Inducing stability conditions

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arXiv:0705.3752

Outline

1. **Stability conditions**
   - Motivations
   - Bridgeland's definition

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2. **Inducing stability conditions**
   - General technique
   - The equivariant case
   - Examples

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3. **Enriques surfaces**
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Motivations

Let X be a smooth (complex) projective variety and let

$$\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X)).$$

With a good definition of stability on $\mathrm{D}^{\mathrm{b}}(X)$ (e.g. Bridgeland's one), one would get:

- a "good" notion of moduli space of stable objects in a derived category (Inaba, Lieblich, Toën-Vaquié, Toda, Arcara-Bertram,...);
- a manifold which should allow one to study algebraic objects related to the derived category. For example:
  - **(a)** t-structures,
  - **(b)** the group of autoequivalences.

Aims of the talk

Problem

Suppose that two smooth projective varieties X and Y are related in some intimate geometric way. Then produce some (maybe weak) relation between the manifolds parametrizing stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ and $\mathrm{D}^{\mathrm{b}}(Y)$ (stability manifolds).

Aim 1: Attack and solve this problem in some interesting special cases.

Aim 2: Relate some connected component of the stability manifold to the description of the group of autoequivalences.

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The definition

For simplicity, we restrict ourselves to the case of stability conditions on derived categories! Any triangulated category (e.g. the equivariant case) would fit.

A stability condition on $\mathrm{D}^{\mathrm{b}}(X)$ is a pair $\sigma = (Z, \mathcal{P})$ where

- $Z : K(\mathrm{D}^{\mathrm{b}}(X)) \to \mathbb{C}$ is a linear map (the central charge)
- $\mathcal{P}(\phi) \subset \mathrm{D}^{\mathrm{b}}(X)$ are full additive subcategories for each $\phi \in \mathbb{R}$

satisfying the following conditions:

The definition

(a) If $0 \neq \mathcal{E} \in \mathcal{P}(\phi)$, then $Z(\mathcal{E}) = m(\mathcal{E}) \exp(i\pi\phi)$ for some $m(\mathcal{E}) \in \mathbb{R}_{>0}$.

(b) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all ϕ.

(c) $\mathrm{Hom}\,(\mathcal{E}_1, \mathcal{E}_2) = 0$ for all $\mathcal{E}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \phi_2$.

(d) Any $0 \neq \mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ admits a Harder–Narasimhan filtration given by a collection of distinguished triangles

$$\mathcal{E}_{i-1} \to \mathcal{E}_i \to \mathcal{A}_i$$

with $\mathcal{E}_0 = 0$ and $\mathcal{E}_n = \mathcal{E}$ such that $\mathcal{A}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \ldots > \phi_n$.

Basic properties (Bridgeland)

To exhibit a stability condition on $\mathrm{D}^{\mathrm{b}}(X)$, it is enough to give

- a bounded t-structure on $\mathrm{D}^{\mathrm{b}}(X)$ with heart $\mathbf{A}$;
- a group homomorphism $Z : K(\mathbf{A}) \to \mathbb{C}$ such that $Z(\mathcal{E}) \in \mathbb{H}$, for all $0 \neq \mathcal{E} \in \mathbf{A}$, and with the Harder–Narasimhan property ($\mathbb{H} := \{z \in \mathbb{C} : z = |z| \exp(i\pi\phi), 0 < \phi \leq 1\}$).

All stability conditions are assumed to be locally-finite. Hence every object in $\mathcal{P}(\phi)$ has a finite Jordan–Hölder filtration. $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X))$ is the set of locally finite stability conditions.

There are two groups acting naturally on $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X))$:

- The group $\mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X))$ of exact autoequivalences of $\mathrm{D}^{\mathrm{b}}(X)$.
- The universal cover $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ of $\mathrm{Gl}_2^+(\mathbb{R})$.

Topological properties (Bridgeland)

1. For each connected component $\Sigma \subseteq \mathrm{Stab}\,(\mathrm{D^b}(X))$ there is a linear subspace $V(\Sigma) \subseteq (K(\mathrm{D^b}(X)) \otimes \mathbb{C})^\vee$ with a well-defined linear topology such that the natural map
$$\mathcal{Z} : \Sigma \longrightarrow V(\Sigma), \qquad (Z, \mathcal{P}) \longmapsto Z$$
is a local homeomorphism.
2. A stability condition such that the central charge factors through the algebraic part of the singular cohomology (denoted $\mathcal{N}(X)$) is numerical.
3. The manifold $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X))$ parametrizing numerical stability conditions is finite dimensional.

