\ \mathsf{F}\ } \mathbf{C}_2,$$

where $\mathbf{C}$ is a (cofibrant) dg category, I is a quasi-equivalence and F is a dg functor. A quasi-functor f in $\mathbf{Hqe}$ between the dg categories $\mathbf{C}_1$ and $\mathbf{C}_2$ induces a functor $\mathrm{H}^0(f)\colon \mathrm{H}^0(\mathbf{C}_1) \to \mathrm{H}^0(\mathbf{C}_2)$, well defined up to isomorphism.

Given a small dg category $\mathbf{C}$, one can consider the dg category $\mathrm{dgMod}(\mathbf{C})$ of *right dg* $\mathbf{C}$*-modules*. A right dg $\mathbf{C}$-module is a dg functor $\mathsf{M}\colon \mathbf{C}^\circ \to \mathrm{dgMod}(\Bbbk)$, where $\mathrm{dgMod}(\Bbbk)$ is the dg category of dg $\Bbbk$-modules. It is known that $\mathrm{H}^0(\mathrm{dgMod}(\mathbf{C}))$ is a triangulated category (see, for example, [17]).

The full dg subcategory of acyclic right dg modules is denoted by $\mathrm{Ac}(\mathbf{C})$, and $\mathrm{H}^0(\mathrm{Ac}(\mathbf{C}))$ is a full triangulated subcategory of the homotopy category $\mathrm{H}^0(\mathrm{dgMod}(\mathbf{C}))$. The objects of $\mathrm{Ac}(\mathbf{C})$ are the dg $\mathbf{C}$-modules M such that the complex $\mathsf{M}(C)$ of $\Bbbk$-modules is acyclic, for all C in $\mathbf{C}$. A right dg $\mathbf{C}$-module is *representable* if it is contained in the image of the Yoneda dg functor

$$\mathsf{Y}^{\mathbf{C}}\colon \mathbf{C} \to \mathrm{dgMod}(\mathbf{C}) \qquad A \mapsto \mathbf{C}(-, A).$$

Notice that in the rest of the paper, we will always use the same symbol to denote the Yoneda functor at the dg category level and the induced one on the corresponding homotopy categories.

The *derived category* of the dg category $\mathbf{C}$ is the Verdier quotient

$$\mathcal{D}(\mathbf{C}) := \mathrm{H}^0(\mathrm{dgMod}(\mathbf{C}))/\mathrm{H}^0(\mathrm{Ac}(\mathbf{C})).$$

Following [10], one could first take the quotient $\mathrm{dgMod}(\mathbf{C})/\mathrm{Ac}(\mathbf{C})$ of the corresponding dg categories. Again by [10], there is a natural exact equivalence

$$\mathrm{H}^0(\mathrm{dgMod}(\mathbf{C}))/\mathrm{H}^0(\mathrm{Ac}(\mathbf{C})) \cong \mathrm{H}^0(\mathrm{dgMod}(\mathbf{C})/\mathrm{Ac}(\mathbf{C})). \tag{2.1}$$

Let us construct some interesting full subcategories of $\mathrm{dgMod}(\mathbf{C})$. A right dg $\mathbf{C}$-module is *free* if it is isomorphic to a direct sum of dg modules of the form $\mathsf{Y}^{\mathbf{C}}(A)[m]$, where $A \in \mathbf{C}$ and $m \in \mathbb{Z}$. A right dg $\mathbf{C}$-module M is *semi-free* if it has a filtration

$$0 = \mathsf{M}_0 \subseteq \mathsf{M}_1 \subseteq \ldots = \mathsf{M} \tag{2.2}$$

such that, M_i is a dg $\mathbf{C}$-module and $\mathsf{M}_i/\mathsf{M}_{i-1}$ is free, for all i, M is the colimit of all M_i's. We denote by $\mathrm{SF}(\mathbf{C})$ the full dg subcategory of semi-free dg modules. Similarly $\mathrm{SF}_{\mathrm{fg}}(\mathbf{C}) \subseteq \mathrm{SF}(\mathbf{C})$ is the full dg subcategory of *finitely generated semi-free* dg modules. This means that, there is n such that $\mathsf{M}_n = \mathsf{M}$ and each $\mathsf{M}_i/\mathsf{M}_{i-1}$ is a finite direct sum of dg modules of the form $\mathsf{Y}^{\mathbf{C}}(A)[m]$. The dg modules which are homotopy equivalent to direct summands of finitely generated semi-free dg modules are called *perfect*. The full dg subcategory consisting of perfect dg modules is denoted by $\mathrm{Perf}^{\,\mathrm{dg}}(\mathbf{C})$.

Remark 2.2. (i) It is well known that, for a dg category $\mathbf{C}$, the homotopy category $\mathrm{H}^0(\mathrm{SF}(\mathbf{C}))$ is triangulated. The dg category $\mathbf{C}$ is called *pretriangulated* if the essential image of the Yoneda functor $\mathsf{Y}^{\mathbf{C}}\colon \mathrm{H}^0(\mathbf{C}) \to \mathrm{H}^0(\mathrm{SF}(\mathbf{C}))$ is a triangulated subcategory.

(ii) Given a dg functor $\mathsf{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$ between two pretriangulated dg categories, the induced functor $\mathrm{H}^0(\mathsf{F})\colon \mathrm{H}^0(\mathbf{C}_1) \to \mathrm{H}^0(\mathbf{C}_2)$ is an exact functor between triangulated categories.

(iii) By [16], there is a natural equivalence of triangulated categories $\mathrm{H}^0(\mathrm{SF}(\mathbf{C})) \cong \mathcal{D}(\mathbf{C})$. We can actually be more precise about it. Indeed, the composition of natural dg functors

$$\mathsf{H}\colon \mathrm{SF}(\mathbf{C}) \hookrightarrow \mathrm{dgMod}(\mathbf{C}) \to \mathrm{dgMod}(\mathbf{C})/\mathrm{Ac}(\mathbf{C})$$

is a quasi-equivalence. So, up to composing with (2.1), $\mathrm{H}^0(\mathsf{H})$ provides the exact equivalence $\mathrm{H}^0(\mathrm{SF}(\mathbf{C})) \cong \mathcal{D}(\mathbf{C})$ mentioned above.

We should keep the following example in mind for the rest of the paper.

Example 2.3. Let $\mathbf{A}$ be a small category which we consider as a dg category according to Example 2.1. As explained in [24, Section 3], an object A in $\mathrm{SF}(\mathbf{C})$ is a complex

$$\ldots \to A^{i-1} \to A^i \to A^{i+1} \to \ldots$$

such that $A_i = \coprod \mathsf{Y}^{\mathbf{C}}(A_j)$ is a free dg module with a filtration as in (2.2). In this situation, it makes sense to define the full subcategory $\mathrm{SF}^{\text{-}}(\mathbf{C})$ of $\mathrm{SF}(\mathbf{C})$ consisting of bounded above complexes.

If we are given a dg functor $\mathsf{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$, there exist dg functors

$$\mathsf{Ind}(\mathsf{F})\colon \mathrm{dgMod}(\mathbf{C}_1) \to \mathrm{dgMod}(\mathbf{C}_2) \qquad \mathsf{Res}(\mathsf{F})\colon \mathrm{dgMod}(\mathbf{C}_2) \to \mathrm{dgMod}(\mathbf{C}_1).$$

While $\mathsf{Res}(\mathsf{F})$ is simply defined by $\mathsf{M} \mapsto \mathsf{M} \circ \mathsf{F}^\circ$, the reader can have a look at [10, Sect. 14] for the explicit definition and properties of $\mathsf{Ind}(\mathsf{F})$. Let us just observe that $\mathsf{Ind}(\mathsf{F})$ preserves semi-free dg modules and $\mathsf{Ind}(\mathsf{F})\colon \mathrm{SF}(\mathbf{C}_1) \to \mathrm{SF}(\mathbf{C}_2)$ is a quasi-equivalence if $\mathsf{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$ is such. Moreover, $\mathsf{Ind}(\mathsf{F})$ commutes with the Yoneda embeddings, up to dg isomorphism.

Example 2.4. Let $\mathbf{C}$ be a dg category and $\mathbf{S}$ a full dg subcategory of $\mathbf{C}$. Denoting by $\mathsf{I}\colon \mathbf{S} \hookrightarrow \mathbf{C}$ the inclusion dg functor, the composition of dg functors

$$\mathbf{C} \xrightarrow{\mathsf{Y}^{\mathbf{C}}} \mathrm{dgMod}(\mathbf{C}) \xrightarrow{\mathsf{Res}(\mathsf{I})} \mathrm{dgMod}(\mathbf{S}) \to \mathrm{dgMod}(\mathbf{S})/\mathrm{Ac}(\mathbf{S})$$

yields, in view of Remark 2.2 (iii), a natural quasi-functor $\mathbf{C} \to \mathrm{SF}(\mathbf{S})$.

Let us give now the key definition for this paper.

Definition 2.5. A *dg enhancement* (or simply an *enhancement*) of a triangulated category $\mathbf{T}$ is a pair $(\mathbf{C}, \mathsf{F})$, where $\mathbf{C}$ is a pretriangulated dg category and $\mathsf{F}\colon \mathrm{H}^0(\mathbf{C}) \to \mathbf{T}$ is an exact equivalence.

A priori, one may have 'different' enhancements for the same triangulated category. To make this precise, we need the following.

Definition 2.6. A triangulated category $\mathbf{T}$ *has a unique enhancement* if, given two enhancements $(\mathbf{C}, \mathsf{F})$ and $(\mathbf{C}', \mathsf{F}')$ of $\mathbf{T}$, there exists a quasi-functor $\mathsf{G}\colon \mathbf{C} \to \mathbf{C}'$ such that $\mathrm{H}^0(\mathsf{G})$ is an exact equivalence.

A concise way to say that a triangulated category $\mathbf{T}$ has a unique enhancements is saying that, for any two enhancements $(\mathbf{C}, \mathsf{F})$ and $(\mathbf{C}', \mathsf{F}')$ of $\mathbf{T}$, the dg categories $\mathbf{C}$ and $\mathbf{C}'$ are isomorphic in **Hqe**. It is clear that the notion of uniqueness of dg enhancements forgets about part of the data in the definition of enhancement. In particular, the equivalence F does not play a role. Nevertheless, there are stronger versions of the notion of uniqueness of dg enhancements. Indeed, we say that $\mathbf{T}$ *has a strongly unique* (respectively, *semi-strongly unique*) *enhancement* if moreover G can be chosen so that there is an isomorphism of exact functors $\mathsf{F} \cong \mathsf{F}' \circ \mathrm{H}^0(\mathsf{G})$ (respectively, there is an isomorphism $\mathsf{F}(C) \cong \mathsf{F}'(\mathrm{H}^0(\mathsf{G})(C))$ in $\mathbf{T}$, for every $C \in \mathbf{C}$).

Example 2.7. (i) If $\mathbf{C}$ is a dg category, $\mathrm{SF}(\mathbf{C})$ and $\mathrm{Perf}^{\mathrm{dg}}(\mathbf{C})$ are enhancements, respectively, of $\mathcal{D}(\mathbf{C})$ and $\mathcal{D}(\mathbf{C})^c$.

(ii) Let $\mathbf{C}$ be a pretriangulated dg category and let $\mathbf{D}$ be a full pretriangulated dg subcategory of $\mathbf{C}$. We mentioned already that, by the main result of [10], we have a natural exact equivalence between the Verdier quotient $\mathrm{H}^0(\mathbf{C})/\mathrm{H}^0(\mathbf{D})$ and $\mathrm{H}^0(\mathbf{C}/\mathbf{D})$. Hence $\mathbf{C}/\mathbf{D}$, with the above equivalence, is an enhancement of $\mathrm{H}^0(\mathbf{C})/\mathrm{H}^0(\mathbf{D})$.

2.2. **Dg enhancements for well generated triangulated categories.** If $\mathbf{C}$ is a small dg category such that $\mathrm{H}^0(\mathbf{C})$ has α-coproducts, we denote by $\mathcal{D}_\alpha(\mathbf{C})$ the *α-continuous derived category* of $\mathbf{C}$, which is defined as the full subcategory of $\mathcal{D}(\mathbf{C})$ with objects those $M \in \mathrm{dgMod}(\mathbf{C})$ such that the natural map

$$(H^*(M))(\coprod_{i\in I} C_i) \longrightarrow \prod_{i\in I}(H^*(M))(C_i)$$

(where the coproduct is intended in $\mathrm{H}^0(\mathbf{C})$) is an isomorphism, for all objects $C_i \in \mathbf{C}$, with $|I| < \alpha$. It is useful to know that $\mathcal{D}_\alpha(\mathbf{C})$ is also equivalent to a quotient of $\mathcal{D}(\mathbf{C})$. More precisely, there is a localizing subcategory $\mathbf{N}$ of $\mathcal{D}(\mathbf{C})$ such that the quotient functor $\mathsf{Q}\colon \mathcal{D}(\mathbf{C}) \to \mathcal{D}(\mathbf{C})/\mathbf{N}$ restricts to an exact equivalence $\mathcal{D}_\alpha(\mathbf{C}) \to \mathcal{D}(\mathbf{C})/\mathbf{N}$ (see [32, Sect. 6] for details).

Remark 2.8. The triangulated category $\mathcal{D}_\alpha(\mathbf{C})$ has an obvious enhancement $\mathrm{SF}_\alpha(\mathbf{C})$ given as the full dg subcategory of $\mathrm{SF}(\mathbf{C})$ whose objects correspond to those in $\mathcal{D}_\alpha(\mathbf{C})$, under the equivalence $\mathrm{H}^0(\mathrm{SF}(\mathbf{C})) \cong \mathcal{D}(\mathbf{C})$ (see Remark 2.2 (iii)). On the other hand, in a similar way, there is a an enahncement $\mathbf{N}'$ of $\mathbf{N}$ and, by Example 2.7 (ii), the composition of dg functors

$$\mathrm{SF}_\alpha(\mathbf{C}) \hookrightarrow \mathrm{SF}(\mathbf{C}) \to \mathrm{SF}(\mathbf{C})/\mathbf{N}'$$

is a quasi-equivalence inducing the exact equivalence $\mathcal{D}_\alpha(\mathbf{C}) \to \mathcal{D}(\mathbf{C})/\mathbf{N}$.

The following result generalizes [32, Theorem 6.4] (since $\mathbf{C}$ is not assumed to be pretriangulated).

