

# Deformations of K3 surfaces and orientation

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# Derived categories and K3 surfaces

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Let  $X$  be a smooth projective complex variety. Denote by  $\mathbf{Coh}(X)$  the abelian category of coherent sheaves on  $X$ .

The main algebraic invariant we are going to study is the **bounded derived category of coherent sheaves**

$$D^b(X) := D^b(\mathbf{Coh}(X)).$$

A **K3 surface** is a smooth compact Kähler (complex) surface  $X$  such that:

- $X$  is simply connected.
- The canonical bundle  $K_X$  is trivial.

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Let  $X$  be a K3 surface.

## Main problem

Describe the group of exact autoequivalences of the triangulated category  $D^b(X)$  or of a first order deformation of it.

## Remark (Orlov)

Such a description is available (in the non-deformed context) when  $X$  is an abelian surface (actually an abelian variety).



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# Geometry: automorphisms

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## Theorem (Torelli Theorem)

Let  $X$  and  $Y$  be K3 surfaces. Suppose that there exists a Hodge isometry

$$g : H^2(X, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z})$$

which maps the class of an ample line bundle on  $X$  into the ample cone of  $Y$ . Then there exists a unique isomorphism  $f : X \xrightarrow{\sim} Y$  such that  $f_* = g$ .

Lattice theory + Hodge structures + ample cone

## Remark

The automorphism is uniquely determined.

# Geometry: diffeomorphisms

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## Theorem (Borcea, Donaldson)

Consider the natural map

$$\rho : \text{Diff}(X) \longrightarrow \text{O}(H^2(X, \mathbb{Z})).$$

Then  $\text{im}(\rho) = \text{O}_+(H^2(X, \mathbb{Z}))$ , where  $\text{O}_+(H^2(X, \mathbb{Z}))$  is the group of orientation preserving isometries.

The orientation is given by the choice of a basis for the 3-dimensional positive space in  $H^2(X, \mathbb{R})$ .

## Remark

The kernel of  $\rho$  is not known!

# Orlov's result

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## Derived Torelli Theorem (Mukai, Orlov)

Let  $X$  and  $Y$  be smooth projective K3 surfaces. Then the following are equivalent:

- 1 There exists an equivalence  $\Phi : D^b(X) \cong D^b(Y)$ .
- 2 There exists a Hodge isometry  $\tilde{H}(X, \mathbb{Z}) \cong \tilde{H}(Y, \mathbb{Z})$ .

The equivalence  $\Phi$  induces an action on cohomology

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\Phi} & D^b(Y) \\ \downarrow v(-)=\text{ch}(-)\cdot\sqrt{\text{td}(X)} & & \downarrow v(-)=\text{ch}(-)\cdot\sqrt{\text{td}(Y)} \\ \tilde{H}(X, \mathbb{Z}) & \xrightarrow{\Phi_H} & \tilde{H}(Y, \mathbb{Z}) \end{array}$$

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## Question

Can we understand better the action induced on cohomology by an equivalence?

**Orientation:** Let  $\sigma$  be a generator of  $H^{2,0}(X)$  and  $\omega$  a Kähler class. Then  $\langle \operatorname{Re}(\sigma), \operatorname{Im}(\sigma), 1 - \omega^2/2, \omega \rangle$  is a positive four-space in  $\tilde{H}(X, \mathbb{R})$  with a natural orientation.

## Problem

The isometry  $j := (\operatorname{id})_{H^0 \oplus H^4} \oplus (-\operatorname{id})_{H^2}$  is not orientation preserving. Is it induced by an autoequivalence?

# Motivation

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There exists an explicit description of the first order deformations of the abelian category of coherent sheaves on a smooth projective variety (Toda).

The existence of equivalences between the derived categories of smooth projective K3 surfaces is detected by the existence of special isometries of the total cohomologies.

## Question

Can we get the same result for derived categories of first order deformations of K3 surfaces using special isometries between 'deformations' of the Hodge and lattice structures on the total cohomologies?

