Derived Torelli Theorem and Orientation

Paolo Stellari

Dipartimento di Matematica “F. Enriques”
Università degli Studi di Milano

Joint work with D. Huybrechts and E. Macrì (math.AG/0608430 + work in progress)

Outline

1. **Derived Torelli Theorem**
   - Motivations
   - The statement
   - Ideas form the proof
   - The conjecture

Outline

Outline

Outline

The problem

Let X be a K3 surface (i.e. a smooth complex compact simply connected surface with trivial canonical bundle).

Main problem

Describe the group $\mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X))$ of exact autoequivalences of the triangulated category

$$\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}_{\textbf{Coh}}(\mathcal{O}_X\text{-}\textbf{Mod}).$$

Remark (Orlov)

Such a description is available when X is an abelian surface (actually an abelian variety).

Geometric case

Torelli Theorem

Let X and Y be K3 surfaces. Suppose that there exists a Hodge isometry

$$g : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$

which maps the class of an ample line bundle on X into the ample cone of Y. Then there exists a unique isomorphism

$$f : X \cong Y$$

such that $f_* = g$.

Lattice theory + Hodge structures + ample cone

Outline

The derived case

Derived Torelli Theorem (Mukai, Orlov)

Let X and Y be smooth projective K3 surfaces.

1. If $\Phi : \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ is an equivalence, then there exists a naturally defined Hodge isometry
$$\Phi_* : \widetilde{H}(X, \mathbb{Z}) \cong \widetilde{H}(Y, \mathbb{Z}).$$
2. Suppose there exists a Hodge isometry $g : \widetilde{H}(X, \mathbb{Z}) \cong \widetilde{H}(Y, \mathbb{Z})$ that preserves the natural orientation of the four positive directions. Then there exists an equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ such that $\Phi_* = g$.

It is not symmetric!

Additional structures

Lattice structure: The Mukai pairing (Euler–Poincaré form up to sign). The lattice is denoted $\widetilde{H}(X,\mathbb{Z})$.

Orientation: Let σ be a generator of $H^{2,0}(X)$ and ω a Kähler class. Then

$$P(X,\sigma,\omega) := \langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma), 1-\omega^2/2, \omega \rangle,$$

is a positive four-space in $\widetilde{H}(X,\mathbb{R})$ with a natural orientation.

Hodge structure: The weight-2 Hodge structure on $H^*(X,\mathbb{Z})$ is

$$\begin{aligned}
\widetilde{H}^{2,0}(X) &:= H^{2,0}(X),\\
\widetilde{H}^{0,2}(X) &:= H^{0,2}(X),\\
\widetilde{H}^{1,1}(X) &:= H^0(X,\mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X,\mathbb{C}).
\end{aligned}$$

Orientation

1. Due to the choice of a basis, the space $P(X, \sigma, \omega)$ comes with a natural orientation.
2. The orientation is independent of the choice of σ_X and ω.
3. It is easy to see that the isometry

$$j := (\mathrm{id})_{H^0 \oplus H^4} \oplus (-\mathrm{id})_{H^2}$$

is not orientation preserving.

Problem

According to the Derived Torelli Theorem, is the isometry j induced by an autoequivalence?

Outline

Ideas from the proof

Definition

$F : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ is of Fourier–Mukai type if there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$ and an isomorphism of functors

$$F \cong \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(-)),$$

where $p : X \times Y \to Y$ and $q : X \times Y \to X$ are the natural projections.

The complex $\mathcal{E}$ is called the kernel of F and a Fourier-Mukai functor with kernel $\mathcal{E}$ is denoted by $\Phi_{\mathcal{E}}$.

Ideas from the proof

Orlov: Every equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ is of Fourier–Mukai type. Generalizable in the following way:

Theorem. (Canonaco-S.)

Let X and Y be smooth projective varieties. Let

$$F : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$$

be an exact functor such that, for any $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}(X)$,

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(Y)}(F(\mathcal{F}), F(\mathcal{G})[j]) = 0 \text{ if } j < 0.$$

Then there exist $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{E}}$. Moreover, $\mathcal{E}$ is uniq. det. up to isomorphism.