Proposition 2.9. *If $\mathbf{C}$ is a small dg category such that $\mathrm{H}^0(\mathbf{C})$ has α-coproducts, then $\mathcal{D}_\alpha(\mathbf{C})$ is α-compactly generated by $\mathbf{G} = \{\mathsf{Y}^{\mathbf{A}}(A) \mid A \in \mathbf{C}\}$.*

Proof. By Proposition 1.6 (whose hypotheses are satisfied due to [32, Theorem 4.9]), $\mathcal{D}(\mathbf{C})/\mathbf{N}$ is α-compactly generated by $\mathsf{Q}(\mathbf{G})$. Hence it is enough to show that $\mathsf{Q}|_{\mathbf{G}}\colon \mathbf{G} \to \mathsf{Q}(\mathbf{G})$ is an equivalence, and for this one can use the same argument as in the proof of [32, Theorem 6.4]. □

The essential step in the proof of [32, Theorem 7.2] can be reformulated and generalized (again, because $\mathbf{S}$ is not assumed to be triangulated) as follows.

Proposition 2.10. *Let $\mathbf{C}$ be a pretriangulated dg category such that $\mathrm{H}^0(\mathbf{C})$ is well generated by a set $\mathbf{S}$ of α-compact generators. If $\mathrm{H}^0(\mathbf{S})$ is closed under α-coproducts, then the natural quasi-functor $\mathbf{C} \to \mathrm{SF}(\mathbf{S})$ (see Example 2.4) induces an exact equivalence $\mathsf{Y}'\colon \mathrm{H}^0(\mathbf{C}) \to \mathcal{D}_\alpha(\mathbf{S})$ (here we regard $\mathbf{S}$ also as a full dg subcategory of $\mathbf{C}$).*

Proof. Looking carefully at the proof of [32, Theorem 7.2], one sees that the same argument used there to prove that $\mathsf{F}\colon \mathbf{T} \to \mathcal{D}_\alpha(\mathbf{A})$ is an equivalence works also for Y'. For this we use the implication (2) $\Longrightarrow$ (3) of Proposition 3.6, which will be proved in the next section, instead of [32, Theorem 5.3] and Proposition 2.9 instead of [32, Theorem 6.4]. □

3. Some abstract results about exact functors

In this section, we go back to the triangulated setting and prove some abstract results about exact functors which will be crucial in the rest of the paper. They generalize and extend some well known results (see, for example, [24, Section 3] and [32]) using the powerful tool of well generation for triangulated categories.

This should be thought of as a rather technical but essential interlude towards the proofs of the main results mentioned in the introduction.

3.1. **Truncations and compatibility of functors.** Let $\mathbf{A}$ be a small category which we see here as a dg category sitting all in degree 0 (see Example 2.1 (i)). Let $\mathcal{D}(\mathbf{A})$ be the dg derived category of $\mathbf{A}$, which naturally identifies with the homotopy category $\mathrm{H}^0(\mathrm{SF}(\mathbf{A}))$ (see Example 2.3 for a discussion about this specific example).

Let $\mathbf{C}$ be a full pretriangulated dg subcategory of $\mathrm{SF}(\mathbf{A})$ which contains the image of the Yoneda functor and such that $\mathrm{H}^0(\mathbf{C})$ is closed under taking stupid truncations $\sigma_{\geq n}$ and $\sigma_{\leq n}$, for all n. Recall that, in general, for a complex

$$A := \{\cdots \to A^j \xrightarrow{d^j} A^{j+1} \xrightarrow{d^{j+1}} \cdots \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to \cdots\}$$

in $\mathrm{dgMod}(\mathbf{A})$, we define the *stupid truncations* $\sigma_{\leq i}A$, $\sigma_{\geq i}A$ as

$$\sigma_{\leq i}A := \{\cdots \to A^j \xrightarrow{d^j} A^{j+1} \xrightarrow{d^{j+1}} \cdots \xrightarrow{d^{i-1}} A^i \to 0 \to \cdots\}$$

$$\sigma_{\geq i}A := \{\cdots \to 0 \to A^i \to A^{i+1} \xrightarrow{d^{i+1}} \cdots \to A^j \xrightarrow{d^j} \cdots\}.$$

Example 3.1. The simplest dg subcategories of $\mathrm{SF}(\mathbf{A})$ with the properties above are the categories $\mathrm{SF}_{\mathrm{fg}}(\mathbf{A})$, $\mathrm{SF}^{-}(\mathbf{A})$ and $\mathrm{Perf}^{\mathrm{dg}}(\mathbf{A})$.

Let $\mathbf{T}$ be any triangulated category and take $\mathsf{F}\colon \mathrm{H}^0(\mathbf{C}) \to \mathbf{T}$ an exact functor such that

(F1) F preserves coproducts;
(F2) $\mathbf{T}$ is α-compactly generated by the set $\mathbf{S} := \{\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)) : A \in \mathbf{A}\}$ of α-compact generators;
(F3) $\mathbf{T}(S_1, S_2[j]) = 0$, for all $S_1, S_2 \in \mathbf{S}$ and all $j < 0$.

In this setting, we can prove the following results which should be compared to Lemma 3.2, Corollary 3.3 and Proposition 3.4 in [24].

Proposition 3.2. *Let F be an exact functor satisfying (F1), (F2) and (F3), where $\mathbf{C}$ is either $\mathrm{SF}(\mathbf{A})$ or $\mathrm{SF}_{\mathrm{fg}}(\mathbf{A})$ or $\mathrm{SF}^{-}(\mathbf{A})$. Take $A \in \mathbf{A}$ and $C \in \mathrm{H}^0(\mathbf{C})$. Then*

$$\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(\sigma_{\geq n}(C))[i]) \cong 0$$

for all integers $i < n$, if $\mathbf{C} = \mathrm{SF}_{\mathrm{fg}}(\mathbf{A}), \mathrm{SF}^{-}(\mathbf{A})$, and for all integers $i < n-1$, if $\mathbf{C} = \mathrm{SF}(\mathbf{A})$.

Proof. For an object $C \in \mathrm{SF}(\mathbf{A})$ with filtration given by $\{\mathsf{M}_j\}$, the corresponding truncation $\sigma_{\geq n}(C)$ has induced filtration $\{\mathsf{M}'_j\}$ with the property that the quotient $\mathsf{M}'_{j+1}/\mathsf{M}'_j$ is isomorphic to a coproduct of objects of the form $\mathsf{Y}^{\mathbf{A}}(A)[s]$, for $A \in \mathbf{A}$ and $s \leq -n$.

Using this and (F1), we get an isomorphism

$$\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(\mathsf{M}'_{j+1}/\mathsf{M}'_j)[i]) \cong \mathbf{T}\left(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \coprod_k \mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A_k))[i+s_k]\right),$$

for $A_k \in \mathbf{A}$ and $s_k \leq -n$. By (F3), we have $\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A_k))[i+s_k]) \cong 0$, for $i < n$. Hence, in view of (F2), we can apply Lemma 1.8 and conclude

$$\mathbf{T}\left(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \coprod_k \mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A_k))[i+s_k]\right) \cong 0.$$

In particular, using induction, we get that $\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(\mathsf{M}'_j)[i]) \cong 0$, for all j and all $i < n$.

If $C \in \mathrm{SF}_{\mathrm{fg}}(\mathbf{A})$, then we are done, since $\sigma_{\geq n}(C) = \mathsf{M}'_j$, for some j. Obviously, if $C \in \mathrm{SF}^{-}(\mathbf{A})$ we can take $\mathsf{M}_i = \sigma_{\geq -i}(C)$. Hence the vanishing of the Hom-spaces in the statement is clear.

If $C \in \mathrm{SF}(\mathbf{A})$ is general, then we have to argue a bit more, using that $\sigma_{\geq n}(C) \cong \mathrm{hocolim}\,(\mathsf{M}'_j)$. Recall that, if we denote by $s_i\colon \mathsf{M}'_i \to \mathsf{M}'_{i+1}$ the inclusion morphism coming from the filtration $\{\mathsf{M}'_j\}$ of $\sigma_{\geq n}(C)$, then $\mathrm{hocolim}\,(\mathsf{M}'_j)$ is, by definition, the cone of the morphism

$$\sum_{i\geq 0}(\mathrm{id}_{\mathsf{M}'_i} - s_i)\colon \bigoplus_{i\geq 0}\mathsf{M}'_i \longrightarrow \bigoplus_{i\geq 0}\mathsf{M}'_i.$$

Thus by (F1) we have an isomorphism

$$\mathsf{F}(\sigma_{\geq n}(C)) \cong \mathsf{F}(\mathrm{hocolim}\,(\mathsf{M}'_j)) \cong \mathrm{hocolim}\,\mathsf{F}(\mathsf{M}'_j)$$

and, for all $i < n-1$, a disinguished triangle

$$\coprod \mathsf{F}(\mathsf{M}'_j)[i] \longrightarrow \mathrm{hocolim}\,\mathsf{F}(\mathsf{M}'_j)[i] \longrightarrow \coprod \mathsf{F}(\mathsf{M}'_j)[i+1].$$

By applying the functor $\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), -)$ to it and using the calculation above, we get that

$$\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(\sigma_{\geq n}(C))[i]) \cong 0,$$

for all integers $i < n-1$. □

Corollary 3.3. *Let F be an exact functor statisfying (F1), (F2) and (F3), where $\mathbf{C}$ is either $\mathrm{SF}(\mathbf{A})$ or $\mathrm{SF}_{\mathrm{fg}}(\mathbf{A})$ or $\mathrm{SF}^{-}(\mathbf{A})$. Then there is an injection*

$$\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(C)) \hookrightarrow \mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(\sigma_{\leq m}(C)))$$

for every $A \in \mathbf{A}$, every $C \in \mathrm{H}^0(\mathbf{C})$ and every integer $m \geq 0$ (respectively $m \geq 1$) if $\mathbf{C} = \mathrm{SF}_{\mathrm{fg}}(\mathbf{A}), \mathrm{SF}^{-}(\mathbf{A})$ (respectively $\mathbf{C} = \mathrm{SF}(\mathbf{A})$). Moreover, the inclusion is an isomorphism for $m > 0$ (respectively $m > 1$).

Proof. For any integer m and any $C \in \mathrm{H}^0(\mathbf{C})$ we have the distinguished triangle

$$\sigma_{\geq m+1}(C) \longrightarrow C \longrightarrow \sigma_{\leq m}(C).$$

By applying the functor $\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(-))$ and by Proposition 3.2 we conclude that

$$\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(\sigma_{\geq m+1}(C))) \cong 0,$$

for $m \geq 0$ (respectively $m \geq 1$). This provides the desired inclusion.

If we assume further that $m > 0$ (respectively $m > 1$), then Proposition 3.2 implies that

$$\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(\sigma_{\geq m+1}(C))[1]) \cong 0,$$

as well. Hence the inclusion is actually an isomorphism. □

Now, let $\mathbf{C}$ be either $\mathrm{SF}^{-}(\mathbf{A})$ or $\mathrm{Perf}^{\,\mathrm{dg}}(\mathbf{A})$, or $\mathrm{SF}_{\mathrm{fg}}(\mathbf{A})$ and let $\mathsf{F}_1, \mathsf{F}_2 : \mathrm{H}^0(\mathbf{C}) \to \mathbf{T}$ be two exact functors satisfying (F1), (F2) and (F3). Assume that there is an isomorphism of exact functors $\theta\colon \mathsf{F}_1 \circ \mathsf{Y}^{\mathbf{A}} \to \mathsf{F}_2 \circ \mathsf{Y}^{\mathbf{A}}$.

Proposition 3.4. *In the situation above, for every $C \in \mathbf{C}$, there exists an isomorphism $\theta_C : \mathsf{F}_1(C) \to \mathsf{F}_2(C)$ such that, for every $A \in \mathbf{A}$ and every $f \in \mathrm{H}^0(\mathbf{C})(\mathsf{Y}^{\mathbf{A}}(A)[k], C)$ with $k \in \mathbb{Z}$, the diagram*

$$\begin{array}{ccc} \mathsf{F}_1(\mathsf{Y}^{\mathbf{A}}(A)[k]) & \xrightarrow{\mathsf{F}_1(f)} & \mathsf{F}_1(C) \\ {\scriptstyle \theta_A[k]}\downarrow & & \downarrow{\scriptstyle \theta_C} \\ \mathsf{F}_2(\mathsf{Y}^{\mathbf{A}}(A)[k]) & \xrightarrow{\mathsf{F}_2(f)} & \mathsf{F}_2(C) \end{array}$$

commutes in $\mathbf{T}$.

Proof. The proof proceeds verbatim as the proof of [24, Proposition 3.4]. The key point to make this argument work in our situation is that, for $C \in \mathrm{SF}^{-}(\mathbf{A})$, then the statements of Proposition 3.2 and Corollary 3.3 are the same as those of [24, Lemma 3.2] and [24, Corollary 3.3], respectively. □

3.2. **A criterion for equivalences.** If $\mathbf{A}$ is an additive category with small coproducts and $\mathbf{B}$ is a full subcategory of $\mathbf{A}$, we denote by $\mathbf{Add}\,\mathbf{B}$ the smallest full subcategory of $\mathbf{A}$ containing $\mathbf{B}$ and closed under coproducts and direct factors. Notice that every object of $\mathbf{Add}\,\mathbf{B}$ is a direct factor of a coproduct of objects of $\mathbf{B}$.

Lemma 3.5. *Let $\mathsf{F} \colon \mathbf{T} \to \mathbf{T}'$ be an exact functor between triangulated categories with small coproducts. Let moreover X and Y_i ($i \in I$) be objects of $\mathbf{T}$ such that the natural morphism $f \colon \coprod_{i\in I} \mathsf{F}(Y_i) \to \mathsf{F}(\coprod_{i\in I} Y_i)$ is an isomorphism. If $\mathsf{F}_{Y_i,X}$ is an isomorphism for every $i \in I$, then $\mathsf{F}_{\coprod_{i\in I} Y_i,X}$ is an isomorphism, too.*

Proof. It is enough to observe that in the commutative diagram

$$\begin{array}{ccccc} \mathbf{T}(\coprod_{i\in I} Y_i, X) & & \xrightarrow{\mathsf{F}_{\coprod_{i\in I} Y_i,X}} & & \mathbf{T}'(\mathsf{F}(\coprod_{i\in I} Y_i), \mathsf{F}(X)) \\ {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle \mathbf{T}'(f,\mathsf{F}(X))} \\ \prod_{i\in I} \mathbf{T}(Y_i, X) & \xrightarrow{\prod_{i\in I} \mathsf{F}_{Y_i,X}} & \prod_{i\in I} \mathbf{T}'(\mathsf{F}(Y_i), \mathsf{F}(X)) & \xleftarrow{g'} & \mathbf{T}'(\coprod_{i\in I} \mathsf{F}(Y_i), \mathsf{F}(X)) \end{array}$$

the natural morphisms g and g' are isomorphisms by the universal property of coproduct, whereas $\mathbf{T}'(f, \mathsf{F}(X))$ and $\prod_{i\in I} \mathsf{F}_{Y_i,X}$ are isomorphisms thanks to the hypotheses. □

The following result is a reformulation and generalization (since $\mathbf{G}$ and $\mathbf{G}'$ are not assumed to be triangulated subcategories) of [32, Theorems 3.5 and 5.3].