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# Hochschild homology and cohomology

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For  $X$  any smooth projective variety, define the **Hochschild homology**

$$\mathrm{HH}_i(X) := \mathrm{Hom}_{\mathrm{D}^b(X \times X)}(\Delta_* \omega_X^\vee[i - \dim(X)], \mathcal{O}_{\Delta_X})$$

and the **Hochschild cohomology**

$$\mathrm{HH}^i(X) := \mathrm{Hom}_{\mathrm{D}^b(X \times X)}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}[i]).$$

On the other hand we put

$$\mathrm{H}\Omega_i(X) := \bigoplus_{q-p=i} H^p(X, \Omega_X^q) \quad \mathrm{H}\mathrm{T}^i(X) := \bigoplus_{p+q=i} H^p(X, \wedge^q \mathcal{T}_X).$$



# Hochschild–Kostant–Rosenberg

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There exist (the [Hochschild–Kostant–Rosenberg](#))  
isomorphisms

$$I_{\text{HKR}}^X : \text{HH}_*(X) \rightarrow \text{H}\Omega_*(X) := \bigoplus_i \text{H}\Omega_i(X)$$

and

$$I_X^{\text{HKR}} : \text{HH}^*(X) \rightarrow \text{H}\text{T}^*(X) := \bigoplus_i \text{H}\text{T}^i(X).$$

One then defines the graded isomorphisms

$$I_K^X = (\text{td}(X)^{1/2} \wedge (-)) \circ I_{\text{HKR}}^X \quad I_X^K = (\text{td}(X)^{-1/2} \lrcorner (-)) \circ I_X^{\text{HKR}}.$$

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- 1 Take a smooth projective variety  $X$ ,  $v \in \mathrm{HH}^2(X)$  and write

$$I_X^{\mathrm{HKR}}(v) = (\alpha, \beta, \gamma) \in \mathrm{HT}^2(X).$$

- 2 Define a sheaf  $\mathcal{O}_X^{(\beta, \gamma)}$  of  $\mathbb{C}[\epsilon]/(\epsilon^2)$ -algebras on  $X$  depending only on  $\beta$  and  $\gamma$ .
- 3 Representing  $\alpha \in H^2(X, \mathcal{O}_X)$  as a Čech 2-cocycle  $\{\alpha_{ijk}\}$  one has an element  $\tilde{\alpha} := \{1 - \epsilon\alpha_{ijk}\}$  which is a Čech 2-cocycle with values in the invertible elements of the center of  $\mathcal{O}_X^{(\beta, \gamma)}$ .

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We get the abelian category

$$\mathbf{Coh}(\mathcal{O}_X^{(\beta, \gamma)}, \tilde{\alpha})$$

of  $\tilde{\alpha}$ -twisted coherent  $\mathcal{O}_X^{(\beta, \gamma)}$ -modules. Set

$$\mathbf{Coh}(X, \nu) := \mathbf{Coh}(\mathcal{O}_X^{(\beta, \gamma)}, \tilde{\alpha}).$$

One also have an isomorphism  $J : \mathbf{HH}^2(X_1) \rightarrow \mathbf{HH}^2(X_1)$   
such that

$$(I_{X_1}^{\text{HKR}} \circ J \circ (I_{X_1}^{\text{HKR}})^{-1})(\alpha, \beta, \gamma) = (\alpha, -\beta, \gamma).$$

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# The Infinitesimal Derived Torelli Theorem

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## Theorem (Macri–S.)

Let  $X_1$  and  $X_2$  be smooth complex projective K3 surfaces and let  $v_i \in \mathrm{HH}^2(X_i)$ , with  $i = 1, 2$ . Then the following are equivalent:

- 1 There exists a Fourier–Mukai equivalence

$$\Phi_{\tilde{\mathcal{E}}} : D^b(X_1, v_1) \xrightarrow{\sim} D^b(X_2, v_2)$$

with  $\tilde{\mathcal{E}} \in D_{\mathrm{perf}}(X_1 \times X_2, -J(v_1) \boxplus v_2)$ .