Ideas form the proof

Using the Chern character one gets the commutative diagram:

$$\begin{array}{ccc}
\mathrm{D}^{\mathrm{b}}(X) & \xrightarrow{\Phi} & \mathrm{D}^{\mathrm{b}}(Y) \\
{\scriptstyle [-]}\downarrow & & \downarrow{\scriptstyle [-]} \\
K(X) & \longrightarrow & K(Y) \\
{\scriptstyle \mathrm{ch}(-)\cdot\sqrt{\mathrm{td}(X)}}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}(-)\cdot\sqrt{\mathrm{td}(Y)}} \\
\widetilde{H}(X,\mathbb{Z}) & \xrightarrow{\Phi_*} & \widetilde{H}(Y,\mathbb{Z})
\end{array}$$

Outline

The statement

Conjecture (Szendröi)

Let X and Y be smooth projective K3 surfaces. Any equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ induces naturally a Hodge isometry $\Phi_* : \widetilde{H}(X,\mathbb{Z}) \to \widetilde{H}(Y,\mathbb{Z})$ preserving the natural orientation of the four positive directions.

Let $\mathrm{O}_+ := \mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$ be the group of orientation preserving Hodge isometries of $\widetilde{H}(X,\mathbb{Z})$.

Using the conjecture, we would get

$$1 \to ? \to \mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X)) \xrightarrow{\Pi} \mathrm{O}_+ \to 1.$$

Outline

The statement

Theorem (Huybrechts-Macrì-S.)

Let X and Y be generic analytic K3 surfaces (i.e. $\mathrm{Pic}\,(X) = \mathrm{Pic}\,(Y) = 0$). If

$$\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y)$$

is an equivalence of Fourier-Mukai type, then up to shift

$$\Phi_{\mathcal{E}} \cong T^{n}_{\mathcal{O}_Y} \circ f_*$$

for some $n \in \mathbb{Z}$ and an isomorphism

$$f : X \xrightarrow{\sim} Y.$$

The functors

Definition

An object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ is a spherical if

$$\mathrm{Hom}\,(\mathcal{E}, \mathcal{E}[i]) \cong \begin{cases} \mathbb{C} & \text{if } i \in \{0, 2\} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\mathcal{O}_X$ is spherical.

The spherical twist $T_{\mathcal{O}_X} : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X)$ that sends $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X)$ to the cone of

$$\bigoplus_i (\mathrm{Hom}\,(\mathcal{O}_X, \mathcal{F}[i]) \otimes \mathcal{O}_X[-i]) \to \mathcal{F}$$

is an orientation preserving equivalence.

Outline

Stability conditions (Bridgeland)

For simplicity, we restrict ourselves to the case of stability conditions on derived categories!

Any triangulated category would fit.

A stability condition on $\mathrm{D}^{\mathrm{b}}(X)$ is a pair $\sigma = (Z, \mathcal{P})$ where

- $Z : \mathcal{N}(X) \otimes \mathbb{C} \to \mathbb{C}$ is a linear map (the central charge; here $\mathcal{N}(X)$ is the sublattice of $\widetilde{H}(X, \mathbb{Z})$ orthogonal to $H^{2,0}(X)$.)
- $\mathcal{P}(\phi) \subset \mathrm{D}^{\mathrm{b}}(X)$ are full additive subcategories for each $\phi \in \mathbb{R}$

satisfying the following conditions:

The definition

(a) If $0 \neq \mathcal{E} \in \mathcal{P}(\phi)$, then $Z(\mathcal{E}) = m(\mathcal{E})\exp(i\pi\phi)$ for some $m(\mathcal{E}) \in \mathbb{R}_{>0}$.

(b) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all ϕ.

(c) $\mathrm{Hom}\,(\mathcal{E}_1, \mathcal{E}_2) = 0$ for all $\mathcal{E}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \phi_2$.

(d) Any $0 \neq \mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ admits a Harder–Narasimhan filtration given by a collection of distinguished triangles

$$\mathcal{E}_{i-1} \to \mathcal{E}_i \to \mathcal{A}_i$$

with $\mathcal{E}_0 = 0$ and $\mathcal{E}_n = \mathcal{E}$ such that $\mathcal{A}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \ldots > \phi_n$.

