

Equivalences of K3 Surfaces and Orientation II The Projective Case

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Let X be a **K3 surface**.

Main problem

Describe the group $\text{Aut}(D^b(X))$ of exact autoequivalences of the triangulated category

$$D^b(X) := D_{\text{Coh}}^b(\mathcal{O}_X\text{-Mod}).$$

Remark (Orlov)

Such a description is available when X is an abelian surface (actually an abelian variety).

Motivations: automorphisms

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Theorem (Torelli Theorem)

Let X and Y be K3 surfaces. Suppose that there exists a Hodge isometry

$$g : H^2(X, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z})$$

which maps the class of an ample line bundle on X into the ample cone of Y . Then there exists a unique isomorphism $f : X \xrightarrow{\sim} Y$ such that $f_* = g$.

Lattice theory + Hodge structures + ample cone

Remark

The automorphism is uniquely determined.

Motivations: diffeomorphisms

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All K3 surfaces are diffeomorphic. Fix X and let
 $\Lambda := H^2(X, \mathbb{Z})$.

Theorem (Borcea, Donaldson)

Consider the natural map

$$\rho : \text{Diff}(X) \longrightarrow O(H^2(X, \mathbb{Z})).$$

Then $\text{im}(\rho) = O_+(H^2(X, \mathbb{Z}))$, where $O_+(H^2(X, \mathbb{Z}))$ is the group of orientation preserving isometries.

Remark

The kernel of ρ is not known!

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Orlov's result

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Derived Torelli Theorem (Mukai, Orlov)

Let X and Y be smooth projective K3 surfaces. Then the following are equivalent:

- 1 There exists an equivalence $\Phi : D^b(X) \cong D^b(Y)$.
- 2 There exists a Hodge isometry $\tilde{H}(X, \mathbb{Z}) \cong \tilde{H}(Y, \mathbb{Z})$.

The equivalence Φ induces an action on cohomology

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\Phi} & D^b(Y) \\ \downarrow v(-)=\text{ch}(-)\cdot\sqrt{\text{td}(X)} & & \downarrow v(-)=\text{ch}(-)\cdot\sqrt{\text{td}(Y)} \\ \tilde{H}(X, \mathbb{Z}) & \xrightarrow{\Phi_H} & \tilde{H}(Y, \mathbb{Z}) \end{array}$$

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Question

Can we understand better the action induced on cohomology by an equivalence?

Orientation: Let σ be a generator of $H^{2,0}(X)$ and ω a Kähler class. Then $\langle \operatorname{Re}(\sigma), \operatorname{Im}(\sigma), 1 - \omega^2/2, \omega \rangle$ is a positive four-space in $\tilde{H}(X, \mathbb{R})$ with a natural orientation.

Problem

The isometry $j := (\operatorname{id})_{H^0 \oplus H^4} \oplus (-\operatorname{id})_{H^2}$ is not orientation preserving. Is it induced by an autoequivalence?

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Main Theorem (Huybrechts–Macrì–S.)

Given a Hodge isometry $g : \tilde{H}(X, \mathbb{Z}) \rightarrow \tilde{H}(Y, \mathbb{Z})$, then there exists an equivalence $\Phi : D^b(X) \rightarrow D^b(Y)$ such that $g = \Phi_H$ if and only if g is orientation preserving.

Szendroi's Conjecture is true: In terms of autoequivalences, this yields a surjective morphism

$$\mathrm{Aut}(D^b(X)) \twoheadrightarrow O_+(\tilde{H}(X, \mathbb{Z})),$$

where $O_+(\tilde{H}(X, \mathbb{Z}))$ is the group of orientation preserving Hodge isometries.

The ‘easy’ implication

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The statement: If g is orientation preserving than it lifts to an equivalence.

- A result of Hosono–Lian–Oguiso–Yau (heavily relaying on Mukai/Orlov’s Derived Torelli Theorem) shows that, up to composing with the isometry j , every isometry can be lifted to an equivalence.
- Since we know that j is not orientation preserving we conclude using the following:

Remark (Huybrechts-S.)

All known equivalences (and autoequivalences) are orientation preserving.

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The non-orientation Hodge isometry

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Take any projective K3 surface X .