- 2 There exists an orientation preserving effective Hodge isometry

$$g : \tilde{H}(X_1, v_1, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(X_2, v_2, \mathbb{Z}).$$

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For  $X$  a K3,  $v \in \mathrm{HH}^2(X)$  and  $\sigma_X$  is a generator for  $\mathrm{HH}_2(X)$ , let

$$w := I_K^X(\sigma_X) + \epsilon I_K^X(\sigma_X \circ v) \in \tilde{H}(X, \mathbb{Z}) \otimes \mathbb{C}[\epsilon]/(\epsilon^2).$$

The free  $\mathbb{Z}[\epsilon]/(\epsilon^2)$ -module of finite rank  $\tilde{H}(X, \mathbb{Z}) \otimes \mathbb{Z}[\epsilon]/(\epsilon^2)$  is endowed with:

- 1 The  $\mathbb{Z}[\epsilon]/(\epsilon^2)$ -linear extension of the generalized Mukai pairing  $\langle -, - \rangle_M$ .
- 2 A weight-2 decomposition on  $\tilde{H}(X, \mathbb{Z}) \otimes \mathbb{C}[\epsilon]/(\epsilon^2)$

$$\tilde{H}^{2,0}(X, v) := \mathbb{C}[\epsilon]/(\epsilon^2) \cdot w \quad \tilde{H}^{0,2}(X, v) := \overline{\tilde{H}^{2,0}(X, v)}$$

$$\text{and } \tilde{H}^{1,1}(X, v) := (\tilde{H}^{2,0}(X, v) \oplus \tilde{H}^{0,2}(X, v))^\perp.$$

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This gives the **infinitesimal Mukai lattice** of  $X$  with respect to  $v$ , which is denoted by  $\tilde{H}(X, v, \mathbb{Z})$ .

$$g : \tilde{H}(X_1, v_1, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(X_2, v_2, \mathbb{Z})$$

which is  $g = g_0 \otimes \mathbb{Z}[\epsilon]/(\epsilon^2)$ , where  $g_0$  is an Hodge isometry of the Mukai lattices is called **effective**.

An effective isometry is **orientation preserving** if  $g_0$  preserves the orientation of the four-space.

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# Deformations

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We just sketch of the implication (i) $\Rightarrow$ (ii).

- Let  $\Phi_{\tilde{\mathcal{E}}} : D^b(X_1, v_1) \xrightarrow{\sim} D^b(X_2, v_2)$  be an equivalence with kernel  $\tilde{\mathcal{E}} \in D_{\text{perf}}(X_1 \times X_2, -J(v_1) \boxplus v_2)$ .
- One shows that the restriction  $\mathcal{E} \in D^b(X_1 \times X_2)$  of  $\tilde{\mathcal{E}}$  is the kernel of a Fourier–Mukai equivalence  $\Phi_{\mathcal{E}} : D^b(X_1) \xrightarrow{\sim} D^b(X_2)$ .
- Using Orlov’s result, take the Hodge isometry  $g_0 := (\Phi_{\mathcal{E}})_H : \tilde{H}(X_1, \mathbb{Z}) \rightarrow \tilde{H}(X_2, \mathbb{Z})$ .

# The isometry

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**Toda:** since  $\tilde{\mathcal{E}}$  is a first order deformation of  $\mathcal{E}$ ,

$$(\Phi_{\mathcal{E}})^{\text{HH}}(v_1) = v_2.$$

## Important!

Assume we know that any Hodge isometry induced by an equivalence  $D^b(X_1) \cong D^b(X_2)$  is orientation preserving.

To conclude and prove that

$$g := g_0 \otimes \mathbb{Z}[\epsilon]/(\epsilon^2) : \tilde{H}(X_1, v_1, \mathbb{Z}) \rightarrow \tilde{H}(X_2, v_2, \mathbb{Z})$$

is an effective orientation preserving Hodge isometry, we need two commutative diagrams.

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Any Fourier–Mukai functor acts on Hochschild homology.