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The good functors

Let $F : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ be an exact functor and assume that, for any $\mathcal{A}, \mathcal{B} \in \mathrm{D}^{\mathrm{b}}(X)$,

$$(*) \qquad \mathrm{Hom}\,(F(\mathcal{A}), F(\mathcal{B})) = 0 \ \text{ implies } \ \mathrm{Hom}\,(\mathcal{A}, \mathcal{B}) = 0.$$

The definition

If $\sigma' = (Z', \mathcal{P}') \in \mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(Y))$, define $\sigma = F^{-1}\sigma' = (Z, \mathcal{P})$ by

$$Z = Z' \circ F_* \qquad \mathcal{P}(\phi) = \{\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X) : F(\mathcal{E}) \in \mathcal{P}'(\phi)\},$$

where $F_* : K(\mathrm{D}^{\mathrm{b}}(X)) \otimes \mathbb{C} \to K(\mathrm{D}^{\mathrm{b}}(Y)) \otimes \mathbb{C}$ is the natural morphism induced by F.

Also Polishchuk!

First properties

Remark

To prove that σ is a locally-finite stability condition, it suffices to prove that HN-filtrations exist.

Lemma

$$\mathrm{Dom}(F^{-1}) := \{\sigma' \in \mathrm{Stab}\,(\mathrm{D^b}(Y)) \,:\, \sigma = F^{-1}\sigma' \in \mathrm{Stab}\,(\mathrm{D^b}(X))\}$$

is closed.

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The setting

Let X be a smooth projective variety over $\mathbb{C}$ with an action of a finite group G.

We denote by $\mathbf{Coh}_G(X)$ the abelian category of G-equivariant coherent sheaves on X, i.e. pairs $(\mathcal{E}, \{\lambda_g\}_{g\in G})$, where

- $\mathcal{E} \in \mathbf{Coh}(X)$;
- for any $g_1, g_2 \in G$, $\lambda_{g_i} : \mathcal{E} \xrightarrow{\sim} g_i^*\mathcal{E}$ is an isomorphism such that $\lambda_{g_1 g_2} = g_2^*(\lambda_{g_1}) \circ \lambda_{g_2}$.

We put $\mathrm{D}^{\mathrm{b}}_G(X) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}_G(X))$.

The setting

Consider the functors

$$\mathrm{Forg}_G : \mathrm{D}^{\mathrm{b}}_G(X) \to \mathrm{D}^{\mathrm{b}}(X)$$

which forgets the G-linearization, and

$$\mathrm{Inf}_G : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}_G(X)$$

defined by

$$\mathrm{Inf}_G(\mathcal{E}) := \left(\bigoplus_{g \in G} g^* \mathcal{E}, \lambda_{\mathrm{nat}} \right),$$

where λ_{nat} is the natural G-linearization.

The first main result

The group G acts on $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$ and $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$.

Hence consider the (possibly empty!) set

$$\Gamma_X := \{\sigma \in \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X)) : g^*\sigma = \sigma, \text{ for any } g \in G\}.$$

Theorem A (M.-M.-S.)

The subset Γ_X of invariant stability conditions in $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$ is a closed submanifold with a closed embedding into $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}_G(X))$ via the forgetful functor.

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Example 1: elliptic curves

If E is an elliptic curve, then the abelian category **Coh**(E) and the function

$$Z(\mathcal{E}) := -\deg(\mathcal{E}) + i \operatorname{rk} \mathcal{E}$$

define a stability condition on $\mathrm{D}^{\mathrm{b}}(E)$. Hence $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(E)) \neq \emptyset$.

Theorem (Bridgeland)

The stability manifold $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(E))$ is naturally isomorphic to $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$.

Example 1: elliptic curves

Consider the action of a finite group G on E. Hence:

- Any $g \in G$ acts as the identity on the even cohomology of E
- $(\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(E)))_G = \mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(E))$.

Theorem A now reads as:

Proposition

$\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(E))$ is embedded as a closed submanifold into $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}_G(E))$.