- Consider the non-orientation preserving Hodge isometry

$$j := (\text{id})_{H^0 \oplus H^4} \oplus (-\text{id})_{H^2}.$$

- Since one implication is already true, to prove the main theorem, it is enough to show that j is not induced by a Fourier–Mukai equivalence.
- We proceed by contradiction assuming that there exists $\mathcal{E} \in D^b(X \times X)$ such that $(\Phi_{\mathcal{E}})_H = j$.

The idea of the proof

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- For some particular K3 surfaces we know that j is not induced by any Fourier–Mukai equivalence: K3 surfaces with trivial Picard group.
- Deform the K3 surface (along a line) in the moduli space such that generically we recover the behaviour of a generic K3 surface.
- Deform the kernel of the equivalence accordingly.
- Derive a contradiction using the generic case.

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Take $R := \mathbb{C}[[t]]$ to be the ring of power series in t with field of fractions $K := \mathbb{C}((t))$.

Define $R_n := \mathbb{C}[[t]]/(t^{n+1})$. Then $\text{Spec}(R_n) \subset \text{Spec}(R_{n+1})$.

For X a smooth projective variety, a **formal deformation** is a proper formal R -scheme

$$\pi : \mathcal{X} \rightarrow \text{Spf}(R)$$

given by an inductive system of schemes $\mathcal{X}_n \rightarrow \text{Spec}(R_n)$ (smooth and proper over R_n) and such that

$$\mathcal{X}_{n+1} \times_{R_{n+1}} \text{Spec}(R_n) \cong \mathcal{X}_n.$$

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There exist sequences

$$\mathbf{Coh}_0(\mathcal{X} \times_R \mathcal{X}') \hookrightarrow \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}') \rightarrow \mathbf{Coh}((\mathcal{X} \times_R \mathcal{X}')_K)$$

$$\mathbf{Coh}_0(\mathcal{X}) \hookrightarrow \mathbf{Coh}(\mathcal{X}) \rightarrow \mathbf{Coh}((\mathcal{X})_K)$$

where $\mathbf{Coh}_0(\mathcal{X} \times_R \mathcal{X}')$ and $\mathbf{Coh}_0(\mathcal{X})$ are the abelian categories of sheaves supported on $X \times X$ and X respectively.

In this setting we also have the sequences

$$D_0^b(\mathcal{X} \times_R \mathcal{X}') \hookrightarrow D_{\mathbf{Coh}}^b(\mathcal{O}_{\mathcal{X} \times_R \mathcal{X}'}\text{-Mod}) \rightarrow D^b((\mathcal{X} \times_R \mathcal{X}')_K)$$

$$D_0^b(\mathcal{X}) \hookrightarrow D_{\mathbf{Coh}}^b(\mathcal{O}_{\mathcal{X}}\text{-Mod}) \rightarrow D^b(\mathcal{X}_K)$$

The key example: the twistor space

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Let us focus now on the case when X is a K3 surface.

Definition

A Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$ is called **very general** if there is no non-trivial integral class $0 \neq \alpha \in H^{1,1}(X, \mathbb{Z})$ orthogonal to ω , i.e. $\omega^\perp \cap H^{1,1}(X, \mathbb{Z}) = 0$.

Take the twistor space $\mathbb{X}(\omega)$ of X determined by the choice of a very general Kähler class $\omega \in \mathcal{K}_X \cap \text{Pic}(X) \otimes \mathbb{R}$:

$$\pi : \mathbb{X}(\omega) \rightarrow \mathbb{P}(\omega).$$

The key example: the twistor space

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Remark

$\mathbb{X}(\omega)$ parametrizes the complex structures 'compatible' with ω .

Choosing a local parameter t around $0 \in \mathbb{P}(\omega)$ we get a formal deformation $\mathcal{X} \rightarrow \mathrm{Spf}(R)$.

More precisely:

$$\mathcal{X}_n := \mathbb{X}(\omega) \times \mathrm{Spec}(R_n),$$

form an inductive system and give rise to a formal R -scheme

$$\pi : \mathcal{X} \rightarrow \mathrm{Spf}(R),$$

which is the **formal neighbourhood of X** in $\mathbb{X}(\omega)$.

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Proposition

If X is a K3 surface and \mathcal{X} is as before, then $D^b(\mathcal{X}_K) \cong D^b(\mathbf{Coh}(\mathcal{X}_K))$. Moreover, $D^b(\mathcal{X}_K)$ is a generic K -linear K3 category.

A K -linear category is a **K3 category** if it contains at least a spherical object and the shift by 2 is the Serre functor.