## Theorem (Macrì–S.)

Let  $X_1$  and  $X_2$  be smooth complex projective varieties and let  $\mathcal{E} \in D^b(X_1 \times X_2)$ . Then the following diagram

$$\begin{array}{ccc} \mathrm{HH}_*(X_1) & \xrightarrow{(\Phi_{\mathcal{E}})_{\mathrm{HH}}} & \mathrm{HH}_*(X_2) \\ \downarrow \mathcal{I}_K^{X_1} & & \downarrow \mathcal{I}_K^{X_2} \\ \tilde{H}(X_1, \mathbb{C}) & \xrightarrow{(\Phi_{\mathcal{E}})_H} & \tilde{H}(X_2, \mathbb{C}) \end{array}$$

commutes.

# Commutativity II

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Using that for K3 surfaces  $H^{0,2}$  is 1-dimensional and the previous result, one get the following commutative diagram (for a Fourier–Mukai equivalence  $\Phi_{\mathcal{E}}$ ):

$$\begin{array}{ccc} \mathrm{HH}^*(X_1) & \xrightarrow{(\Phi_{\mathcal{E}})^{\mathrm{HH}}} & \mathrm{HH}^*(X_2) \\ (-) \circ \sigma_{X_1} \downarrow & & \downarrow (-) \circ (\Phi_{\mathcal{E}})_{\mathrm{HH}}(\sigma_{X_1}) \\ \mathrm{HH}_*(X_1) & \xrightarrow{(\Phi_{\mathcal{E}})_{\mathrm{HH}}} & \mathrm{HH}_*(X_2) \\ I_K^{X_1} \downarrow & & \downarrow I_K^{X_2} \\ \tilde{H}(X_1, \mathbb{C}) & \xrightarrow{(\Phi_{\mathcal{E}})_H} & \tilde{H}(X_2, \mathbb{C}), \end{array}$$

where  $\sigma_{X_1}$  is a generator of  $\mathrm{HH}_2(X_1)$ .

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We go back to the original problem of describing the group of exact autoequivalences of the derived category of a K3 surface.

## Remarks

- 1 To conclude the previous argument involving (first order) deformations, we need to prove that any equivalence induces an orientation preserving Hodge isometry.
- 2 The (quite involved) proof of this result will use deformation of kernels.

# The statement

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## Main Theorem (Huybrechts–Macrì–S.)

Given a Hodge isometry  $g : \tilde{H}(X, \mathbb{Z}) \rightarrow \tilde{H}(Y, \mathbb{Z})$ , then there exists an equivalence  $\Phi : D^b(X) \rightarrow D^b(Y)$  such that  $g = \Phi_H$  if and only if  $g$  is orientation preserving.

**Szendroi's Conjecture is true:** In terms of autoequivalences, this yields a surjective morphism

$$\mathrm{Aut}(D^b(X)) \twoheadrightarrow O_+(\tilde{H}(X, \mathbb{Z})),$$

where  $O_+(\tilde{H}(X, \mathbb{Z}))$  is the group of orientation preserving Hodge isometries.

# The ‘easy’ implication

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**The statement:** If  $g$  is orientation preserving than it lifts to an equivalence.

- A result of Hosono–Lian–Oguiso–Yau (heavily relaying on Mukai/Orlov’s Derived Torelli Theorem) shows that, up to composing with the isometry  $j$ , every isometry can be lifted to an equivalence.
- Since we know that  $j$  is not orientation preserving we conclude using the following:

## Remark (Huybrechts-S.)

All known equivalences (and autoequivalences) are orientation preserving.



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# The non-orientation Hodge isometry

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Take any projective K3 surface  $X$ .

- Consider the non-orientation preserving Hodge isometry

$$j := (\text{id})_{H^0 \oplus H^4} \oplus (-\text{id})_{H^2}.$$

- Since one implication is already true, to prove the main theorem, it is enough to show that  $j$  is not induced by a Fourier–Mukai equivalence.
- We proceed by contradiction assuming that there exists  $\mathcal{E} \in D^b(X \times X)$  such that  $(\Phi_{\mathcal{E}})_H = j$ .