Proposition 3.6. *Let $\mathbf{T}$ and $\mathbf{T}'$ be triangulated categories with small coproducts and let α be an infinite regular cardinal. Let moreover $\mathbf{G} \subset \mathbf{T}$ and $\mathbf{G}' \subset \mathbf{T}'$ be full subcategories closed under α-coproducts such that $\mathbf{T}$ (respectively $\mathbf{T}'$) is α-compactly generated by $\mathbf{G}$ (respectively $\mathbf{G}'$). If $\mathsf{F} \colon \mathbf{T} \to \mathbf{T}'$ is an exact functor whose restriction induces an essentially surjective functor $\mathbf{G} \to \mathbf{G}'$, then the following conditions are equivalent.*

1. F *preserves coproducts and $\mathsf{F}_{G,H[n]}$ is an isomorphism for every $G, H \in \mathbf{G}$ and every $n \in \mathbb{Z}$.*
2. $\mathsf{F}_{G,X}$ *is an isomorphism for every $G \in \mathbf{G}$ and every $X \in \mathbf{T}$.*
3. F *is an equivalence.*

Proof. Obviously (3) implies both (1) and (2), and we start by proving that (1) and (2) together imply (3). To see this, denote by $\mathbf{S}$ the full subcategory of $\mathbf{T}$ with objects those $Y \in \mathbf{T}$ such that $\mathsf{F}_{Y,X}$ is an isomorphism for every $X \in \mathbf{T}$. It is easy to see that $\mathbf{S}$ is a triangulated subcategory of $\mathbf{T}$, and Lemma 3.5 implies that it is also localizing (since F preserves coproducts by (1)). As $\mathbf{S}$ contains $\mathbf{G}$ by (2) and $\mathbf{G}$ generates $\mathbf{T}$, it follows that $\mathbf{S} = \mathbf{T}$, namely F is fully faithful. Then F is also essentially surjective because its essential image is a localizing subcategory of $\mathbf{T}'$ containing $\mathbf{G}'$ and $\mathbf{G}'$ generates $\mathbf{T}'$.

So it remains to prove that (1) and (2) are equivalent. As a first step, we are going to show that, assuming (1) or (2), the restriction of F induces an equivalence $\mathsf{F}_0 \colon \mathbf{Add}\,\mathbf{G} \to \mathbf{Add}\,\mathbf{G}'$. To this purpose, we start by observing that in any case the restriction of F induces an equivalence $\mathbf{G} \to \mathbf{G}'$. Then, given objects G and G_i $(i \in I)$ of $\mathbf{G}$, there are natural isomorphisms

$$\mathbf{T}(G, \coprod_{i\in I} G_i) \cong \operatorname*{colim}_{J\subset I, |J|<\alpha} \mathbf{T}(G, \coprod_{i\in J} G_i) \cong \operatorname*{colim}_{J\subset I, |J|<\alpha} \mathbf{T}'(\mathsf{F}(G), \coprod_{i\in J} \mathsf{F}(G_i)) \cong \mathbf{T}'(\mathsf{F}(G), \coprod_{i\in I} \mathsf{F}(G_i)).$$

Here the first (respectively last) isomorphism is due to the fact that G (respectively $\mathsf{F}(G)$) is α-small, whereas the middle one uses that $\mathbf{G}$ and $\mathbf{G}'$ are closed under α-coproducts and that F induces an equivalence $\mathbf{G} \to \mathbf{G}'$. Denoting by $f \colon \coprod_{i\in I} \mathsf{F}(G_i) \to \mathsf{F}(\coprod_{i\in I} G_i)$ the natural morphism, it is straightforward to check that the above isomorphism makes the diagram

$$\begin{array}{ccc} \mathbf{T}(G, \coprod_{i\in I} G_i) & \xrightarrow{\quad\cong\quad} & \mathbf{T}'(\mathsf{F}(G), \coprod_{i\in I} \mathsf{F}(G_i)) \\ & {\scriptstyle \mathsf{F}_{G,\coprod_{i\in I} G_i}} \searrow \quad \swarrow {\scriptstyle \mathbf{T}'(\mathsf{F}(G),f)} & \\ & \mathbf{T}'(\mathsf{F}(G), \mathsf{F}(\coprod_{i\in I} G_i)) & \end{array}$$

commute. Now, assuming (1), f is an isomorphism, hence $\mathsf{F}_{G,\coprod_{i\in I} G_i}$ is an isomorphism. This immediately implies that $\mathsf{F}_{G,X}$ is an isomorphism for every $G \in \mathbf{G}$ and every $X \in \mathbf{Add}\,\mathbf{G}$. On the other hand, if (2) holds, $\mathsf{F}_{G,\coprod_{i\in I} G_i}$ is an isomorphism, hence $\mathbf{T}'(\mathsf{F}(G), f)$ is an isomorphism. As $\mathbf{G}'$ generates $\mathbf{T}'$ and every object of $\mathbf{G}'$ is isomorphic to $\mathsf{F}(G)$ for some $G \in \mathbf{G}$, this actually implies that f is an isomorphism (namely, F preserves coproducts of objects of $\mathbf{G}$). It is then clear that the restriction of F induces a functor $\mathsf{F}_0 \colon \mathbf{Add}\,\mathbf{G} \to \mathbf{Add}\,\mathbf{G}'$. Moreover, $\mathsf{F}_{\coprod_{i\in I} G_i, X}$ is an isomorphism for every $G_i \in \mathbf{G}$ and every $X \in \mathbf{Add}\,\mathbf{G}$ by Lemma 3.5. From this it is again very easy to deduce that $\mathsf{F}_{Y,X}$ is an isomorphism for every $X, Y \in \mathbf{Add}\,\mathbf{G}$, which means that F_0 is fully faithful. In order to see that F_0 is also essentially surjective, consider $X' \in \mathbf{Add}\,\mathbf{G}'$. There exists $Y' \in \mathbf{Add}\,\mathbf{G}'$ such that $Z' := X' \coprod Y'$ is a coproduct of objects of $\mathbf{G}'$, hence $Z' \cong \mathsf{F}_0(Z)$ with Z a coproduct of objects of $\mathbf{G}$. Let $e' \colon Z' \to Z'$ be the (idempotent) morphism defined as the composition of the projection $Z' \twoheadrightarrow X'$ and of the inclusion $X' \hookrightarrow Z'$. As F_0 is fully faithful, there exists a unique morphism $e \colon Z \to Z$ such that $e' = \mathsf{F}(e)$, and e is idempotent, as well. Since $\mathbf{T}$ is Karoubian (having countable coproducts), e determines a decomposition $Z \cong X \coprod Y$ such that $X' \cong \mathsf{F}(X)$ and $Y' \cong \mathsf{F}(Y)$. This concludes the proof that F_0 is an equivalence.

The proof that (2) implies (1) now works as in the 3rd step of the proof of [32, Theorem 3.5].

As for the proof that (1) implies (2), let $\mathbf{S}$ be the full subcategory of $\mathbf{T}$ with objects those $X \in \mathbf{T}$ such that $\mathsf{F}_{G,X[n]}$ is an isomorphism for every $G \in \mathbf{G}$ and every $n \in \mathbb{Z}$. It is easy to see that $\mathbf{S}$ is a triangulated subcategory of $\mathbf{T}$ and $\mathbf{G} \subseteq \mathbf{S}$ by hypothesis. Since $\mathbf{G}$ generates $\mathbf{T}$, it is enough to show that $\mathbf{S}$ is localizing in order to conclude that $\mathbf{S} = \mathbf{T}$. Thus, given $X_i \in \mathbf{S}$ $(i \in I)$, we have to prove that $\mathsf{F}_{G,(\coprod_{i\in I} X_i)[n]}$ is an isomorphism for every $G \in \mathbf{G}$ and every $n \in \mathbb{Z}$. We can clearly assume that $n = 0$, and then the proof works as in the 2nd step of the proof of [32, Proposition 5.2]. □

4. Uniqueness of enhancements: the first criterion

This section is completely devoted to the proof of Theorem C. Hence, let $\mathbf{A}$ be a small category which we see here as a dg category sitting all in degree 0 and let $\mathbf{L}$ be a localizing subcategory of $\mathcal{D}(\mathbf{A})$. We will always assume that

(a) The quotient $\mathcal{D}(\mathbf{A})/\mathbf{L}$ is a well generated triangulated category;
(b) $\mathcal{D}(\mathbf{A})/\mathbf{L}(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_1)), \mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_2))[i]) \cong 0$, for all $A_1, A_2 \in \mathbf{A}$ and all integers $i < 0$;

as in the hypotheses of Theorem C.

4.1. **The quasi-functor.** Assume that there exists an equivalence $\mathsf{F}\colon \mathcal{D}(\mathbf{A})/\mathbf{L} \to \mathrm{H}^0(\mathbf{C})$, for some pretriangulated dg category $\mathbf{C}$. Consider the composition of functors

$$\mathsf{G}\colon \mathbf{A} \xrightarrow{\mathsf{Y}^{\mathbf{A}}} \mathcal{D}(\mathbf{A}) \xrightarrow{\mathsf{Q}} \mathcal{D}(\mathbf{A})/\mathbf{L} \xrightarrow{\mathsf{F}} \mathrm{H}^0(\mathbf{C}).$$

Consider the subcategories $\mathbf{S} = \{\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)) : A \in \mathbf{A}\}$ of $\mathcal{D}(\mathbf{A})/\mathbf{L}$ and $\mathbf{S}' = \{\mathsf{G}(A) : A \in \mathbf{A}\}$ of $\mathrm{H}^0(\mathbf{C})$.

Lemma 4.1. (i) *There exists a regular cardinal α such that $\mathbf{S}$ and $\mathbf{S}'$ are α-compact generators of the triangulated categories $\mathcal{D}(\mathbf{A})/\mathbf{L}$ and $\mathrm{H}^0(\mathbf{C})$, respectively.*

(ii) *For α as in* (i) *and I, J sets such that $|I|, |J| < \alpha$, we have*

$$\mathrm{H}^0(\mathbf{C})\left(\coprod_{i\in I} S_i, \coprod_{j\in J} S_j[k_j]\right) \cong 0,$$

for $S_i, S_j \in \mathbf{S}'$ and $k_j < 0$.

Proof. Since the images under $\mathsf{Y}^{\mathbf{A}}$ of the objects of $\mathbf{A}$ form a set of compact generators of $\mathcal{D}(\mathbf{A})$ (see [24, Example 1.9]), by (a) there exists a regular cardinal α such that $\mathbf{S}$ is all contained in $(\mathcal{D}(\mathbf{A})/\mathbf{L})^{\alpha}$. But then $\mathcal{D}(\mathbf{A})^{\alpha}$ is mapped to $(\mathcal{D}(\mathbf{A})/\mathbf{L})^{\alpha}$ by Q. This is because Q preserves coproducts and $\mathcal{D}(\mathbf{A})^{\alpha}$ is the smallest α-localizing subcategory of $\mathcal{D}(\mathbf{A})$ containing the objects of $\mathbf{A}$ (see [20, Lemma 5]).

Hence we can apply Proposition 1.6 and conclude that $\mathbf{S}$ is a set of α-compact generators for $\mathcal{D}(\mathbf{A})/\mathbf{L}$. The statement about $\mathbf{S}'$ is now obvious because F is an equivalence. This concludes the proof of (i).

As for (ii), observe that, by (b) and the fact that F is an equivalence, we have

$$\mathrm{H}^0(\mathbf{C})\,(S_i, S_j[k_j]) \cong 0,$$

for all $i \in I$, $j \in J$ and all $S_i, S_j \in \mathbf{S}'$. By (i) $\mathbf{S}'$ is a set of α-compact generators and we can apply Lemma 1.8, getting

$$\mathrm{H}^0(\mathbf{C})\left(S_i, \coprod_{j\in J} S_j[k_j]\right) \cong 0,$$

for all $i \in I$. Using that

$$\mathrm{H}^0(\mathbf{C})\left(\coprod_{i\in I} S_i, \coprod_{j\in J} S_j[k_j]\right) \cong \prod_{i\in I} \mathrm{H}^0(\mathbf{C})\left(S_i, \coprod_{j\in J} S_j[k_j]\right),$$

we get (ii). □

Consider the closures $\overline{\mathbf{S}}$ and $\overline{\mathbf{S}}'$ in $\mathcal{D}(\mathbf{A})/\mathbf{L}$ and $\mathrm{H}^0(\mathbf{C})$ of $\mathbf{S}$ and $\mathbf{S}'$, respectively, under α-coproducts.

Remark 4.2. By [20, Theorem C], the closure under α-coproducts of a set of α-compact generators for an α-compactly generated triangulated category $\mathbf{T}$ is again a set of α-compact generators for $\mathbf{T}$. Thus $\overline{\mathbf{S}}$ and $\overline{\mathbf{S}}'$ above are sets of α-compact generators for $\mathcal{D}(\mathbf{A})/\mathbf{L}$ and $\mathrm{H}^0(\mathbf{C})$ respectively.