A K3 category is **generic** if, up to shift, it contains only one spherical object.

Remark

In this setting, the unique spherical object is $(\mathcal{O}_{\mathcal{X}})_K$, the image of $\mathcal{O}_{\mathcal{X}}$.

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As before, given $\mathcal{F} \in D_{\mathbf{Coh}}^b(\mathcal{O}_{\mathcal{X} \times_R \mathcal{X}'})$ -Mod, we denote by \mathcal{F}_K the natural image in the category $D^b((\mathcal{X} \times_R \mathcal{X}')_K)$.

Proposition

Let $\tilde{\mathcal{E}} \in D^b(\mathcal{X} \times_R \mathcal{X}')$ be such that $\mathcal{E} = i^* \tilde{\mathcal{E}}$. Then $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}_K$ are kernels of Fourier–Mukai equivalences.

Here we denoted by $i : X \times X \rightarrow \mathcal{X} \times_R \mathcal{X}'$ the natural inclusion.

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The equivalence $\Phi_{\mathcal{E}}$ induces a morphism

$$\Phi_{\mathcal{E}}^{\mathrm{HH}} : \mathrm{HH}^2(X) \rightarrow \mathrm{HH}^2(X).$$

Proposition

Let $v_1 \in H^1(X, \mathcal{I}_X)$ be the Kodaira–Spencer class of first order deformation given by a twistor space $\mathbb{X}(\omega)$ as above. Then

$$v'_1 := \Phi_{\mathcal{E}}^{\mathrm{HH}}(v_1) \in H^1(X, \mathcal{I}_X).$$

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Let \mathcal{X}'_1 be the first order deformation corresponding to v'_1 .

Using results of Toda one gets the following conclusion

Proposition (Toda)

For v_1 and v'_1 as before, there exists $\mathcal{E}_1 \in D^b(\mathcal{X}_1 \times_{R_1} \mathcal{X}'_1)$ such that

$$i_1^* \mathcal{E}_1 = \mathcal{E}_0 := \mathcal{E}.$$

Here $i_1 : \mathcal{X}_0 \times_{\mathbb{C}} \mathcal{X}_0 \hookrightarrow \mathcal{X}'_1 \times_{R_1} \mathcal{X}'_1$ is the natural inclusion.

Hence there is a first order deformation of \mathcal{E} .

Higher order deformations

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More generally

We construct, at any order n , an analytic deformation \mathcal{X}'_n such that there exists $\mathcal{E}_n \in \mathbf{D}^b(\mathcal{X}_n \times_{R_n} \mathcal{X}'_n)$, with

$$i_n^* \mathcal{E}_n = \mathcal{E}_{n-1}.$$

Main difficulties

- 1 Write the obstruction to deforming complexes in terms of Atiyah–Kodaira classes.
- 2 Show that the obstruction is zero.

Our approach imitates the first order case (using relative Hochschild homology).

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Use the generic analytic case

There exist integers n and m such that the Fourier–Mukai equivalence

$$T_{(\mathcal{O}_{\mathcal{X}})_K}^n \circ \Phi_{\mathcal{E}_K}[m]$$

has kernel $\mathcal{G} \in \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}')$.

Remark

This shows that the autoequivalences of the derived category $D^b(\mathcal{X}_K)$ behaves like the derived category of a complex K3 surface with trivial Picard group.

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Definition

A K -rational point of $\pi : \mathcal{X} \rightarrow \mathrm{Spf}(R)$ is an integral formal subscheme $\mathcal{Z} \subseteq \mathcal{X}$ which is flat of relative dimension zero and such that $\pi|_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathrm{Spf}(R)$ is an isomorphism.

- One constructs a locally finite stability condition σ on $D^b(\mathcal{X}_K)$ such that, if $\mathcal{F} \in D^b(\mathcal{X}_K)$ is σ -stable and semi-rigid with $\mathrm{End}_{\mathcal{X}_K}(\mathcal{F}) \cong K$, then up to shift \mathcal{F} is a K -rational point.

An object $\mathcal{F} \in D^b(\mathcal{X}_K)$ is **semi-rigid** if $\mathrm{Ext}_K^1(\mathcal{F}, \mathcal{F}) \cong K^{\oplus 2}$.