# The idea of the proof

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- **Huybrechts–Macrì–S.:** For some particular K3 surfaces we know that  $j$  is not induced by any Fourier–Mukai equivalence: K3 surfaces with trivial Picard group.
- Deform the K3 surface in the moduli space such that generically we recover the behaviour of a generic K3 surface.
- Deform the kernel of the equivalence accordingly.
- Derive a contradiction using the generic case.

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# Formal deformations

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Take  $R := \mathbb{C}[[t]]$  to be the ring of power series in  $t$  with field of fractions  $K := \mathbb{C}((t))$ .

Define  $R_n := \mathbb{C}[[t]]/(t^{n+1})$ . Then  $\text{Spec}(R_n) \subset \text{Spec}(R_{n+1})$ .

For  $X$  a smooth projective variety, a **formal deformation** is a proper formal  $R$ -scheme

$$\pi : \mathcal{X} \rightarrow \text{Spf}(R)$$

given by an inductive system of schemes  $\mathcal{X}_n \rightarrow \text{Spec}(R_n)$  (smooth and proper over  $R_n$ ) and such that

$$\mathcal{X}_{n+1} \times_{R_{n+1}} \text{Spec}(R_n) \cong \mathcal{X}_n.$$

# The categories

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There exist sequences

$$\mathbf{Coh}_0(\mathcal{X} \times_R \mathcal{X}') \hookrightarrow \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}') \rightarrow \mathbf{Coh}((\mathcal{X} \times_R \mathcal{X}')_K)$$

$$\mathbf{Coh}_0(\mathcal{X}) \hookrightarrow \mathbf{Coh}(\mathcal{X}) \rightarrow \mathbf{Coh}(\mathcal{X}_K)$$

where  $\mathbf{Coh}_0(\mathcal{X} \times_R \mathcal{X}')$  and  $\mathbf{Coh}_0(\mathcal{X})$  are the abelian categories of sheaves supported on  $X \times X$  and  $X$  respectively.

In this setting we also have the sequences

$$D_0^b(\mathcal{X} \times_R \mathcal{X}') \hookrightarrow D_{\mathbf{Coh}}^b(\mathcal{O}_{\mathcal{X} \times_R \mathcal{X}'}\text{-Mod}) \rightarrow D^b((\mathcal{X} \times_R \mathcal{X}')_K)$$

$$D_0^b(\mathcal{X}) \hookrightarrow D_{\mathbf{Coh}}^b(\mathcal{O}_{\mathcal{X}}\text{-Mod}) \rightarrow D^b(\mathcal{X}_K)$$

# The key example: the twistor space

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Let us focus now on the case when  $X$  is a K3 surface.

## Definition

A Kähler class  $\omega \in H^{1,1}(X, \mathbb{R})$  is called **very general** if there is no non-trivial integral class  $0 \neq \alpha \in H^{1,1}(X, \mathbb{Z})$  orthogonal to  $\omega$ , i.e.  $\omega^\perp \cap H^{1,1}(X, \mathbb{Z}) = 0$ .

Take the twistor space  $\mathbb{X}(\omega)$  of  $X$  determined by the choice of a very general Kähler class  $\omega \in \mathcal{K}_X \cap \text{Pic}(X) \otimes \mathbb{R}$ :

$$\pi : \mathbb{X}(\omega) \rightarrow \mathbb{P}(\omega).$$

# The key example: the twistor space

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## Remark

$\mathbb{X}(\omega)$  parametrizes the complex structures 'compatible' with  $\omega$ .

Choosing a local parameter  $t$  around  $0 \in \mathbb{P}(\omega)$  we get a formal deformation  $\mathcal{X} \rightarrow \mathrm{Spf}(R)$ .

More precisely:

$$\mathcal{X}_n := \mathbb{X}(\omega) \times \mathrm{Spec}(R_n),$$

form an inductive system and give rise to a formal  $R$ -scheme

$$\pi : \mathcal{X} \rightarrow \mathrm{Spf}(R),$$

which is the **formal neighbourhood of  $X$**  in  $\mathbb{X}(\omega)$ .