Thinking of $\overline{\mathbf{S}}'$ as a full dg subcategory of $\mathbf{C}$, we have then that G factors through

$$\mathsf{G}'\colon \mathbf{A} \longrightarrow \mathrm{H}^0(\overline{\mathbf{S}}').$$

Given Lemma 4.1 (ii), we have a quasi-equivalence $\tau_{\leq 0}(\overline{\mathbf{S}}') \to \mathrm{H}^0(\overline{\mathbf{S}}')$ and the inclusion dg functor $\tau_{\leq 0}(\overline{\mathbf{S}}') \to \overline{\mathbf{S}}'$. Recall that the dg category $\tau_{\leq 0}(\overline{\mathbf{S}}')$ has the same objects as $\overline{\mathbf{S}}'$ while, for S_1 and S_2 in $\overline{\mathbf{S}}'$, we have

$$\tau_{\leq 0}(\overline{\mathbf{S}}')\,(S_1, S_2) := \tau_{\leq 0}\left(\overline{\mathbf{S}}'\,(S_1, S_2)\right).$$

For a complex of $\Bbbk$-modules

$$A := \{\cdots \to A^j \xrightarrow{d^j} A^{j+1} \xrightarrow{d^{j+1}} \cdots \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to \cdots\},$$

we have

$$\tau_{\leq i}A := \{\cdots \to A^j \xrightarrow{d^j} A^{j+1} \xrightarrow{d^{j+1}} \cdots \xrightarrow{d^{i-1}} \ker d^i \to 0 \to \cdots\}.$$

Putting all together we get a quasi-functor

$$\mathsf{G}''\colon \mathbf{A} \longrightarrow \overline{\mathbf{S}}'$$

which, in turn, provides a quasi-functors

$$\mathsf{Ind}(\mathsf{G}'')\colon \mathrm{SF}(\mathbf{A}) \longrightarrow \mathrm{SF}(\overline{\mathbf{S}}').$$

Thus, we can compose $\mathsf{Ind}(\mathsf{G}'')$ with the quotient dg functor $\mathrm{SF}(\overline{\mathbf{S}}') \to \mathrm{SF}(\overline{\mathbf{S}}')/\mathbf{N}'$, where $\mathbf{N}'$ was defined in Section 2.2. In view of Remark 2.8, we finally get the quasi-functor

$$\mathsf{K}\colon \mathrm{SF}(\mathbf{A}) \longrightarrow \mathrm{SF}_\alpha(\overline{\mathbf{S}}').$$

and, by passing to the homotopy categories, the exact functor $\widetilde{\mathsf{K}} := \mathrm{H}^0(\mathsf{K})\colon \mathcal{D}(\mathbf{A}) \longrightarrow \mathcal{D}_\alpha(\overline{\mathbf{S}}')$.

On the other hand, we can proceed differently and take the exact functor

$$\widetilde{\mathsf{L}}\colon \mathcal{D}(\mathbf{A}) \xrightarrow{\mathsf{Q}} \mathcal{D}(\mathbf{A})/\mathbf{L} \xrightarrow{\mathsf{F}} \mathrm{H}^0(\mathbf{C}) \xrightarrow{\mathsf{Y}'} \mathcal{D}_\alpha(\overline{\mathbf{S}}'),$$

where Y' is the equivalence from Proposition 2.10, in view of Remark 4.2.

The following result will be used later.

Lemma 4.3. (i) *There exists an isomorphism of exact functors* $\widetilde{\mathsf{K}} \circ \mathsf{Y}^{\mathbf{A}} \xrightarrow{\sim} \widetilde{\mathsf{L}} \circ \mathsf{Y}^{\mathbf{A}}$.

(ii) *The functors* $\widetilde{\mathsf{K}}$ *and* $\widetilde{\mathsf{L}}$ *satisfy* (F1)–(F3) *for* $\mathbf{C} = \mathrm{SF}(\mathbf{A})$.

Proof. Part (i) is obvious from the definitions of $\widetilde{\mathsf{K}}$ and $\widetilde{\mathsf{L}}$. It is clear that (F1)–(F3) hold for $\widetilde{\mathsf{L}}$ while (F1) and (F3) certainly hold for $\widetilde{\mathsf{K}}$. Thus the only non-trivial part of (ii) consists in showing that $\widetilde{\mathsf{K}}$ satisfies (F2). But for this, we just apply (i). $\square$

4.2. **The proof of Theorem C.** Let $\mathbf{C}$ and $\mathsf{F}\colon \mathcal{D}(\mathbf{A})/\mathbf{L} \to \mathrm{H}^0(\mathbf{C})$ be as in Section 4.1. Denote by $\mathbf{L}'$ the full dg subcategory of $\mathrm{SF}(\mathbf{A})$ such that $\mathrm{H}^0(\mathbf{L}') \cong \mathbf{L}$ under the equivalence $\mathrm{H}^0(\mathrm{SF}(\mathbf{A})) \cong \mathcal{D}(\mathbf{A})$.

Lemma 4.4. *The quasi-functor* K *factors through the quotient dg functor* $\mathrm{SF}(\mathbf{A}) \to \mathrm{SF}(\mathbf{A})/\mathbf{L}'$.

Proof. The proof is very similar to the one of [24, Lemma 5.2] with the required adjustments due to the more general setting we are working in. More precisely, in view of the main result of [10], it is enough to show that $\widetilde{\mathsf{K}}$ factors through the quotient $\mathcal{D}(\mathbf{A})/\mathbf{L}$ and thus that $\widetilde{\mathsf{K}}(L) \cong 0$, for all L in $\mathbf{L}$.

By Lemma 4.1 (i) and Lemma 4.3 (ii), $\{\widetilde{\mathsf{K}}(\mathsf{Y}^{\mathbf{A}}(A)) : A \in \mathbf{A}\}$ forms a set of α-compact generators for $\mathcal{D}_\alpha(\overline{\mathbf{S}}')$. Thus we have just to show that

$$\mathcal{D}_\alpha(\overline{\mathbf{S}}')(\widetilde{\mathsf{K}}(\mathsf{Y}^{\mathbf{A}}(A)), \widetilde{\mathsf{K}}(L)[k]) \cong 0,$$

for all $A \in \mathbf{A}$, $L \in \mathbf{L}$ and $k \in \mathbb{Z}$.

By Corollary 3.3, for $m > k$, we have an isomorphism

$$\mathcal{D}_\alpha(\overline{\mathbf{S}}')(\widetilde{\mathsf{K}}(\mathsf{Y}^{\mathbf{A}}(A)), \widetilde{\mathsf{K}}(L)[k]) \cong \mathcal{D}_\alpha(\overline{\mathbf{S}}')(\widetilde{\mathsf{K}}(\mathsf{Y}^{\mathbf{A}}(A)), \widetilde{\mathsf{K}}(\sigma_{\leq m+1}(L))[k]).$$

By applying Proposition 3.4 to $\sigma_{\leq m+1}(L)$ we get an isomorphism $\widetilde{\mathsf{K}}(\sigma_{\leq m+1}(L)) \cong \widetilde{\mathsf{L}}(\sigma_{\leq m+1}(L))$.

This, together with Corollary 3.3 applied to $\widetilde{\mathsf{L}}$ as above, yields a sequence of isomorphisms

$$\begin{aligned}\mathcal{D}_\alpha(\overline{\mathbf{S}}')(\widetilde{\mathsf{K}}(\mathsf{Y}^{\mathbf{A}}(A)), \widetilde{\mathsf{K}}(L)[k]) &\cong \mathcal{D}_\alpha(\overline{\mathbf{S}}')(\widetilde{\mathsf{K}}(\mathsf{Y}^{\mathbf{A}}(A)), \widetilde{\mathsf{L}}(\sigma_{\leq m+1}(L))[k]) \\ &\cong \mathcal{D}_\alpha(\overline{\mathbf{S}}')(\widetilde{\mathsf{L}}(\mathsf{Y}^{\mathbf{A}}(A)), \widetilde{\mathsf{L}}(L)[k]).\end{aligned}$$

The latter Hom-space is obviously trivial, because it is naturally isomorphic to the Hom-space $\mathcal{D}(\mathbf{A})/\mathbf{L}(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{Q}(L)[k])$. $\square$

Hence, we get a quasi-functor $\mathsf{K}'\colon \mathrm{SF}(\mathbf{A})/\mathbf{L}' \to \mathrm{SF}_\alpha(\overline{\mathbf{S}}')$. If we show that it defines an isomorphism in **Hqe**, Theorem C would follow immediately, taking into account Proposition 2.10. This is the content of the next proposition.

Proposition 4.5. *In the above situation,* $\mathsf{K}'\colon \mathrm{SF}(\mathbf{A})/\mathbf{L}' \to \mathrm{SF}_\alpha(\overline{\mathbf{S}}')$ *defines an isomorphism in* **Hqe**.

Proof. Setting $\widetilde{\mathsf{K}}' := \mathrm{H}^0(\mathsf{K}')\colon \mathcal{D}(\mathbf{A})/\mathbf{L} \to \mathcal{D}_\alpha(\overline{\mathbf{S}}')$, by Lemma 4.1 and the implication (1) $\Longrightarrow$ (3) of Proposition 3.6, we just need to show that $\widetilde{\mathsf{K}}'_{\coprod_{i\in I} S_i, \coprod_{j\in J} S_j[k]}$ is an isomorphism, for all sets I and J with $|I|, |J| < \alpha$, all $S_i, S_j \in \mathbf{S}$ and all $k \in \mathbb{Z}$. Under our assumptions, the quotient functor $\mathsf{Q}\colon \mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{A})/\mathbf{L}$ has a right adjoint Q^R (see Remark 1.5). Thus, setting $B := \mathsf{Q}^R(\coprod_{j\in J} S_j)$, we can just check that $\widetilde{\mathsf{K}}_{\coprod_{i\in I} \mathsf{Y}^{\mathbf{A}}(A_i), B[k]}$ is an isomorphism, for all $A_i \in \mathbf{A}$ and all $k \in \mathbb{Z}$. By Lemma 3.5 it is actually enough to show that $\widetilde{\mathsf{K}}_{\mathsf{Y}^{\mathbf{A}}(A), B[k]}$ is an isomorphism for all A in $\mathbf{A}$ and all $k \in \mathbb{Z}$.

Now the proof proceeds as in [24, Lemma 5.3]. Let us outline the argument here for the convenience of the reader. The functors Q, $\widetilde{\mathsf{K}}$ and $\widetilde{\mathsf{L}}$ satisfy (F1)–(F3). In particular, Corollary 3.3 applies and gives an isomorphism

$$\mathcal{D}(\mathbf{A})/\mathbf{L}\left(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{Q}(B)[k]\right) \cong \mathcal{D}(\mathbf{A})/\mathbf{L}\left(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{Q}(\sigma_{\leq m+1}(B))[k]\right),$$

for $m > k$. Moreover, this is compatible with the natural isomorphism

$$\mathcal{D}(\mathbf{A})(\mathsf{Y}^{\mathbf{A}}(A), B[k]) \cong \mathcal{D}(\mathbf{A})(\mathsf{Y}^{\mathbf{A}}(A), \sigma_{\leq m+1}(B)[k]).$$

As $\mathsf{Q}_{\mathsf{Y}^{\mathbf{A}}(A),B[k]}$ is an isomorphism (by definition of B and because $\mathsf{Q} \circ \mathsf{Q}^R \cong \mathrm{id}$), it follows that $\mathsf{Q}_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\leq m+1}(B)[k]}$ is an isomorphism, too.

The same argument applies to the functor $\widetilde{\mathsf{K}}$ and then it is enough to check that $\widetilde{\mathsf{K}}_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\leq m+1}(B)[k]}$ is an isomorphism, for all A in $\mathbf{A}$ and all $k \in \mathbb{Z}$. To this purpose, consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{D}(\mathbf{A})(\mathsf{Y}^{\mathbf{A}}(A), \sigma_{\leq m+1}(B)[k]) & \xrightarrow{\widetilde{\mathsf{K}}_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\leq m+1}(B)[k]}} & \mathcal{D}_\alpha(\overline{\mathbf{S}}')\left(\widetilde{\mathsf{K}}(\mathsf{Y}^{\mathbf{A}}(A)), \widetilde{\mathsf{K}}(\sigma_{\leq m+1}(B))[k]\right) \\
& \searrow^{\widetilde{\mathsf{L}}_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\leq m+1}(B)[k]}} & \downarrow \gamma \\
& & \mathcal{D}_\alpha(\overline{\mathbf{S}}')\left(\widetilde{\mathsf{L}}(\mathsf{Y}^{\mathbf{A}}(A)), \widetilde{\mathsf{L}}(\sigma_{\leq m+1}(B))[k]\right),
\end{array}$$

where the existence of an isomorphism γ is ensured by Proposition 3.4. Since $\widetilde{\mathsf{L}} = \mathsf{Y}' \circ \mathsf{F} \circ \mathsf{Q}$, the fact (which we observed above) that $\mathsf{Q}_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\leq m+1}(B)[k]}$ is an isomorphism implies that also $\widetilde{\mathsf{L}}_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\leq m+1}(B)[k]}$ is an isomorphism, taking into account that $\mathsf{Y}' \circ \mathsf{F}$ is an equivalence. In conclusion, $\widetilde{\mathsf{K}}_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\leq m+1}(B)[k]}$ is an isomorphism as well. □

5. Uniqueness of enhancements: the derived category of a Grothendieck category

In this section, we prove Theorem A and discuss some geometric applications of this abstract criterion for Grothendieck categories.

5.1. **The abstract result.**

Let $\mathbf{C}$ be a Grothendieck category and let $\mathbf{A}$ be a full subcategory of $\mathbf{C}$ whose objects form a set of generators of $\mathbf{C}$. Set

$$\mathbf{M} := \mathrm{Mod}(\mathbf{A}),$$

where $\mathrm{Mod}(\mathbf{A})$ is the abelian category of additive functors $\mathbf{A}^\circ \to \mathrm{Mod}(\Bbbk)$ and $\mathrm{Mod}(\Bbbk)$ is the abelian category of $\Bbbk$-modules. We will denote by $\mathsf{S}\colon \mathbf{C} \to \mathbf{M}$ the natural functor defined by

$$\mathsf{S}(C)(A) := \mathbf{C}(A, C),$$

for $C \in \mathbf{C}$ and $A \in \mathbf{A}$.

We can first prove the following result which should be compared to [24, Theorem 7.4].