Stability conditions (Bridgeland)

- The non-zero objects in the category $\mathcal{P}(\phi)$ are the semistable objects of phase ϕ . The objects $\mathcal{A}_i$ in (d) are the semistable factors of $\mathcal{E}$.
- The minimal objects of $\mathcal{P}(\phi)$ are called stable of phase ϕ.
- The category $\mathcal{P}((0, 1])$ is called the heart of σ.

Stability conditions (Bridgeland)

To exhibit a stability condition on $\mathrm{D}^{\mathrm{b}}(X)$, it is enough to give

- a bounded t-structure on $\mathrm{D}^{\mathrm{b}}(X)$ with heart $\mathbf{A}$;
- a group homomorphism $Z : K(\mathbf{A}) \to \mathbb{C}$ such that $Z(\mathcal{E}) \in \mathbb{H}$, for all $0 \neq \mathcal{E} \in \mathbf{A}$, and with the Harder–Narasimhan property ($\mathbb{H} := \{z \in \mathbb{C} : z = |z| \exp(i\pi\phi),\ 0 < \phi \leq 1\}$).

All stability conditions are assumed to be locally-finite. Hence every object in $\mathcal{P}(\phi)$ has a finite Jordan–Hölder filtration. $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X))$ is the manifold parametrizing locally finite stability conditions.

The group $\mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X))$ of exact autoequivalences of $\mathrm{D}^{\mathrm{b}}(X)$ acts on $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X))$.

Stability conditions: the generic case

Consider the open subset

$$R := \mathbb{C} \setminus \mathbb{R}_{\geq -1} = R_+ \cup R_- \cup R_0,$$

where the sets are defined in the natural way:

- $R_+ := R \cap (\mathbb{H} \setminus \mathbb{R}_{<0})$,
- $R_- := R \cap (-\mathbb{H} \setminus \mathbb{R}_{<0})$,
- $R_0 := R \cap \mathbb{R}$.

Given $z = u + iv \in R$, take the subcategories

$$\mathcal{F}(z), \mathcal{T}(z) \subset \mathbf{Coh}(X)$$

defined as follows:

Stability conditions: the generic case

1. If $z \in R_+ \cup R_0$ then $\mathcal{F}(z)$ and $\mathcal{T}(z)$ are respectively the full subcategories of all torsion free sheaves and torsion sheaves.
2. If $z \in R_-$ then $\mathcal{F}(z)$ is trivial and $\mathcal{T}(z) = \mathbf{Coh}(X)$.

Now define abelian subcategories as follows:

- If $z \in R_+ \cup R_0$, we put

$$\mathcal{A}(z) := \left\{ \mathcal{E} \in \mathrm{D^b}(X) : \begin{array}{l} \bullet\ H^0(\mathcal{E}) \in \mathcal{T}(z) \\ \bullet\ H^{-1}(\mathcal{E}) \in \mathcal{F}(z) \\ \bullet\ H^i(\mathcal{E}) = 0 \text{ oth.} \end{array} \right\}.$$

- If $z \in R_-$, let $\mathcal{A}(z) = \mathbf{Coh}(X)$.

Stability conditions: the generic case

Proposition (Bridgeland)

$\mathcal{A}(z)$ is the heart of a bounded t-structure for any $z \in R$.

For any $z = u + iv \in R$ we define the function

$$\begin{array}{rcl} Z : \mathcal{A}(z) & \to & \mathbb{C} \\ \mathcal{E} & \mapsto & \langle v(\mathcal{E}), (1, 0, z) \rangle = -u \cdot r - s - i(r \cdot v), \end{array}$$

where $v(\mathcal{E}) = (r, 0, s)$ is the Mukai vector of $\mathcal{E}$.

Lemma

For any $z \in R$ the function Z defines a stability function on $\mathcal{A}(z)$ which has the Harder-Narasimhan property.

Stability conditions: the generic case

Proposition

For any $\sigma \in \mathrm{Stab}(\mathrm{D^b}(X))$, there is $n \in \mathbb{Z}$ such that $T^n_{\mathcal{O}_X}(\mathcal{O}_p)$ is stable in σ, for any closed point $p \in X$.

Definition

An object $\mathcal{E} \in \mathrm{D^b}(X)$ is semi-rigid if $\mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{E}, \mathcal{E}[1]) \cong \mathbb{C}^{\oplus 2}$.