Example 1: elliptic curves

Geigle and Lenzing consider the case of elliptic curves E and involutions ι and weighted projective lines C such that the following categories are equivalent:

- **Coh**$([E/\langle\iota\rangle])$ of the stack $[E/\langle\iota\rangle]$;
- **Coh**(C).

The "interpretation"

A Mirror Symmetry interpretation should relate $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}_{\langle\iota\rangle}(E))$ to the unfolding space of the elliptic singularity corresponding to C. The embedded closed submanifold $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(E))$ should be the deformation space of the elliptic curve describing the singularity.

Example 2: K3 and abelian surfaces

Let X be an abelian or a K3 surface.

Fix $\omega, \beta \in \mathrm{NS}(X) \otimes \mathbb{Q}$ with ω in the ample cone.

Take the slope function μ_ω associated to ω.

Define the categories

- $\mathcal{T}(\omega, \beta)$ consisting of sheaves whose torsion-free part have μ_ω-semistable Harder–Narasimhan factors with slope greater than $\omega \cdot \beta$
- $\mathcal{F}(\omega, \beta)$ consisting of torsion-free sheaves whose μ_ω-semistable Harder–Narasimhan factors have slope smaller or equal to $\omega \cdot \beta$.

Example 2: K3 and abelian surfaces

Next consider the abelian category

$$\mathcal{A}(\omega,\beta) := \left\{ \mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X) : \begin{array}{l} \bullet\ \mathcal{H}^i(\mathcal{E}) = 0 \text{ for } i \notin \{-1,0\}, \\ \bullet\ \mathcal{H}^{-1}(\mathcal{E}) \in \mathcal{F}(\omega,\beta), \\ \bullet\ \mathcal{H}^0(\mathcal{E}) \in \mathcal{T}(\omega,\beta) \end{array} \right\}.$$

and the $\mathbb{C}$-linear map

$$Z_{\omega,\beta} : \mathcal{N}(X) \to \mathbb{C}, \qquad \mathcal{E} \longmapsto \langle \exp(\beta + i\omega), v(\mathcal{E}) \rangle,$$

where $v(\mathcal{E})$ is the Mukai vector of $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ and $\langle -, - \rangle$ is the Mukai pairing.

Proposition (Bridgeland)

If $\omega \cdot \omega > 2$, the pair $(Z_{\omega,\beta}, \mathcal{A}(\omega,\beta))$ defines a stability condition.

Example 2: K3 and abelian surfaces

Bridgeland considered the connected component

$$\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(X)) \subseteq \mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X))$$

containing $(Z_{\omega,\beta}, \mathcal{A}(\omega, \beta))$ with ω and β as above.

Theorem (Bridgeland, Huybrechts-Macrì-S.)

If X is an abelian surface, then $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(X))$ is the unique connected component of maximal dimension. Moreover it is simply connected.

Example 2: K3 and abelian surfaces

Let A be an abelian surface and $\mathrm{Km}(A)$ the associated Kummer surface.

$\mathrm{Km}(A)$ is the minimal resolution of the quotient $A/\langle\iota\rangle$, where $\iota : A \xrightarrow{\sim} A$ is the involution such that $\iota(a) = -a$.

By its very definition, $\iota^* : \mathcal{N}(A) \xrightarrow{\sim} \mathcal{N}(A)$ is the identity.

Hence Γ_A is open and closed in $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A))$ and, if non-empty, Γ_A is a connected component.

Proposition

$\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A))$ is realized as a closed submanifold of $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A)))$.

Further perspectives (Toda, M.-M.-S.)

Problem

Define stability conditions on X, algebraic of dimension 3 and with trivial K_X.

Take a K3 surface or an abelian surface X with an involution $\iota_1 : X \to X$:

- Suppose that the derived category $\mathrm{D}^{\mathrm{b}}([X/\iota_1])$ of the quotient stack $[X/\iota_1]$ is equivalent to the derived category of a weighted projective space.
- One gets a description of $\mathrm{D}^{\mathrm{b}}([X/\iota_1])$ in terms of quivers.

Example

Take $X := E \times E$, with E elliptic curve and $\iota_1 := \iota \times \iota$.

Further perspectives (Toda, M.-M.-S.)

Take an elliptic curve E with an involution $\iota_2 : E \to E$.