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- Using this stability condition, one proves that there are integers n and m such that the Fourier–Mukai equivalence

$$T_{(\mathcal{O}_{\mathcal{X}})_K}^n \circ \Phi_{\mathcal{E}_K}[m]$$

send K -rational points to K -rational points.

- One shows that if a Fourier–Mukai equivalence sends K -rational points to K -rational points, then its kernel \mathcal{G} is a sheaf, i.e.

$$\mathcal{G} \in \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}').$$

Key ingredients

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Sketch of the proof

In the previous proof we use that $(\mathcal{O}_X)_K$ is the unique, up to shift, spherical object in $D^b(\mathcal{X}_K)$.

In particular, we use that given a locally finite stability condition σ on $D^b(\mathcal{X}_K)$, there exists an integer n such that in the stability condition $T_{(\mathcal{O}_X)_K}^n(\sigma)$ all K -rational points are stable with the same phase.

Remark

Notice that for our proof we use stability conditions in a very mild form. We just use a specific stability condition in which we can classify all semi-rigid stable objects.

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Properties of \mathcal{G}

- 1 $\mathcal{G}_0 := i^*\mathcal{G}$ is a sheaf in $\mathbf{Coh}(X \times X)$.
- 2 The natural morphism

$$(\Phi_{\mathcal{G}_0})_H : H^*(X, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$$

is such that $(\Phi_{\mathcal{G}_0})_H = (\Phi_{\mathcal{E}})_H = j$.

For the second part, we show that \mathcal{G}_0 and \mathcal{E} induce the same action on the Grothendieck groups and have the same Mukai vector!

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The contradiction is now obtained using the following lemma:

Lemma

If $\mathcal{G} \in \mathbf{Coh}(X \times X)$, then $(\Phi_{\mathcal{G}})_H \neq j$.

Warning!

We have not proved that \mathcal{E} is a (shift of a) sheaf! We have just proved that the action in cohomology is the same as the one of a sheaf!

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There exists an explicit description of the first order deformations of the abelian category of coherent sheaves on a smooth projective variety (Toda).

The existence of equivalences between the derived categories of smooth projective K3 surfaces is detected by the existence of special isometries of the total cohomologies.

Question

Can we get the same result for derived categories of first order deformations of K3 surfaces using special isometries between 'deformations' of the Hodge and lattice structures on the total cohomologies?

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Hochschild–Kostant–Rosenberg

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The Hochschild–Kostant–Rosenberg isomorphism

$$\mathbf{L}\Delta_X^* \mathcal{O}_{\Delta_X} \xrightarrow{\sim} \bigoplus_i \Omega_X^i[i]$$

yields the isomorphisms

$$I_{\text{HKR}}^X : \mathbb{H}_*(X) \rightarrow \mathbb{H}\Omega_*(X) := \bigoplus_i \mathbb{H}\Omega_i(X)$$

and

$$I_X^{\text{HKR}} : \mathbb{H}^*(X) \rightarrow \mathbb{H}\mathbb{T}^*(X) := \bigoplus_i \mathbb{H}\mathbb{T}^i(X).$$

One then defines the graded isomorphisms

$$I_K^X = (\text{td}(X)^{1/2} \wedge (-)) \circ I_{\text{HKR}}^X \quad I_X^K = (\text{td}(X)^{-1/2} \lrcorner (-)) \circ I_X^{\text{HKR}}.$$

The categorical setting (Toda)

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Given a smooth projective variety X and for any $v \in \mathrm{HH}^2(X)$,
Toda constructed explicitly the abelian category

$$\mathbf{Coh}(X, v)$$

which is the first order deformation of $\mathbf{Coh}(X)$ in the
direction v .

One also has an isomorphism $J : \mathrm{HH}^2(X_1) \rightarrow \mathrm{HH}^2(X_1)$
such that

$$(I_{X_1}^{\mathrm{HKR}} \circ J \circ (I_{X_1}^{\mathrm{HKR}})^{-1})(\alpha, \beta, \gamma) = (\alpha, -\beta, \gamma).$$

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Theorem (Macri–S.)

Let X_1 and X_2 be smooth complex projective K3 surfaces and let $v_i \in \mathrm{HH}^2(X_i)$, with $i = 1, 2$. Then the following are equivalent:

- 1 There exists a Fourier–Mukai equivalence

$$\Phi_{\tilde{\mathcal{E}}} : D^b(X_1, v_1) \xrightarrow{\sim} D^b(X_2, v_2)$$

with $\tilde{\mathcal{E}} \in D_{\mathrm{perf}}(X_1 \times X_2, -J(v_1) \boxplus v_2)$.