Lemma

If $z \in \mathbb{R}_{<0}$, then the only semi-rigid stable objects in $\mathcal{A}(z)$ are the skyscraper sheaves.

The proof

Consider an equivalence of Fourier–Mukai type $\Phi : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$.

(a) Take the distinguished stability condition

$$\sigma = \sigma_{Z=(u,v=0)}$$

constructed before. Let

$$\tilde{\sigma} := \Phi_{\mathcal{E}}(\sigma).$$

(b) We have seen that, there exists an integer n such that all skyscraper sheaves $\mathcal{O}_p$ are stable of the same phase in the stability condition $T^n_{\mathcal{O}_Y}(\tilde{\sigma})$.

The proof

(c) The composition $\Psi := T^n_{\mathcal{O}_Y} \circ \Phi_{\mathcal{E}}$ has the properties:

1. It sends the stability condition σ to a stability condition σ' for which all skyscraper sheaves are stable of the same phase.
2. Up to shifting the kernel $\mathcal{F}$ of Ψ sufficiently, we can assume that $\phi_{\sigma'}(\mathcal{O}_y) \in (0, 1]$ for all closed points $y \in Y$.

Thus, the heart $\mathcal{P}'((0, 1])$ of the t-structure associated to σ' (identified with $\mathcal{A}(z)$) contains as stable objects the images $\Psi(\mathcal{O}_p)$ of all closed points $p \in X$ and all skyscraper sheaves $\mathcal{O}_y$.

The proof

(d) We observed that the only semi-rigid stable objects in $\mathcal{A}(z)$ are the skyscraper sheaves. Hence, for all $p \in X$ there exists a point $y \in Y$ such that $\Psi(\mathcal{O}_p) \cong \mathcal{O}_y$. Therefore Ψ is a composition of f_*, for some isomorphism

$$f : X \xrightarrow{\sim} Y,$$

and the tensorization by a line bundle.

(e) But there are no non-trivial line bundles on Y.

Concluding remarks

There are some important features in the proof:

Proposition

Up to shifts, $\mathcal{O}_X$ is the only spherical sheaf in the category $\mathrm{D}^{\mathrm{b}}(X)$.

Theorem (Huybrechts-Macrì-S.)

The manifold parametrizing numerical stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ is connected and simply-connected.

This proves a conjecture by Bridgeland in the generic analytic case.

Outline

The non-orienatation Hodge isometry

Take any projective K3 surface X.

We have already remarked that the isometry

$$j := (\mathrm{id})_{H^0 \oplus H^4} \oplus (-\mathrm{id})_{H^2}$$

is not orientation preserving.

Since any orientation preserving Hodge isometry lifts to an equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X)$ (due to HLOY and Huybrechts-S.), to prove the conjecture, it is enough to prove that j is not induced by a Fourier–Mukai equivalence.

We proceed by contradiction assuming that there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times X)$ such that $(\Phi_{\mathcal{E}})_* = j$.

The twistor space

Definition

A Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$ is called very general if there is no non-trivial integral class $0 \neq \alpha \in H^{1,1}(X, \mathbb{Z})$ orthogonal to ω, i.e. $\omega^{\perp} \cap H^{1,1}(X, \mathbb{Z}) = 0$.

Take the twistor space $\mathbb{X}(\omega)$ of X determined by the choice of a very general Kähler class $\omega \in \mathcal{K}_X \cap \mathrm{Pic}\,(X) \otimes \mathbb{R}$. Hence we get a complex deformation

$$\pi : \mathbb{X}(\omega) \to \mathbb{P}(\omega).$$

Take $R := \mathbb{C}[[t]]$ to be the ring of power series in t with residue field $K := \mathbb{C}((t))$.

The twistor space

If $R_n := k[[t]]/t^{n+1}$, then the infinitesimal neighbourhoods

$$\mathcal{X}_n := \mathbb{X}(\omega) \times \mathrm{Spec}\,(R_n),$$

form an inductive system and give rise to a formal R-scheme

$$\pi : \mathcal{X} \to \mathrm{Spf}(R),$$

which is the formal neighbourhood of X in $\mathbb{X}(\omega)$.