# Equivalences

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As before, given  $\mathcal{F} \in D_{\mathbf{Coh}}^b(\mathcal{O}_{\mathcal{X} \times_R \mathcal{X}'})$ -Mod, we denote by  $\mathcal{F}_K$  the natural image in the category  $D^b((\mathcal{X} \times_R \mathcal{X}')_K)$ .

## Proposition

Let  $\tilde{\mathcal{E}} \in D^b(\mathcal{X} \times_R \mathcal{X}')$  be such that  $\mathcal{E} = i^* \tilde{\mathcal{E}}$ . Then  $\tilde{\mathcal{E}}$  and  $\tilde{\mathcal{E}}_K$  are kernels of Fourier–Mukai equivalences.

Here we denoted by  $i : X \times X \rightarrow \mathcal{X} \times_R \mathcal{X}'$  the natural inclusion.

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# The first order deformation

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The equivalence  $\Phi_\varepsilon$  induces a morphism

$$\Phi_\varepsilon^{\mathrm{HH}} : \mathrm{HH}^2(X) \rightarrow \mathrm{HH}^2(X).$$

## Proposition

Let  $v_1 \in H^1(X, \mathcal{I}_X)$  be the Kodaira–Spencer class of first order deformation given by a twistor space  $\mathbb{X}(\omega)$  as above. Then

$$v'_1 := \Phi_\varepsilon^{\mathrm{HH}}(v_1) \in H^1(X, \mathcal{I}_X).$$

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Let  $\mathcal{X}'_1$  be the first order deformation corresponding to  $v'_1$ .

Using results of Toda one gets the following conclusion

## Proposition (Toda)

For  $v_1$  and  $v'_1$  as before, there exists  $\mathcal{E}_1 \in D^b(\mathcal{X}_1 \times_{R_1} \mathcal{X}'_1)$  such that

$$i_1^* \mathcal{E}_1 = \mathcal{E}_0 := \mathcal{E}.$$

Here  $i_1 : \mathcal{X}_0 \times_{\mathbb{C}} \mathcal{X}_0 \hookrightarrow \mathcal{X}'_1 \times_{R_1} \mathcal{X}'_1$  is the natural inclusion.

Hence there is a first order deformation of  $\mathcal{E}$ .

# Higher order deformations

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## More generally

We construct, at any order  $n$ , a deformation  $\mathcal{X}'_n$  such that there exists  $\mathcal{E}_n \in D^b(\mathcal{X}_n \times_{\mathbb{R}_n} \mathcal{X}'_n)$ , with

$$i_n^* \mathcal{E}_n = \mathcal{E}_{n-1}.$$

## Main difficulties

- 1 Write the obstruction to deforming complexes in terms of Atiyah–Kodaira classes (Huybrechts–Thomas).
- 2 Show that the obstruction is zero.

Our approach imitates the first order case (using relative Hochschild homology).

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# The generic fiber

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## Use the generic analytic case

There exist integers  $n$  and  $m$  such that the Fourier–Mukai equivalence

$$\Phi_{(\mathcal{I}_{\Delta_{\mathcal{X}}}[1])_K}^n \circ \Phi_{\mathcal{E}_K}[m]$$

has kernel  $\mathcal{G}_K \in \mathbf{Coh}((\mathcal{X} \times_R \mathcal{X}')_K)$ , for  $\mathcal{G} \in \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}')$ .

## Remark

This shows that the autoequivalences of the derived category  $D^b(\mathcal{X}_K)$  behaves like the derived category of a complex K3 surface with trivial Picard group (Huybrechts–Macrì–S.).

# The conclusion

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## Properties of $\mathcal{G}$

- 1  $\mathcal{G}_0 := i^*\mathcal{G}$  is a sheaf in  $\mathbf{Coh}(X \times X)$ .
- 2 The natural morphism

$$(\Phi_{\mathcal{G}_0})_H : H^*(X, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$$

is such that  $(\Phi_{\mathcal{G}_0})_H = (\Phi_{\mathcal{E}})_H = j$ .

## Lemma

If  $\mathcal{G}_0 \in \mathbf{Coh}(X \times X)$ , then  $(\Phi_{\mathcal{G}_0})_H \neq j$ .