- One can "easily" construct stability conditions on $\mathrm{D}^{\mathrm{b}}([(X \times E/(\iota_1 \times \iota_2)])$ in terms of quivers.

Goal

Apply the previous procedure of inducing stability conditions to construct stability conditions on $X \times E$ using stability conditions on $\mathrm{D}^{\mathrm{b}}([(X \times E)/(\iota_1 \times \iota_2)])$.

Warning!

One may need to deform a bit the "easy" examples of stability conditions on $\mathrm{D}^{\mathrm{b}}([(X \times E)/(\iota_1 \times \iota_2)])$ to lift them to $X \times E$.

Enriques surfaces

Let Y be an Enriques surface. Moreover, let

- $\pi : X \to Y$ be its universal cover;
- $\iota : X \to X$ be the fixed-point-free involution such that $Y = X/G$, where G is now the group generated by ι.

In this special setting:

- $\mathbf{Coh}(Y)$ is naturally isomorphic to the abelian category $\mathbf{Coh}_G(X)$;
- $\mathrm{D^b}(Y) \cong \mathrm{D^b_G}(X)$.

Enriques surfaces: the second main result

Theorem B (M.-M.-S.)

There exist a connected component $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y))$ of $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y))$ naturally embedded into $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ as a closed submanifold and a natural homomorphism

$$\mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(Y)) \to \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G/G$$

whose image contains the index-2 subgroup of G-equivariant orientation preserving Hodge isometries quotiented by G.

Moreover, if Y is generic, the category $\mathrm{D}^{\mathrm{b}}(Y)$ does not contain spherical objects and $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y))$ is isomorphic to the distinguished connected component $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$.

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A few ideas from the proof

1. Γ_X is non-empty. Indeed,
   - choose $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbb{R}$ invariant for the action of ι^*
   - so $\iota^* \sigma_{\omega,\beta} = \sigma_{\omega,\beta}$.
2. Given the map $\mathrm{Forg}_G^{-1} : \Gamma_X \to \mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(Y))$, by Theorem A, $\Sigma(Y) := \mathrm{Forg}_G^{-1}(\Gamma_X \cap \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(X)))$ is closed.

Moreover, the following diagram commutes

$$\begin{array}{ccccc}
\Gamma_X \cap \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(X)) & \xrightarrow{\mathrm{Forg}_G^{-1}} & \Sigma(Y) & \xrightarrow{\mathrm{Inf}_G^{-1}} & \Gamma_X \cap \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(X)) \\
\downarrow & & \downarrow{\scriptstyle \mathcal{Z}} & & \downarrow \\
(\mathcal{N}(X) \otimes \mathbb{C})^{\vee}_G & \xrightarrow{\mathrm{Forg}^{\vee}_{G*}} & (\mathcal{N}(Y) \otimes \mathbb{C})^{\vee} & \xrightarrow{\mathrm{Inf}^{\vee}_{G*}} & (\mathcal{N}(X) \otimes \mathbb{C})^{\vee}_G.
\end{array}$$

A few ideas from the proof

Using the morphism Inf_G^{-1} in the previous diagram, one also shows that $\Sigma(Y)$ is open.

We define

$$\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y)) \subseteq \Sigma(Y)$$

to be the (non-empty) connected component containing the images of the stability conditions

$$(Z_{\omega,\beta}, \mathcal{A}(\omega, \beta))$$

with G-invariant $\omega, \beta \in \mathrm{NS}(X) \otimes \mathbb{Q}$ (previous example!).

An example: abelian, K3 and Enriques surfaces

Take two non-isogenous elliptic curves E_1 and E_2.

- Choose two order-2 points $e_1 \in E_1$ and $e_2 \in E_2$.
- The abelian surface $A := E_1 \times E_2$ has an involution ι defined by

$$\iota : (z_1, z_2) \longmapsto (-z_1 + e_1, z_2 + e_2).$$

- The induced involution $\tilde{\iota} : \mathrm{Km}(A) \to \mathrm{Km}(A)$ has no fixed points.

An example: abelian, K3 and Enriques surfaces

Let Y be the Enriques surface $\mathrm{Km}(A)/\langle\tilde{\iota}\rangle$. Combining Theorem B and the example about Kummer surfaces we obtain the following:

Proposition

There exist a connected component

$$\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y)) \subseteq \mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y))$$

and embeddings

$$\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A)) \hookrightarrow \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y)) \hookrightarrow \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A)))$$

of closed submanifolds.