Outline

The first order deformation

The equivalence $\Phi_{\mathcal{E}}$ induces a morphim

$$\Phi_{\mathcal{E}}^{HH} : HH^2(X) \to HH^2(X).$$

Proposition

Let $v_1 \in H^1(X, \mathcal{T}_X)$ be the Kodaira–Spencer class of first order deformation given by a twistor space $\mathbb{X}(\omega)$ as above. Then

$$v_1' := \Phi_{\mathcal{E}}^{HH}(v_1) \in H^1(X, \mathcal{T}_X).$$

The first order deformation

Let $\mathcal{X}_1'$ be the first order deformation corresponding to v_1'.

Using results of Toda one gets the following conclusion

Proposition (Toda)

For v_1 and v_1' as before, there exists $\mathcal{E}_1 \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}_1 \times_{R_1} \mathcal{X}_1')$ such that

$$i_1^*\mathcal{E}_1 = \mathcal{E}_0 := \mathcal{E}.$$

Here $i_1 : \mathcal{X}_0 \times_{\mathbb{C}} \mathcal{X}_0 \hookrightarrow \mathcal{X}_1' \times_{R_1} \mathcal{X}_1'$ is the natural inclusion.

Hence there is a first order deformation of $\mathcal{E}$.

Higher order deformations

Work in progress (. . . almost concluded)

Construct at any order n, an analytic deformation $\mathcal{X}'_n$ such that there exists $\mathcal{E}_n \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}_n \times_{R_n} \mathcal{X}'_n)$, with

$$i_n^* \mathcal{E}_n = \mathcal{E}_{n-1}.$$

Problems

1. Rewrite Lieblich-Lowen's obstruction for deforming complexes in terms of Atiyah–Kodaira classes.
2. Show that the obstruction is zero.

Our approach imitates the first order case.

Outline

Equivalences go to equivalences

There exists a sequence

$$\mathbf{Coh}_0(\mathcal{X} \times_R \mathcal{X}') \hookrightarrow \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}') \to \mathbf{Coh}((\mathcal{X} \times_R \mathcal{X}')_K),$$

where $\mathbf{Coh}_0(\mathcal{X} \times_R \mathcal{X}')$ is the abelian category of sheaves on $\mathcal{X} \times_R \mathcal{X}'$ supported on $X \times X$.

Proposition

Let $\widetilde{\mathcal{E}} \in \mathrm{D}^{\mathrm{b}}(\mathcal{X} \times_R \mathcal{X}')$ be such that $\mathcal{E} = i^*\widetilde{\mathcal{E}}$ (here $i : X \times X \to \mathcal{X} \times_R \mathcal{X}'$ is the inclusion). Then $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{E}}_K$ are kernels of Fourier–Mukai equivalences.

Here $\widetilde{\mathcal{E}}_K$ is the image via the natural functor in

$$\mathrm{D}^{\mathrm{b}}((\mathcal{X} \times_R \mathcal{X}')_K) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}((\mathcal{X} \times_R \mathcal{X}')_K)).$$

The generic fiber

Proposition

The triangulated category $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(\mathcal{X}_K))$ is a generic K3 category, i.e. [2] is the Serre functor and $(\mathcal{O}_{\mathcal{X}})_K$ is, up to shifts, the unique spherical object.

Use the generic analytic case

Hence, reasoning as the analytic generic case, one can compose $\Phi_{\mathcal{E}_K}$ with some power of the spherical twist by $(\mathcal{O}_{\mathcal{X}})_K$ getting a Fourier–Mukai equivalence $\Phi_{\mathcal{G}_K}$ where $\mathcal{G} \in \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}')$.

The conclusion

Properties of $\mathcal{G}$

1. $\mathcal{G}_0 := i^*\mathcal{G}$ is a sheaf in $\mathbf{Coh}(X \times X)$.
2. The natural morphism

$$(\Phi_{\mathcal{G}_0})_* : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$$

is such that $(\Phi_{\mathcal{G}_0})_* = (\Phi_{\mathcal{E}})_* = j$.

Notice that $\mathcal{G}_0$ and $\mathcal{E}$ have the same Mukai vector!

The conclusion

The contradiction is now obtained using the following lemma:

Lemma

If $\mathcal{F} \in \mathbf{Coh}(X \times X)$, then $(\Phi_{\mathcal{F}})_* \neq j$.

Open question

Which is the kernel of the map $\mathrm{Aut}\,(\mathrm{D^b}(X)) \to \mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$?