Proposition 5.1. *The functor* $\mathsf{S}\colon \mathbf{C} \to \mathbf{M}$ *admits a left adjoint* $\mathsf{T}\colon \mathbf{M} \to \mathbf{C}$. *Moreover,* T *is exact,* $\mathsf{T} \circ \mathsf{S} \cong \mathrm{id}_{\mathbf{C}}$, $\mathbf{N} := \ker \mathsf{T}$ *is a localizing Serre subcategory of* $\mathbf{M}$ *and* T *induces an equivalence* $\mathsf{T}'\colon \mathbf{M}/\mathbf{N} \to \mathbf{C}$ *such that* $\mathsf{T} \cong \mathsf{T}' \circ \Pi$, *where* $\Pi\colon \mathbf{M} \to \mathbf{M}/\mathbf{N}$ *is the projection functor.*

Proof. In [9, Theorem 2.2] the analogous statement is proved for the functor $\mathsf{S}'\colon \mathbf{C} \to \text{MOD-}R$, which we are going to define. Consider the object $U := \coprod_{A\in\mathbf{A}} A$ of $\mathbf{C}$ and denote, for every $A \in \mathbf{A}$, by $\iota_A\colon A \hookrightarrow U$ and $\rho_A\colon U \twoheadrightarrow A$ the natural inclusion and projection morphisms, respectively. Let S be the ring (with unit) $\mathbf{C}(U,U)$ and R the subring of S consisting of those $s \in S$ such that $s \circ \iota_A \neq 0$ only for a finite number of $A \in \mathbf{A}$. Notice that R is a ring with unit if and only if $\mathbf{A}$ has a finite number of objects, in which case obviously $R = S$. Let moreover MOD-R be the full subcategory of $\mathrm{Mod}(R)$ having as objects those $P \in \mathrm{Mod}(R)$ such that $PR = P$ (clearly MOD-$R = \mathrm{Mod}(R) = \mathrm{Mod}(S)$ if $\mathbf{A}$ has a finite number of objects). Then S' is simply given as the composition of $\mathbf{C}(U,-)\colon \mathbf{C} \to \mathrm{Mod}(S)$ with the natural functor $\mathrm{Mod}(S) \to \text{MOD-}R$ defined on objects by $P \mapsto PR$. To deduce our statement from [9, Theorem 2.2] it is therefore enough to show that there is an equivalence of categories $\mathsf{E}\colon \mathbf{M} \to \text{MOD-}R$ such that $\mathsf{S}' \cong \mathsf{E} \circ \mathsf{S}$.

In order to define E, consider first an object M of $\mathbf{M}$, namely a $\Bbbk$-linear functor $M\colon \mathbf{A}^\circ \to \mathrm{Mod}(\Bbbk)$. As a $\Bbbk$-module $\mathsf{E}(M)$ is just $\coprod_{A\in\mathbf{A}} M(A)$, whereas the R-module structure is defined as follows. Given $r \in R$ and $m \in \mathsf{E}(M)$ with components $m_A \in M(A)$ for every $A \in \mathbf{A}$, the element $mr \in \mathsf{E}(M)$ has components $(mr)_A = \sum_{B\in\mathbf{A}} M(\rho_B \circ r \circ \iota_A)(m_B)$. It is easy to prove that this actually defines an object $\mathsf{E}(M)$ of MOD-R. As for morphisms, given $M, M' \in \mathrm{Mod}(\mathbf{A})$ and a natural transormation $\gamma\colon M \to M'$, the morphism of R-modules $\mathsf{E}(\gamma)\colon \mathsf{E}(M) \to \mathsf{E}(M')$ sends m to m', where $m'_A := \gamma(A)(m_A)$ for every $A \in \mathbf{A}$. It is not difficult to check that this really defines a functor $\mathsf{E}\colon \mathbf{M} \to \text{MOD-}R$ and that $\mathsf{S}' \cong \mathsf{E} \circ \mathsf{S}$.

It remains to prove that E is an equivalence. It is clear by definition that E is faithful. As for fullness, given $M, M' \in \mathrm{Mod}(\mathbf{A})$ and a morphism $\phi\colon \mathsf{E}(M) \to \mathsf{E}(M')$ in MOD-R, it is easy to see that $\phi = \mathsf{E}(\gamma)$, where $\gamma\colon M \to M'$ is the natural transformation defined as follows. For every $A \in \mathbf{A}$ and for every $a \in M(A)$, denoting by m the element of $\mathsf{E}(M)$ such that $m_A = a$ and $m_B = 0$ for $A \neq B \in \mathbf{A}$, we set $\gamma(A)(a) := \phi(m)_A$. Finally, E is essentially surjective because it is not difficult to prove that for every $P \in \text{MOD-}R$ we have $P \cong \mathsf{E}(M)$ with $M \in \mathbf{M}$ defined in the following way. Setting $r_f := \iota_B \circ f \circ \rho_A \in R$ for every morphism $f\colon A \to B$ of $\mathbf{A}$, we define $M(A) := Pr_{\mathrm{id}_A}$ for every $A \in \mathbf{A}$, whereas $M(f)\colon M(B) = Pr_{\mathrm{id}_B} \to M(A) = Pr_{\mathrm{id}_A}$ for every morphism $f\colon A \to B$ of $\mathbf{A}$ is given by $pr_{\mathrm{id}_B} \mapsto pr_f = (pr_f)r_{\mathrm{id}_A}$ for every $p \in P$. □

Remark 5.2. It should be noted that while, by Proposition 5.1, the functor T is exact, S is only left-exact in general. On the other hand, the fact that $\mathsf{T} \circ \mathsf{S} \cong \mathrm{id}_{\mathbf{C}}$ implies that S is fully faithful.

Passing from $\mathbf{C}$ to its derived category $\mathrm{D}(\mathbf{C})$, we observe that the functors T, T' and Π being exact, we can denote by the same letters the corresponding derived functors.

Denote by $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ the full triangulated subcategory of $\mathrm{D}(\mathbf{M})$ consisting of complexes with cohomology in $\mathbf{N}$. Let moreover $\pi\colon \mathrm{D}(\mathbf{M}) \to \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ be the projection functor and denote by $\mathsf{Y}^{\mathbf{A}}\colon \mathbf{A} \hookrightarrow \mathbf{M}$ the Yoneda embedding.

Corollary 5.3. *The functor Π induces an equivalence $\Pi'\colon \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}) \to \mathrm{D}(\mathbf{M}/\mathbf{N})$ such that $\Pi \cong \Pi' \circ \pi$. Moreover, denoting by $\varphi\colon \mathrm{D}(\mathbf{C}) \to \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ a quasi-inverse of $\mathsf{T}' \circ \Pi'$, we have $\varphi|_{\mathbf{A}} \cong \pi|_{\mathbf{M}} \circ \mathsf{Y}^{\mathbf{A}}$.*

Proof. By Proposition 5.1, $\Pi\colon \mathbf{M} \to \mathbf{M}/\mathbf{N}$ admits a right adjoint, so the first part of the statement follows from [19, Lemma 5.9]. Hence the diagram

$$\begin{array}{ccccc} & & \mathrm{D}(\mathbf{M}) & & \\ & \swarrow^{\pi} & \downarrow{\scriptstyle \Pi} & \searrow^{\mathsf{T}} & \\ \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}) & \xrightarrow[\Pi']{} & \mathrm{D}(\mathbf{M}/\mathbf{N}) & \xrightarrow[\mathsf{T}']{} & \mathrm{D}(\mathbf{C}) \end{array}$$

commutes up to isomorphism. By the above commutativity, the second part of the statement reduces to proving that the inclusion $\mathbf{A} \hookrightarrow \mathbf{C}$ is isomorphic to $\mathsf{T} \circ \mathsf{Y}^{\mathbf{A}}$, which is clear, since $\mathsf{Y}^{\mathbf{A}} \cong \mathsf{S}|_{\mathbf{A}}$ by definition and $\mathsf{T} \circ \mathsf{S} \cong \mathrm{id}_{\mathbf{C}}$. □

We are now ready to prove our first result.

Proof of Theorem A. Given a Grothendieck category $\mathbf{C}$, by Proposition 5.1 and Corollary 5.3 we know that $\mathrm{D}(\mathbf{C}) \cong \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M})$, for $\mathbf{M}$ and $\mathbf{N}$ defined as above.

Consider $\mathbf{A}$, defined as above and consisting of a set of generators of $\mathbf{C}$, as a small dg category all sitting in degree 0. It is clear that there is a natural equivalence $\mathrm{D}(\mathbf{M}) \cong \mathcal{D}(\mathbf{A})$. By setting $\mathbf{L}$ to be the full localizing subcategory of $\mathcal{D}(\mathbf{A})$ which is the image of $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ under the above equivalence, we have that $\mathrm{D}(\mathbf{C}) \cong \mathcal{D}(\mathbf{A})/\mathbf{L}$.

Let us observe the following:

(a) The quotient $\mathcal{D}(\mathbf{A})/\mathbf{L}$ is a well generated triangulated category. This is because $\mathrm{D}(\mathbf{C})$, which is naturally equivalent to $\mathcal{D}(\mathbf{A})/\mathbf{L}$, is well generated by Example 1.4 and well generation is obviously preserved under equivalences.

(b) Consider the objects $\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A))$, for A in $\mathbf{A}$. By Corollary 5.3, they are mapped to objects in the abelian category $\mathbf{C}$, by the composition of the equivalences $\mathcal{D}(\mathbf{A})/\mathbf{L} \cong \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}) \cong \mathrm{D}(\mathbf{C})$ described above. This means that

$$\mathcal{D}(\mathbf{A})/\mathbf{L}(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_1)), \mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_2))[i]) \cong 0,$$

for all $A_1, A_2 \in \mathbf{A}$ and all integers $i < 0$.

In particular, the assumptions of Theorem C are satisfied and we can apply this result concluding that the triangulated category $\mathcal{D}(\mathbf{A})/\mathbf{L}$ (and hence $\mathrm{D}(\mathbf{C})$) has a unique enhancement. □

5.2. **The geometric examples.** We discuss now some geometric incarnations of Theorem A. There are certainly many interesting geometric triangulated categories which are equivalent to the derived category of a Grothendieck category and which are not considered here. So we do not claim that our list of applications is complete. Notice that, beyond the geometric situations studied in [24] and described in the introduction, the uniqueness of enhancements has been investigated in other cases, e.g. for the derived categories of supported quasi-coherent sheaves in special situations (see [8, Lemma 4.6]).

Algebraic stacks. Let X be an algebraic stack. For general facts about these geometric objects, we refer to [22] and [37].

We can consider the abelian categories $\mathrm{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules on X and $\mathbf{Qcoh}(X)$ of quasi-coherent $\mathcal{O}_X$-modules on X. The fact that $\mathbf{Qcoh}(X)$ is a Grothendieck category is proved in [37, Tag 06WU]. Passing to the derived categories, we can consider $\mathrm{D}(\mathbf{Qcoh}(X))$ and the full triangulated subcategory $\mathrm{D}_{\mathrm{qc}}(X)$ of $\mathrm{D}(\mathrm{Mod}(\mathcal{O}_X))$ consisting of complexes with quasi-coherent cohomology. The relation between these two triangulated categories is delicate, as pointed out in [13, Theorem 1.2].

We then have the following.

Corollary 5.4. *If X is an algebraic stack, then $\mathrm{D}(\mathbf{Qcoh}(X))$ has a unique enhancement. If X is also quasi-compact and with quasi-finite affine diagonal, then $\mathrm{D}_{\mathrm{qc}}(X)$ has a unique enhancement.*

Proof. The first part of the statement is an obvious consequence of Theorem A. For the second part, observe that, by [14, Theorem A], the category $\mathrm{D}_{\mathrm{qc}}(X)$ is compactly generated by a single object. Hence, by [13, Theorem 1.1], the natural functor $\mathrm{D}(\mathbf{Qcoh}(X)) \to \mathrm{D}_{\mathrm{qc}}(X)$ is an equivalence. □

Remark 5.5. The above result specializes to the case of schemes. In particular, $\mathrm{D}(\mathbf{Qcoh}(X))$ has a unique enhancement for any scheme X. If X is quasi-compact and semi-separated (i.e. the diagonal is affine), then $\mathrm{D}(\mathbf{Qcoh}(X)) \cong \mathrm{D}_{\mathrm{qc}}(X)$ (see [2, Corollary 5.5]) and the same uniqueness result holds for $\mathrm{D}_{\mathrm{qc}}(X)$. This extends vastly the results in [24], where the uniqueness results for both categories are proved only for quasi-compact, semi-separated schemes with enough locally free sheaves. This means that for any finitely presented sheaf F there is an epimorphism $E \twoheadrightarrow F$ in $\mathbf{Qcoh}(X)$, where E is locally free of finite type.

As observed in [24, Remark 7.7], we should recall here that we can simply assume that X is a semi-separated scheme rather that separated, because the proof of [2, Corollary 5.5] works for a semi-separated scheme as well.

Twisted sheaves. Let X be a scheme and pick $\alpha \in H^2_{\text{ét}}(X, \mathcal{O}_X^*)$, i.e. element in the Brauer group $\mathrm{Br}(X)$ of X. We may represent α by a Čech 2-cocycle $\{\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$ with $X = \bigcup_{i\in I} U_i$ an appropriate open cover in the étale topology. An *α-twisted quasi-coherent sheaf* E consists of pairs $(\{E_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$ such that the E_i are quasi-coherent sheaves on U_i and $\varphi_{ij} : E_j|_{U_i\cap U_j} \to E_i|_{U_i\cap U_j}$ are isomorphisms satisfying the following conditions:

- $\varphi_{ii} = \mathrm{id}$;
- $\varphi_{ji} = \varphi_{ij}^{-1}$;
- $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$.

We denote by $\mathbf{Qcoh}(X, \alpha)$ the abelian category of such α-twisted quasi-coherent sheaves on X.

It is proved in [23, Proposition 2.1.3.3] that this definition coincides with the alternative one in terms of quasi-coherent sheaves on the gerbe $\mathcal{X} \to X$ on X associated to α. In particular, by [1, Proposition 3.2], if X is a quasi-compact and quasi-separated scheme, then the abelian category $\mathbf{Qcoh}(X, \alpha)$ is a Grothendieck abelian category. It is then clear from Theorem A that we can deduce the following.

Corollary 5.6. *If X is a quasi-compact and quasi-separated scheme and $\alpha \in \mathrm{Br}(X)$, then the triangulated category $\mathrm{D}(\mathbf{Qcoh}(X, \alpha))$ has a unique enhancement.*

6. Uniqueness of enhancements: the category of compact objects

In this section we prove Theorem B. This needs some preparation. In particular, using the arguments in Section 5.1, we construct an equivalence $\mathcal{D}(\mathbf{A})/\mathbf{L} \cong \mathrm{D}(\mathbf{C})$, for some localizing subcategory $\mathbf{L}$ of $\mathcal{D}(\mathbf{A})$ and reduce to the criterion for uniqueness due to Lunts and Orlov (see [24, Theorem 2]). Verifying that the assumptions of Lunts–Orlov's result are satisfied is the main and most delicate task of this section.