- 2 There exists an orientation preserving effective Hodge isometry

$$g : \tilde{H}(X_1, v_1, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(X_2, v_2, \mathbb{Z}).$$

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For X a K3, $v \in \mathrm{HH}^2(X)$ and σ_X is a generator for $\mathrm{HH}_2(X)$, let

$$w := l_K^X(\sigma_X) + \epsilon l_K^X(\sigma_X \circ v) \in \tilde{H}(X, \mathbb{Z}) \otimes \mathbb{Z}[\epsilon]/(\epsilon^2).$$

The free $\mathbb{Z}[\epsilon]/(\epsilon^2)$ -module of finite rank $\tilde{H}(X, \mathbb{Z}) \otimes \mathbb{Z}[\epsilon]/(\epsilon^2)$ is endowed with:

- 1 The $\mathbb{Z}[\epsilon]/(\epsilon^2)$ -linear extension of the generalized Mukai pairing $\langle -, - \rangle_M$.
- 2 A weight-2 decomposition on $\tilde{H}(X, \mathbb{Z}) \otimes \mathbb{C}[\epsilon]/(\epsilon^2)$

$$\tilde{H}^{2,0}(X, v) := \mathbb{C}[\epsilon]/(\epsilon^2) \cdot w \quad \tilde{H}^{0,2}(X, v) := \overline{\tilde{H}^{2,0}(X, v)}$$

$$\text{and } \tilde{H}^{1,1}(X, v) := (\tilde{H}^{2,0}(X, v) \oplus \tilde{H}^{0,2}(X, v))^\perp.$$

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This gives the infinitesimal Mukai lattice of X with respect to v , which is denoted by $\tilde{H}(X, v, \mathbb{Z})$.

The isometry

$$g : \tilde{H}(X_1, v_1, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(X_2, v_2, \mathbb{Z})$$

which can be decomposed as $g = g_0 + \epsilon g_1$, where g_0 is an Hodge isometry of the Mukai lattices is called **effective**.

An effective isometry is **orientation preserving** if g_0 preserves the orientation of the four-space.

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Commutativity I

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The first key ingredient (of independent interest) is the following:

Theorem (Macri–S.)

Let X_1 and X_2 be smooth complex projective varieties and let $\mathcal{E} \in D^b(X_1 \times X_2)$. Then the following diagram

$$\begin{array}{ccc} \mathrm{HH}_*(X_1) & \xrightarrow{(\Phi_{\mathcal{E}})_{\mathrm{HH}}} & \mathrm{HH}_*(X_2) \\ \downarrow \scriptstyle \mathcal{I}_K^{X_1} & & \downarrow \scriptstyle \mathcal{I}_K^{X_2} \\ \tilde{H}(X_1, \mathbb{C}) & \xrightarrow{(\Phi_{\mathcal{E}})_H} & \tilde{H}(X_2, \mathbb{C}) \end{array}$$

commutes.

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Using that for K3 surfaces $H^{0,2}$ is 1-dimensional and the previous result, one get the following commutative diagram (for a Fourier–Mukai equivalence $\Phi_{\mathcal{E}}$):

$$\begin{array}{ccc} \mathrm{HH}^*(X_1) & \xrightarrow{(\Phi_{\mathcal{E}})^{\mathrm{HH}}} & \mathrm{HH}^*(X_2) \\ \downarrow (-) \circ \sigma_{X_1} & & \downarrow (-) \circ (\Phi_{\mathcal{E}})_{\mathrm{HH}}(\sigma_{X_1}) \\ \mathrm{HH}_*(X_1) & \xrightarrow{(\Phi_{\mathcal{E}})_{\mathrm{HH}}} & \mathrm{HH}_*(X_2) \\ \downarrow I_K^{X_1} & & \downarrow I_K^{X_2} \\ \tilde{H}(X_1, \mathbb{C}) & \xrightarrow{(\Phi_{\mathcal{E}})_H} & \tilde{H}(X_2, \mathbb{C}), \end{array}$$

where σ_{X_1} is a generator of $\mathrm{HH}_2(X_1)$.

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Using the previous commutativities, we could also clarify the proof of our Main Theorem.

In particular, one could simplify the hypothesis about the choice of the Kähler class giving rise to the twistor space.