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The statement

Geometric Torelli Theorem

The geometry (and automorphism group) of an Enriques surface Y is governed by the Hodge isometries of the second cohomology group of its universal cover.

The existence of the natural homomorphism

$$\Pi : \mathrm{Aut}\,(\mathrm{D^b}(Y)) \to \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G/G$$

in Theorem B is the analogue on the level of $\mathrm{D^b}(Y)$.

The morphism

Define G_Δ to be the group generated by the involution $\iota \times \iota$ on $X \times X$.

Consider the following set of objects:

1. $\mathrm{Ker}^{G_\Delta}(\mathrm{D}^{\mathrm{b}}(X)) := \{(\mathcal{G}, \lambda) \in \mathrm{D}^{\mathrm{b}}_{G_\Delta}(X \times X) : \Phi_{\mathcal{G}} \in \mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X))\}$
2. $\mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X))_G := \{\Phi \in \mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X)) : \iota^* \circ \Phi \circ \iota^* \cong \Phi\}$.

A few ideas from the proof

Due to a remark by Ploog, the functors Forg_G and Inf_G are 2 : 1 and fit into the diagram

$$\begin{array}{ccc} & \mathrm{Ker}^{G_\Delta}(\mathrm{D^b}(X)) & \\ \swarrow^{\mathrm{Forg}_G} & & \searrow^{\mathrm{Inf}_G} \\ \mathrm{Aut}\,(\mathrm{D^b}(X))_G & & \mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}_G(X)) = \mathrm{Aut}\,(\mathrm{D^b}(Y)). \end{array}$$

This yields a natural surjective homomorphism

$$\mathrm{Lift} : \mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}_G(X)) \to \mathrm{Aut}\,(\mathrm{D^b}(X))_G/G.$$

Compose with the natural map

$$\mathrm{Aut}\,(\mathrm{D^b}(X))_G/G \to \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G/G.$$

Orientation

Lattice structure: The Mukai pairing (Euler–Poincaré form up to sign). The lattice is denoted $\widetilde{H}(X,\mathbb{Z})$.

Orientation: Let σ be a generator of $H^{2,0}(X)$ and ω a Kähler class. Then

$$P(X,\sigma,\omega) := \langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma), 1-\omega^2/2, \omega\rangle,$$

is a positive four-space in $\widetilde{H}(X,\mathbb{R})$ with a natural orientation.

Hodge structure: The weight-2 Hodge structure on $H^*(X,\mathbb{Z})$ is

$$\begin{aligned}
\widetilde{H}^{2,0}(X) &:= H^{2,0}(X),\\
\widetilde{H}^{0,2}(X) &:= H^{0,2}(X),\\
\widetilde{H}^{1,1}(X) &:= H^0(X,\mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X,\mathbb{C}).
\end{aligned}$$

Orientation

- We will denote by $\mathrm{O}(\widetilde{H}(X,\mathbb{Z}))$ ($\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$) the group of (orientation preserving) Hodge isometries of $\widetilde{H}(X,\mathbb{Z})$.
- The subgroups consisting of the equivariant isometries are denoted by $\mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G$ and $\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))_G$.

At the very end the proof boils down to the following:

Proposition (Huybrechts-S.)

All known autoequivalences of $\mathrm{D}^{\mathrm{b}}(X)$ are orientation preserving.

The connected component

- Define the open subset $\mathcal{P}(X) \subseteq \mathcal{N}(X) \otimes \mathbb{C}$ consisting of those vectors whose real and imaginary parts span a positive definite two plane in $\mathcal{N}(X) \otimes \mathbb{R}$.
- Denote by $\mathcal{P}^+(X)$ one of the two connected components of $\mathcal{P}(X)$.
- If $\Delta(X)$ is the set of vectors in $\mathcal{N}(X)$ with self-intersection -2, following Bridgeland, consider
$$\mathcal{P}_0^+(X) := \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^{\perp}.$$
- Define $\mathcal{P}_0^+(Y) := \mathrm{Forg}_{G*}^{\vee}(\mathcal{P}_0^+(X))_G$.