6.1. **The first reduction.** If $\mathbf{C}$ is a Grothendieck category and $\mathbf{A}$ is a set of generators of $\mathbf{C}$ which we think of as a full subcategory of $\mathbf{C}$, we know from Section 5.1 that there is a pair of adjoint functors

$$\mathsf{T}\colon \mathbf{M} \to \mathbf{C} \qquad \mathsf{S}\colon \mathbf{C} \to \mathbf{M}$$

where $\mathbf{M} := \mathrm{Mod}(\mathbf{A})$.

In the following we need to know how T is precisely defined. In order to explain this, first we fix some notation. For $M \in \mathbf{M}$, let $(\mathsf{Y}^{\mathbf{A}} \downarrow M)$ be the comma category whose objects are pairs (A, a) with $A \in \mathbf{A}$ and $a \in \mathbf{M}(\mathsf{Y}^{\mathbf{A}}(A), M)$, and whose morphisms are given by

$$(\mathsf{Y}^{\mathbf{A}} \downarrow M)((A', a'), (A, a)) := \{f \in \mathbf{A}(A', A) \mid a' = a \circ \mathsf{Y}^{\mathbf{A}}(f)\}.$$

Observe that, by Yoneda's lemma, $\mathbf{M}(\mathsf{Y}^{\mathbf{A}}(A), M)$ can be identified with $M(A)$ and that, in this way, the above equality $a' = a \circ \mathsf{Y}^{\mathbf{A}}(f)$ becomes $a' = M(f)(a)$; in what follows we will freely use these identifications. Denoting by $\mathsf{F}_M\colon (\mathsf{Y}^{\mathbf{A}} \downarrow M) \to \mathbf{A}$ the forgetful functor, it is well known that

$$M \cong \varinjlim((\mathsf{Y}^{\mathbf{A}} \downarrow M) \xrightarrow{\mathsf{F}_M} \mathbf{A} \xrightarrow{\mathsf{Y}^{\mathbf{A}}} \mathbf{M}).$$

Since T (being a left adjoint) preserves colimits, we obtain

$$\mathsf{T}(M) \cong \varinjlim((\mathsf{Y}^{\mathbf{A}} \downarrow M) \xrightarrow{\mathsf{F}_M} \mathbf{A} \subseteq \mathbf{C}).$$

More explicitly, by (the dual version of) [26, Theorem 2, p. 109], this colimit can be described as follows. Consider the objects of $\mathbf{C}$

$$Y_M := \coprod_{(A,a) \in (\mathsf{Y}^{\mathbf{A}} \downarrow M)} A, \qquad X_M := \coprod_{(f\colon (A',a') \to (A,a)) \in \mathrm{Mor}(\mathsf{Y}^{\mathbf{A}} \downarrow M)} A',$$

and denote by $\iota_{(A,a)}\colon A \hookrightarrow Y_M$ (for every object (A, a) of $(\mathsf{Y}^{\mathbf{A}} \downarrow M)$) and $\iota_f\colon A' \hookrightarrow X_M$ (for every morphism $f\colon (A', a') \to (A, a)$ of $(\mathsf{Y}^{\mathbf{A}} \downarrow M)$) the natural morphisms. In conclusion, we have:

Lemma 6.1. *There is a natural isomorphism*

$$\mathsf{T}(M) \cong \mathrm{coker}(\alpha_M\colon X_M \to Y_M),$$

where, for every morphism $f\colon (A', a') \to (A, a)$ *of* $(\mathsf{Y}^{\mathbf{A}} \downarrow M)$,

$$\alpha_M \circ \iota_f := \iota_{(A',a')} - \iota_{(A,a)} \circ f. \tag{6.1}$$

Setting $\mathbf{N} := \ker \mathsf{T}$, by Corollary 5.3 there is an equivalence $\varphi\colon \mathrm{D}(\mathbf{C}) \longrightarrow \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ such that $\varphi|_{\mathbf{A}} \cong \pi|_{\mathbf{M}} \circ \mathsf{Y}^{\mathbf{A}}$. As we pointed out in Section 5.1, the quotient $\mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ is naturally equivalent to $\mathcal{D}(\mathbf{A})/\mathbf{L}$. For this, we think of $\mathbf{A}$ as a dg category sitting in degree 0 and we take $\mathbf{L}$ to be the localizing subcategory corresponding to $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ under the natural equivalence $\mathrm{D}(\mathbf{M}) \cong \mathcal{D}(\mathbf{A})$. In particular, φ can be thought of as an equivalence

$$\mathrm{D}(\mathbf{C}) \longrightarrow \mathcal{D}(\mathbf{A})/\mathbf{L},$$

such that A in $\mathbf{A}$, seen as a subcategory of $\mathbf{C}$, is mapped to $\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A))$, where $\mathsf{Q}\colon \mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{A})/\mathbf{L}$ is the quotient functor. As a consequence,

$$\mathcal{D}(\mathbf{A})/\mathbf{L}(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_1)), \mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_2))[i]) \cong 0,$$

for all $A_1, A_2 \in \mathbf{A}$ and all integers $i < 0$.

Consider now the following result.

Theorem 6.2 ([24], Theorem 2)**.** *Let* $\mathbf{A}$ *be a small category and let* $\mathbf{L}$ *be a localizing subcategory of* $\mathcal{D}(\mathbf{A})$ *such that:*

(a) $\mathbf{L}^c = \mathbf{L} \cap \mathcal{D}(\mathbf{A})^c$ *and* $\mathbf{L}$ *is generated by* $\mathbf{L}^c$*;*
(b) $\mathcal{D}(\mathbf{A})/\mathbf{L}(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_1)), \mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_2))[i]) \cong 0$*, for all* $A_1, A_2 \in \mathbf{A}$ *and all integers* $i < 0$*.*

Then $(\mathcal{D}(\mathbf{A})/\mathbf{L})^c$ *has a unique enhancement.*

By the above discussion, (b) is verified. If we could prove that $\mathrm{D}_{\mathbf{N}}(\mathbf{M})^c = \mathrm{D}_{\mathbf{N}}(\mathbf{M}) \cap \mathrm{D}(\mathbf{M})^c$, then this theorem would immediately imply that $\mathrm{D}(\mathbf{C})$ has a unique enhancement and the proof of Theorem B would be complete. This delicate check will be the content of the next section.

6.2. **Verifying the main assumption.** We do not expect item (a) of Theorem 6.2 to hold true in general. This is the reason we need the further assumptions (1)–(4) in Theorem B. For the convenience of the reader, we list them again here.

(1) $\mathbf{A}$ is closed under finite direct sums.
(2) Every object of $\mathbf{A}$ is noetherian in $\mathbf{C}$.
(3) If $f\colon A' \twoheadrightarrow A$ is an epimorphism of $\mathbf{C}$ with $A, A' \in \mathbf{A}$, then $\ker f \in \mathbf{A}$.
(4) For every $A \in \mathbf{A}$ there exists $N(A) > 0$ such that $\mathrm{D}(\mathbf{C})(A, A'[N(A)]) = 0$ for every $A' \in \mathbf{A}$.

Remark 6.3. If $f\colon \coprod_{i\in I} C_i \to C$ is a morphism in $\mathbf{C}$ and B is a noetherian subobject of C such that $B \subseteq \operatorname{im} f$, then there exists a finite subset I' of I such that $B \subseteq f(\coprod_{i\in I'} C_i)$ (for otherwise we could find elements $i_1, i_2, \ldots$ in I such that $f(\coprod_{j=1}^n C_{i_j}) \cap B$ for $n > 0$ form a strictly increasing sequence of subobjects of B).

Lemma 6.4. *Assume that conditions* (1) *and* (2) *are satisfied. If* $f\colon C \twoheadrightarrow A$ *is an epimorphism of* $\mathbf{C}$ *with* $A \in \mathbf{A}$*, then there exists a morphism* $g\colon A' \to C$ *with* $A' \in \mathbf{A}$ *such that* $f \circ g\colon A' \twoheadrightarrow A$ *is again an epimorphism of* $\mathbf{C}$*.*

Proof. Given f as in the statement, there exists an epimorphism $g'\colon \coprod_{i\in I} A_i \twoheadrightarrow C$ (so that $f \circ g'$ is also an epimorphism) with $A_i \in \mathbf{A}$ for every $i \in I$ (because the objects of $\mathbf{A}$ form a set of generators of $\mathbf{C}$). As A is noetherian in $\mathbf{C}$ by condition (2), Remark 6.3 implies that there exists a finite subset I' of I such that, setting $A' := \coprod_{i\in I'} A_i$ (which is an object of $\mathbf{A}$ thanks to condition (1)) and $g := g'|_{A'}$, the composition $f \circ g\colon A' \twoheadrightarrow A$ is an epimorphism of $\mathbf{C}$. □

Proposition 6.5. *If conditions* (1) *and* (2) *are satisfied, then* $\mathbf{N}$ *coincides with the full subcategory* $\mathbf{N}'$ *of* $\mathbf{M}$ *having as objects those* $M \in \mathbf{M}$ *satisfying the following property: for every object* (A, a) *of* $(\mathsf{Y}^{\mathbf{A}} \downarrow M)$ *there exists an epimorphism* $f\colon A' \twoheadrightarrow A$ *of* $\mathbf{C}$ *with* $A' \in \mathbf{A}$ *such that* $a \circ \mathsf{Y}^{\mathbf{A}}(f) = 0$*.*

Proof. Given $M \in \mathbf{N}'$, we have to prove that $\mathsf{T}(M) \cong 0$. By Lemma 6.1, this is true if and only if α_M is an epimorphism. So, given a morphism $g\colon Y_M \to C$ in $\mathbf{C}$ such that $g \circ \alpha_M = 0$, we need to show that $g = 0$. Now, if g is given by morphisms $g_{(A,a)} \colon A \to C$ for every $(A, a) \in (\mathsf{Y}^{\mathbf{A}} \downarrow M)$, then $g \circ \alpha_M = 0$ is equivalent, by (6.1), to $g_{(A',a')} = g_{(A,a)} \circ f$ for every morphism $f \colon (A', a') \to (A, a)$ of $(\mathsf{Y}^{\mathbf{A}} \downarrow M)$. Since $M \in \mathbf{N}'$, for every $(A, a) \in (\mathsf{Y}^{\mathbf{A}} \downarrow M)$ there exists an epimorphism $f \colon A' \twoheadrightarrow A$ of $\mathbf{C}$ with $A' \in \mathbf{A}$ such that $a \circ \mathsf{Y}^{\mathbf{A}}(f) = 0$. Then $f, 0 \colon (A', 0) \to (A, a)$ are morphisms of $(\mathsf{Y}^{\mathbf{A}} \downarrow M)$, whence

$$g_{(A,a)} \circ f = g_{(A',0)} = g_{(A,a)} \circ 0 = 0.$$

As f is an epimorphism, we conclude that $g_{(A,a)} = 0$, thus proving that $g = 0$.

Conversely, assume that $N \in \mathbf{N}$, and fix an object (A, a) of $(\mathsf{Y}^{\mathbf{A}} \downarrow N)$. Since α_N is an epimorphism (again by Lemma 6.1) and A is a noetherian object of $\mathbf{C}$, by Remark 6.3 we can find a finite number of distinct morphisms of $(\mathsf{Y}^{\mathbf{A}} \downarrow N)$, say $f_i \colon (A'_i, a'_i) \to (A_i, a_i)$ for $i = 1, \dots, n$, such that, setting

$$A'_0 := \coprod_{i=1}^{n} A'_i \subset X_N,$$

we have $\iota_{(A,a)}(A) \subseteq \alpha_N(A'_0)$. Moreover,

$$\alpha_N(A'_0) \subseteq A_0 := \coprod_{(A',a') \in I} A' \subset Y_N,$$

where I is the (finite) subset of the objects of $(\mathsf{Y}^{\mathbf{A}} \downarrow N)$ consisting of those (A', a') which are equal to (A'_i, a'_i) or (A_i, a_i) for some $i = 1, \dots, n$. Note that $A_0, A'_0 \in \mathbf{A}$ by condition (1). In the cartesian diagram

$$\begin{array}{ccc} B & \xrightarrow{f'} & A \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle \iota_{(A,a)}} \\ A'_0 & \xrightarrow[\alpha_N|_{A'_0}]{} & A_0 \end{array}$$

in $\mathbf{C}$ the morphism f' is an epimorphism because $\iota_{(A,a)}(A) \subseteq \alpha_N(A'_0)$. So, by Lemma 6.4, there exists a morphism $k \colon A' \to B$ with $A' \in \mathbf{A}$ such that $f := f' \circ k \colon A' \twoheadrightarrow A$ is an epimorphism of $\mathbf{C}$. Setting also $g := g' \circ k \colon A' \to A'_0$ and denoting by

$$a_0 \colon \mathsf{Y}^{\mathbf{A}}(A_0) \cong \coprod_{(A',a') \in I} \mathsf{Y}^{\mathbf{A}}(A') \to N$$

the morphism of $\mathbf{M}$ whose components are given by a' for every $(A', a') \in I$, the diagram

$$\begin{array}{ccccc} \mathsf{Y}^{\mathbf{A}}(A') & \xrightarrow{\mathsf{Y}^{\mathbf{A}}(f)} & \mathsf{Y}^{\mathbf{A}}(A) & \xrightarrow{a} & N \\ {\scriptstyle \mathsf{Y}^{\mathbf{A}}(g)}\downarrow & & \downarrow{\scriptstyle \mathsf{Y}^{\mathbf{A}}(\iota_{(A,a)})} & \nearrow{\scriptstyle a_0} & \\ \mathsf{Y}^{\mathbf{A}}(A'_0) & \xrightarrow[\mathsf{Y}^{\mathbf{A}}(\alpha_N|_{A'_0})]{} & \mathsf{Y}^{\mathbf{A}}(A_0) & & \end{array}$$

commutes in $\mathbf{M}$. As $\mathsf{Y}^{\mathbf{A}}(A'_0) \cong \coprod_{i=1}^{n} \mathsf{Y}^{\mathbf{A}}(A'_i)$ and

$$a_0 \circ \mathsf{Y}^{\mathbf{A}}(\alpha_N \circ \iota_{f_i}) = a'_i - a_i \circ \mathsf{Y}^{\mathbf{A}}(f_i) = 0$$

for every $i = 1, \dots, n$ (by (6.1) and by definition of morphism in $(\mathsf{Y}^{\mathbf{A}} \downarrow N)$), we obtain that $a_0 \circ \mathsf{Y}^{\mathbf{A}}(\alpha_N|_{A'_0}) = 0$. This clearly implies that $a \circ \mathsf{Y}^{\mathbf{A}}(f) = 0$, which proves that $N \in \mathbf{N}'$. □

Theorem 6.6. *Assume that conditions* (1)*,* (2)*,* (3) *and* (4) *are satisfied. Then* $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ *is generated by* $\mathrm{D}_{\mathbf{N}}(\mathbf{M}) \cap \mathrm{D}(\mathbf{M})^c$*.*

Proof. In the triangulated category $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$, consider an object

$$M = (\cdots \to M^0 \xrightarrow{m^0} M^1 \to \cdots)$$

such that $M \not\cong 0$, we must find a morphism $0 \neq x \colon P \to M$ with P in $\mathrm{D}_{\mathbf{N}}(\mathbf{M}) \cap \mathrm{D}(\mathbf{M})^c$. Setting $N^i := H^i(M)$, by definition $N^i \in \mathbf{N}$ for every $i \in \mathbb{Z}$ and $N^i \neq 0$ for at least one i. Without loss of generality we can assume that $N^0 \neq 0$, hence there exists $(A^0, \bar{a}^0) \in (\mathsf{Y}^{\mathbf{A}} \downarrow N^0)$ with $\bar{a}^0 \neq 0$.

We claim that we can find a complex

$$A = (0 \to A^{-n} \xrightarrow{d^{-n}} \cdots \xrightarrow{d^{-1}} A^0 \to 0)$$

of $\mathbf{A} \subseteq \mathbf{C}$ with $n = N(A^0)$ such that $H^i(A) = 0$ for every $i \neq -n$. Here $N(A^0)$ is the integer whose existence is prescribed by (4) applied to A^0. Furthermore, we will show that there is a morphism $a \colon \mathsf{Y}^{\mathbf{A}}(A) \to M$ of complexes of $\mathbf{M}$ (with components $a^i \colon \mathsf{Y}^{\mathbf{A}}(A^i) \to M^i$) such that, denoting by $p^i \colon \ker m^i \twoheadrightarrow N^i$ the natural projection morphism for every $i \in \mathbb{Z}$, $p^0 \circ a^0 = \bar{a}^0$. Notice that, since $m^0 \circ a^0 = 0$, we can regard a^0 as a morphism $\mathsf{Y}^{\mathbf{A}}(A^0) \to \ker m^0$. Moreover, observe that for such a complex A the objects $K^i := \ker d^i$ of $\mathbf{C}$ are actually in $\mathbf{A}$. Indeed, this is clear for $i \geq 0$ or $i < -n$, whereas for $-n \leq i < 0$ there is a short exact sequence

$$0 \longrightarrow K^i \xrightarrow{j^i} A^i \longrightarrow K^{i+1} \longrightarrow 0$$

in $\mathbf{C}$ (because $H^{i+1}(A) = 0$), hence one can prove that $K^i \in \mathbf{A}$ by descending induction on i using condition (3).

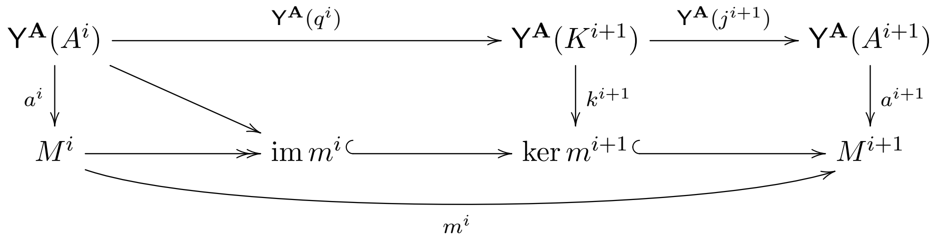
In order to prove the claim, we define the morphisms a^i and d^i again by descending induction on i. For $i = 0$, we can find $a^0 \colon \mathsf{Y}^{\mathbf{A}}(A^0) \to \ker m^0 \subseteq M^0$ such that $p^0 \circ a^0 = \bar{a}^0$ because p^0 is an epimorphism in $\mathbf{M}$. As for the inductive step, assume that $-n \leq i < 0$ and that suitable $a^{i'}$ and $d^{i'}$ have already been defined for $i' > i$. There exists (unique) $k^{i+1} \colon \mathsf{Y}^{\mathbf{A}}(K^{i+1}) \to \ker m^{i+1}$ such that the diagram

$$\begin{array}{ccccc}
\mathsf{Y}^{\mathbf{A}}(K^{i+1}) & \xrightarrow{\mathsf{Y}^{\mathbf{A}}(j^{i+1})} & \mathsf{Y}^{\mathbf{A}}(A^{i+1}) & \xrightarrow{\mathsf{Y}^{\mathbf{A}}(d^{i+1})} & \mathsf{Y}^{\mathbf{A}}(A^{i+2}) \\
\downarrow k^{i+1} & & \downarrow a^{i+1} & & \downarrow a^{i+2} \\
\ker m^{i+1} & \hookrightarrow & M^{i+1} & \xrightarrow[m^{i+1}]{} & M^{i+2}
\end{array}$$

commutes (because $d^{i+1} \circ j^{i+1} = 0$ and the square on the right commutes by induction). Consider the object $(K^{i+1}, p^{i+1} \circ k^{i+1})$ of $(\mathsf{Y}^{\mathbf{A}} \downarrow N^{i+1})$. Since $N^{i+1} \in \mathbf{N}$, by Proposition 6.5 there exists an epimorphism $q^i \colon A^i \twoheadrightarrow K^{i+1}$ such that $p^{i+1} \circ k^{i+1} \circ \mathsf{Y}^{\mathbf{A}}(q^i) = 0$. So

$$k^{i+1} \circ \mathsf{Y}^{\mathbf{A}}(q^i) \colon \mathsf{Y}^{\mathbf{A}}(A^i) \to \ker m^{i+1}$$

factors through $\operatorname{im} m^i \hookrightarrow \ker m^{i+1}$, and there exists a morphism a^i such that the diagram



commutes. Then, setting $d^i := j^{i+1} \circ q^i$, we clearly have

$$H^{i+1}(A) = 0 \quad \text{and} \quad a^{i+1} \circ \mathsf{Y}^{\mathbf{A}}(d^i) = m^i \circ a^i,$$

thus completing the proof of the inductive step.

As $A \cong K^{-n}[n]$ in $\mathrm{D}(\mathbf{C})$ and $K^{-n} \in \mathbf{A}$, the natural morphism of complexes $l \colon A^0 \to A$ defined by $l^0 = \mathrm{id}_{A_0}$ is 0 in $\mathrm{D}(\mathbf{C})(A^0, A) \cong \mathrm{D}(\mathbf{C})(A^0, K^{-n}[n])$ by condition (4). Thus we can find a complex C of $\mathbf{C}$ and a quasi-isomorphism $r \colon C \to A^0$ such that $l \circ r \sim 0$, where $\sim$ denotes homotopy of morphisms of complexes. As $H^i(C)$ is isomorphic to an object of $\mathbf{A}$ for every $i \in \mathbb{Z}$ and is 0 for $i > 0$, there exists a quasi-isomorphism $s : B \to C$ with $B^i \in \mathbf{A}$ for every $i \in \mathbb{Z}$ and $B^i = 0$ for $i > 0$: this follows for instance from [39, Lemma 1.9.5] (applied with F the inclusion of $\mathbf{A}$ in $\mathbf{C}$ and $\mathcal{C}$ the full subcategory of the category of complexes in $\mathbf{C}$ having as objects the complexes whose cohomologies are bounded above and isomorphic to objects of $\mathbf{A}$), whose key condition 1.9.5.1 is satisfied due to Lemma 6.4. Then $t := r \circ s \colon B \to A^0$ is also a quasi-isomorphism and $l \circ t \sim 0$. It is straightforward to check that t factors through a quasi-isomorphism $\tilde{t} \colon \tilde{B} := \tau_{\geq -n} B \to A^0$ and that $l \circ \tilde{t} \sim 0$, too. Hence, denoting by $\tilde{A}$ the mapping cone of $\tilde{t}$ and by $u \colon A^0 \to \tilde{A}$ the natural inclusion, there exists a morphism of complexes $f \colon \tilde{A} \to A$ such that $f \circ u \sim l$.

Now we can take $P := \mathsf{Y}^{\mathbf{A}}(\tilde{A})$ and $x := a \circ \mathsf{Y}^{\mathbf{A}}(f) \colon P \to M$ (or, better, its image in $\mathrm{D}(\mathbf{M})$). Indeed, $x \circ \mathsf{Y}^{\mathbf{A}}(u) \sim a \circ \mathsf{Y}^{\mathbf{A}}(l) = a^0$, which implies

$$H^0(x \circ \mathsf{Y}^{\mathbf{A}}(u)) = H^0(a^0) = \bar{a}^0 \neq 0.$$

Therefore $x \circ \mathsf{Y}^{\mathbf{A}}(u) \neq 0$, whence $x \neq 0$ in $\mathrm{D}(\mathbf{M})$. Moreover, $P \in \mathrm{D}(\mathbf{M})^c$ because $\mathrm{D}(\mathbf{M})^c$ is a triangulated subcategory of $\mathrm{D}(\mathbf{M})$ containing the image of $\mathsf{Y}^{\mathbf{A}}$, and $\tilde{A}$ is a bounded complex of objects of $\mathbf{A}$. Finally, as $P \cong \mathsf{S}(\tilde{A})$ (see Corollary 5.3), we have $\mathsf{T}(P) \cong \tilde{A}$. Remembering that T is exact and observing that $\tilde{A}$ is an acyclic complex (being the mapping cone of the quasi-isomorphism $\tilde{t}$), we conclude that

$$\mathsf{T}(H^i(P)) \cong H^i(\mathsf{T}(P)) \cong H^i(\tilde{A}) = 0$$

for every $i \in \mathbb{Z}$, which means that $P \in \mathrm{D}_{\mathbf{N}}(\mathbf{M})$. □

An easy application of the above result is the following.

Corollary 6.7. *Assume that conditions* (1), (2), (3) *and* (4) *are satisfied. Then*

(i) $\mathrm{D}_{\mathbf{N}}(\mathbf{M})^c = \mathrm{D}_{\mathbf{N}}(\mathbf{M}) \cap \mathrm{D}(\mathbf{M})^c$;
(ii) *The quotient functor* $\mathrm{D}(\mathbf{M}) \to \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ *sends* $\mathrm{D}(\mathbf{M})^c$ *to* $(\mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}))^c$;
(iii) *The induced functor* $\mathrm{D}(\mathbf{M})^c/\mathrm{D}_{\mathbf{N}}(\mathbf{M})^c \to (\mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}))^c$ *is fully faithful and identifies* $(\mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}))^c$ *with the idempotent completion of* $\mathrm{D}(\mathbf{M})^c/\mathrm{D}_{\mathbf{N}}(\mathbf{M})^c$.

Proof. One just combines Theorem 6.6 and Theorem 1.3. □

Theorem 6.6 and part (i) of Corollary 6.7 imply that (a) of Theorem 6.2 is satisfied, in our specific situation. Hence the proof of Theorem B is complete.

Remark 6.8. It should be noted that, under the same assumptions (1)–(4) in Theorem B, one can actually prove that the triangulated category $\mathrm{D}(\mathbf{C})^c$ has a semi-strongly unique enhancement. This result follows again from Theorem 6.6 and part (i) of Corollary 6.7 using [24, Theorem 6.4], rather than Theorem 6.2.

6.3. **The geometric examples.** In this section we describe an easy geometric application of Theorem B in the case of perfect complexes on some algebraic stacks. For this we need to recall some definitions.

Let R be a commutative ring. A complex $P \in \mathrm{D}(\mathrm{Mod}(R))$ is perfect if it is quasi-isomorphic to a bounded complex of projective R-modules of finite presentation. Following [14], if X is an algebraic stack, a complex $P \in \mathrm{D}_{\mathrm{qc}}(X)$ is *perfect* if for any smooth morphism $\mathrm{Spec}(R) \to X$, where R is a commutative ring, the complex of R-modules $\mathbf{R}\Gamma(\mathrm{Spec}(R), P|_{\mathrm{Spec}(R)})$ is perfect. We denote by $\mathbf{Perf}(X)$ the full subcategory of $\mathrm{D}_{\mathrm{qc}}(X)$ consisting of perfect complexes.

A quasi-compact and quasi-separated algebraic stack X is *concentrated* if $\mathbf{Perf}(X) \subseteq \mathrm{D}_{\mathrm{qc}}(X)^c$. On the other hand, if X has also quasi-finite affine diagonal, then the other inclusion $\mathrm{D}_{\mathrm{qc}}(X)^c \subseteq \mathbf{Perf}(X)$ holds as well, as a direct consequence of [14, Theorem A]. Moreover, we already observed in the proof of Corollary 5.4 that, under the same assumptions, the natural functor $\mathrm{D}(\mathbf{Qcoh}(X)) \to \mathrm{D}_{\mathrm{qc}}(X)$ is an equivalence.

Summing up, if X is a concentrated algebraic stack with quasi-finite affine diagonal, then there is a natural equivalence

$$\mathbf{Perf}(X) \cong \mathrm{D}(\mathbf{Qcoh}(X))^c. \tag{6.2}$$

When a stack X has the property that $\mathbf{Qcoh}(X)$ is generated, as a Grothendieck category, by a set of objects contained in $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$, we say that X *has enough perfect coherent sheaves.*

Example 6.9. Suppose that a scheme X has enough locally free sheaves, according to the definition given in Remark 5.5. This yields a set of generators of $\mathbf{Qcoh}(X)$ contained in $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$. Indeed, we can take a set of representatives for the isomorphism classes of locally free sheaves, as every sheaf in $\mathbf{Qcoh}(X)$ is a filtered colimit of finitely presented $\mathcal{O}_X$-modules (see [12, 9.4.9]). Hence a scheme with enough locally free sheaves has enough perfect coherent sheaves as well.

As an application of Theorem B, we get the following.