The connected component

Let $\Sigma(Y) := \mathrm{Forg}_G^{-1}(\Gamma_X \cap \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D^b}(X)))$.

Let $\mathrm{Aut}^0(\mathrm{D^b}(Y))$ be the subgroup of those autoequivalences preserving $\Sigma(Y)$ and inducing the identity on cohomology via the morphisms Π.

Proposition

The morphism $\mathcal{Z} : \Sigma(Y) \to \mathcal{N}(Y) \otimes \mathbb{C}$ defines a covering map onto $\mathcal{P}_0^+(Y)$ such that

$$\mathrm{Aut}^0(Y) := \mathrm{Aut}^0(\mathrm{D^b}(Y))/\langle(-) \otimes \omega_Y\rangle$$

acts as the group of deck transformations.

A conjecture

Conjecture

The group $\mathrm{Aut}(\mathrm{D}^b(Y))$ preserves $\Sigma(Y)$ and, moreover, $\Sigma(Y)$ is connected and simply connected.

From the previous conjecture we get:

$$1 \to \pi_1(\mathcal{P}_0^+(Y)) \to \mathrm{Aut}^0(Y) \to \mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))_G/G \to 1.$$

Remark: work in progress with Huybrechts and Macrì

Try to solve a similar problem for K3 surfaces (this would conclude also in the Enriques case).

Outline

Generic Enriques surfaces

Let Y be a generic Enriques surface.

Remark

Using the surjectivity of the period map for Enriques and K3 surfaces one proves that the universal cover X of a generic Enriques surface Y has Picard number 10.

Remark

In the above setting, X does not contain rational curves. Hence Y does not contain rational curves neither.

Spherical objects

Definition

An object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(Y)$ such that $\mathcal{E} \cong \mathcal{E} \otimes \omega_Y$ is

1. spherical if $\mathrm{Hom}\,(\mathcal{E}^\bullet, \mathcal{E}^\bullet[i]) \cong \mathbb{C}$ if $i \in \{0, \dim\ Y\}$ and it is trivial otherwise.
2. rigid if $\mathrm{Hom}\,(\mathcal{E}^\bullet, \mathcal{E}^\bullet[1]) = 0$.

To complete the proof of Theorem B:

Proposition

Let Y be a generic Enriques surface. Then

1. $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X)) \subseteq \mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ is isomorphic to $\Sigma(Y)$.
2. $\mathrm{D}^{\mathrm{b}}(Y)$ does not contain spherical objects.

A remark

Generic Enriques surfaces have no spherical objects but plenty of rigid objects.

For K3 surfaces, spherical objects are always present (at least in the untwisted case).

As was proved in collaboration with Huybrechts and Macrì, the only way to reduce drastically the number of rigid and spherical objects is to pass to twisted or generic analytic K3 surfaces.

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Canonical bundles

We consider now an easy example where the information we get is much less.

- Consider the canonical bundle

$$\pi : \omega_{\mathbb{P}^N} \longrightarrow \mathbb{P}^N$$

  over the projective space $\mathbb{P}^N$. And let X be the total space.

- Let $i : \mathbb{P}^N \hookrightarrow X$ denote the zero-section and C its image.

- Let $\mathrm{D}^{\mathrm{b}}_0(X) := \mathrm{D}^{\mathrm{b}}_C(\mathbf{Coh}(X))$, the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))$ whose objects have cohomology sheaves supported on C.

The general case

Denote by $\mathrm{Stab}(X)$ the stability manifold of $\mathrm{D}^{\mathrm{b}}_0(X)$.

Proposition (M.-M.-S.)

There is an open subset of $\mathrm{Stab}(X)$ embedded into $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^N))$.

Remark

The functor i_* induces stability conditions from $\mathrm{Stab}(X)$ to $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^N))$ but the behaviour is not so nice.

Outline

The case $N = 1$

For $N = 1$, the best result we get is the following:

Theorem C (M.-M.-S.)

An open subset of $\mathrm{Stab}\,(\mathrm{D}^b(\mathbb{P}^1))$ embeds into $\mathrm{Stab}\,(X)$ as a fundamental domain for the action of the autoequivalences group.

Remark

As a by-product we get a simple proof of the connectedness and simply-connectedness of the space $\mathrm{Stab}\,(X)$.

This was previously proved by Okada and, more generally, by Ishii-Uehara-Ueda.