Proposition 6.10. *Let X be a noetherian concentrated algebraic stack with quasi-finite affine diagonal and enough perfect coherent sheaves. Then $\mathbf{Perf}(X)$ has a unique enhancement.*

Proof. Consider the isomorphism classes of objects in $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$. It is clear that they form a set. Indeed, since X is quasi-compact, the isomorphism classes of objects in $\mathbf{Perf}(X)$ form a set. Define then $\mathbf{A}$ to be the full subcategory of $\mathbf{Qcoh}(X)$ whose set of objects is obtained by taking a representative in each isomorphism class of objects in $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$. Since, by assumption, a subset of $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$ generates $\mathbf{Qcoh}(X)$, $\mathbf{A}$ does the same.

Let us now observe that $\mathbf{A}$ satisfies (1)–(4) in Theorem B. Indeed, (1) is obvious and (2) holds true because X is noetherian. To prove (3), observe that the kernel is defined in $\mathbf{Coh}(X)$ up to isomorphism and moreover, the kernel of an epimorphism $A \twoheadrightarrow A'$ in $\mathbf{A}$ is isomorphic to the shift of the cone of f in $\mathbf{Perf}(X)$. Hence it is (up to isomorphism) an object in $\mathbf{A}$. Finally, since X is concentrated, (4) is verified as well, in view of [14, Remark 4.12].

At this point, the result follows directly from Theorem B and (6.2). □

As a direct consequence, we get the following.

Corollary 6.11. *If X is a noetherian semi-separated scheme with enough locally free sheaves, then* $\mathbf{Perf}(X)$ *has a unique enhancement.*

Proof. A scheme that is noetherian is concentrated (see [4, Theorem 3.1.1]). By Example 6.9 and Proposition 6.10, the result is then clear. □

7. Applications

In this section we discuss two easy applications of the circle of ideas concerning the uniqueness of enhancements for the category of perfect complexes. The first one is about a uniqueness result for the enhancements of the bounded derived category of coherent sheaves. The second one concerns some basic questions related to exact functors between the categories of perfect complexes or the complexes of quasi-coherent sheaves.

7.1. **The bounded derived category of coherent sheaves.** Assume again that X is a noetherian semi-separated scheme with enough locally free sheaves. Let $\mathbf{A}$ be a full subcategory of $\mathbf{Qcoh}(X)$ whose objects are obtained by picking a representative in each isomorphism class of objects in $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$. As we observed in the proof of Corollary 6.11, $\mathbf{A}$ is a set of generators of $\mathbf{Qcoh}(X)$. Hence, we can apply the discussion in Section 5.1, getting a natural equivalence

$$\mathrm{D}(\mathbf{Qcoh}(X)) \cong \mathcal{D}(\mathbf{A})/\mathbf{L}, \tag{7.1}$$

where $\mathbf{L}$ is an explicit localizing subcategory of $\mathcal{D}(\mathbf{A})$. Remember that, under this equivalence, every object $A \in \mathbf{A}$ is mapped to $\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A))$, where, as usual, $\mathsf{Q}\colon \mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{A})/\mathbf{L}$ denotes the quotient functor (see the discussion in Section 6.1 about this point). Since $\mathbf{A} \subseteq \mathrm{D}(\mathbf{Qcoh}(X))^c$, in view of (6.2), it follows from [24, Remark 1.20] that $\mathbf{S} := \{\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)) : A \in \mathbf{A}\}$ is a set of compact generators of $\mathcal{D}(\mathbf{A})/\mathbf{L}$.

Following [24, Section 8], we say that an object B in $\mathcal{D}(\mathbf{A})/\mathbf{L}$ is *compactly approximated by the objects in* $\mathbf{S}$ if

1. There is $m \in \mathbb{Z}$ such that, for any $S \in \mathbf{S}$, we have $\mathcal{D}(\mathbf{A})/\mathbf{L}(S, B[i]) \cong 0$ when $i < m$;
2. For any $k \in \mathbb{Z}$, there are P_k in $(\mathcal{D}(\mathbf{A})/\mathbf{L})^c$ and a morphism $f_k\colon P_k \to B$ such that, for every $S \in \mathbf{S}$, the canonical map
$$\mathcal{D}(\mathbf{A})/\mathbf{L}\,(S, P_k[i]) \longrightarrow \mathcal{D}(\mathbf{A})/\mathbf{L}\,(S, B[i])$$
is an isomorphism when $i \geq k$.

We denote by $(\mathcal{D}(\mathbf{A})/\mathbf{L})^{ca}$ the full subcategory of $\mathcal{D}(\mathbf{A})/\mathbf{L}$ consisting of the objects which are compactly approximated by $\mathbf{S}$.

Denote by $\mathrm{D}^b(X)$ the bounded derived category of the abelian category $\mathbf{Coh}(X)$ of coherent sheaves on X. The same argument as in the proof of [24, Proposition 8.9] applies in our setting and we have that the equivalence (7.1) induces an equivalence

$$\mathrm{D}^b(X) \cong (\mathcal{D}(\mathbf{A})/\mathbf{L})^{ca}. \tag{7.2}$$

Consider now the following result.

Theorem 7.1 ([24], Theorem 8.8)**.** *Let* $\mathbf{A}$ *be a small category and let* $\mathbf{L}$ *be a localizing subcategory of* $\mathcal{D}(\mathbf{A})$ *such that:*

(a) $\mathbf{L}^c = \mathbf{L} \cap \mathcal{D}(\mathbf{A})^c$ *and* $\mathbf{L}$ *is generated by* $\mathbf{L}^c$*.*

(b) $\mathcal{D}(\mathbf{A})/\mathbf{L}(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_1)), \mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_2))[i]) \cong 0$*, for all* $A_1, A_2 \in \mathbf{A}$ *and all integers* $i < 0$*;*

Then $(\mathcal{D}(\mathbf{A})/\mathbf{L})^{ca}$ *has a unique enhancement.*

This has the following easy consequence.

Corollary 7.2. *If* X *is a noetherian semi-separated scheme with enough locally free sheaves, then* $\mathrm{D}^b(X)$ *has a unique enhancement.*

Proof. The proofs of Proposition 6.10 and Corollary 6.11 actually show that, with these assumptions on X and our choice of $\mathbf{A}$, hypotheses (a) and (b) of Theorem 6.2 are satisfied. As they coincide with (a) and (b) in Theorem 7.1, we conclude by (7.2). □

7.2. **Fourier–Mukai functors.** Assume that X_1 and X_2 are noetherian schemes. Given $\mathcal{E} \in \mathrm{D}(\mathbf{Qcoh}(X_1 \times X_2))$, we define the exact functor $\Phi_{\mathcal{E}}\colon \mathrm{D}(\mathbf{Qcoh}(X_1)) \to \mathrm{D}(\mathbf{Qcoh}(X_2))$ as

$$\Phi_{\mathcal{E}}(-) := \mathbf{R}(p_2)_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_1^*(-)),$$

where $p_i\colon X_1 \times X_2 \to X_i$ is the natural projection.

Definition 7.3. An exact functor $\mathsf{F}\colon \mathrm{D}(\mathbf{Qcoh}(X_1)) \to \mathrm{D}(\mathbf{Qcoh}(X_2))$ ($\mathsf{G}\colon \mathbf{Perf}(X_1) \to \mathbf{Perf}(X_2)$, respectively) is a *Fourier–Mukai functor* (or of *Fourier–Mukai type*) if there exists an object $\mathcal{E} \in \mathrm{D}(\mathbf{Qcoh}(X_1 \times X_2))$ and an isomorphism of exact functors $\mathsf{F} \cong \Phi_{\mathcal{E}}$ ($\mathsf{G} \cong \Phi_{\mathcal{E}}$, respectively).

These functors are ubiquitous in algebraic geometry (see [7] for a survey on the subject) and for a long while it was believed that all exact functors between $\mathrm{D}^b(X_1)$ and $\mathrm{D}^b(X_2)$, with X_i a smooth projective scheme, had to be of Fourier–Mukai type. A beautiful counterexample by Rizzardo and Van den Bergh [34] showed this expectation to be false. Moreover, if X_1 and X_2 are not projective it is not even clear if the celebrated result of Orlov [31] asserting that all equivalences between $\mathrm{D}^b(X_1)$ and $\mathrm{D}^b(X_2)$ are of Fourier–Mukai type holds true.

A much weaker question can be now formulated as follows. For two triangulated categories $\mathbf{D}_1$ and $\mathbf{D}_2$, we denote by $\mathbf{Eq}(\mathbf{D}_1, \mathbf{D}_2)$ the set of isomorphism classes of exact equivalences between $\mathbf{D}_1$ and $\mathbf{D}_2$. When $\mathbf{D}_i$ is either $\mathrm{D}(\mathbf{Qcoh}(X_i))$ or $\mathbf{Perf}(X_i)$, for X_i a noetherian scheme, we can further define the subset $\mathbf{Eq}^{\mathrm{FM}}(\mathbf{D}_1, \mathbf{D}_2)$ consisting of equivalences of Fourier–Mukai type.

As an application of the results in the previous section, we get the following.

Proposition 7.4. *Let X_1 and X_2 be noetherian semi-separated schemes with enough locally free sheaves. Then* $\mathbf{Eq}(\mathbf{Perf}(X_1), \mathbf{Perf}(X_2)) \neq \emptyset$ *if and only if* $\mathbf{Eq}(\mathrm{D}(\mathbf{Qcoh}(X_1)), \mathrm{D}(\mathbf{Qcoh}(X_2))) \neq \emptyset$. *Moreover, each of the two equivalent conditions implies* $\mathbf{Eq}(\mathrm{D}^b(X_1), \mathrm{D}^b(X_2)) \neq \emptyset$.

Proof. In view of (6.2), an exact equivalence $\mathrm{D}(\mathbf{Qcoh}(X_1)) \to \mathrm{D}(\mathbf{Qcoh}(X_2))$ restricts to an exact equivalence $\mathbf{Perf}(X_1) \to \mathbf{Perf}(X_2)$, since clearly compact objects are preserved. Hence, $\mathbf{Eq}(\mathrm{D}(\mathbf{Qcoh}(X_1)), \mathrm{D}(\mathbf{Qcoh}(X_2))) \neq \emptyset$ implies that the same is true for the categories of perfect complexes.

On the other hand, assume that $\mathbf{Eq}(\mathbf{Perf}(X_1), \mathbf{Perf}(X_2)) \neq \emptyset$. Denoting by $\mathrm{Perf}^{\mathrm{dg}}(X_i)$ a dg enhancement of $\mathbf{Perf}(X_i)$, for $i = 1, 2$, by Corollary 6.11 $\mathrm{Perf}^{\mathrm{dg}}(X_1) \cong \mathrm{Perf}^{\mathrm{dg}}(X_2)$ in $\mathbf{Hqe}$. This clearly implies that there is an exact equivalence between $\mathcal{D}(\mathrm{Perf}^{\mathrm{dg}}(X_1))$ and $\mathcal{D}(\mathrm{Perf}^{\mathrm{dg}}(X_2))$. By [24, Proposition 1.16] (see also the proof of [24, Corollary 9.13]), there is an exact equivalence between $\mathcal{D}(\mathrm{Perf}^{\mathrm{dg}}(X_i))$ and $\mathrm{D}(\mathbf{Qcoh}(X_i))$, for $i = 1, 2$. Thus $\mathbf{Eq}(\mathrm{D}(\mathbf{Qcoh}(X_1)), \mathrm{D}(\mathbf{Qcoh}(X_2))) \neq \emptyset$.

As for the last statement, assume (without loss of generality by the previous part) that there is F in $\mathbf{Eq}(\mathrm{D}(\mathbf{Qcoh}(X_1)), \mathrm{D}(\mathbf{Qcoh}(X_2)))$. By [35, Proposition 6.9], the functor F sends the subcategory $\mathrm{D}^b(\mathbf{Qcoh}(X_1))$ of cohomologically bounded complexes to $\mathrm{D}^b(\mathbf{Qcoh}(X_2))$. By using the same argument as above, we see that F induces an exact equivalence

$$\mathrm{D}^b(\mathbf{Qcoh}(X_1))^c \longrightarrow \mathrm{D}^b(\mathbf{Qcoh}(X_2))^c.$$

Then we conclude that $\mathbf{Eq}(\mathrm{D}^b(X_1), \mathrm{D}^b(X_2)) \neq \emptyset$, since $\mathrm{D}^b(\mathbf{Qcoh}(X_i))^c \cong \mathrm{D}^b(X_i)$, for $i = 1, 2$, by [35, Corollary 6.16]. □

Notice that, if we assume further that $X_1 \times X_2$ is noetherian and that any complex in $\mathbf{Perf}(X_i)$ is isomorphic to a bounded complex of vector bundles, then [38, Corollary 8.12] and [25, Theorem 1.1] imply that

$$\mathbf{Eq}(\mathbf{Perf}(X_1), \mathbf{Perf}(X_2)) \neq \emptyset \text{ iff } \mathbf{Eq}^{\mathrm{FM}}(\mathbf{Perf}(X_1), \mathbf{Perf}(X_2)) \neq \emptyset$$

and

$$\mathbf{Eq}(\mathrm{D}(\mathbf{Qcoh}(X_1)), \mathrm{D}(\mathbf{Qcoh}(X_2))) \neq \emptyset \text{ iff } \mathbf{Eq}^{\mathrm{FM}}(\mathrm{D}(\mathbf{Qcoh}(X_1)), \mathrm{D}(\mathbf{Qcoh}(X_2))) \neq \emptyset.$$

Hence Proposition 7.4 can be reformulated in terms of the sets of Fourier–Mukai equivalences.

Remark 7.5. By using the observation in Remark 6.8 and the strategy in the proof of [24, Corollary 9.12], we can make the above remarks more precise, when dealing with perfect complexes. Indeed, pick $\mathsf{F} \in \mathbf{Eq}(\mathbf{Perf}(X_1), \mathbf{Perf}(X_2))$, for X_i noetherian semi-separated with enough locally free sheaves and such that $X_1 \times X_2$ is noetherian and any complex in $\mathbf{Perf}(X_i)$ is isomorphic to a bounded complex of vector bundles. Then there exists $\mathsf{G} \in \mathbf{Eq}^{\mathrm{FM}}(\mathbf{Perf}(X_1), \mathbf{Perf}(X_2))$ such that $\mathsf{F}(C) \cong \mathsf{G}(C)$, for any C in $\mathbf{Perf}(X_1)$.

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