

# Derived categories and cubic hypersurfaces

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UNIVERSITÀ  
DEGLI STUDI  
DI MILANO

Roma, February 2011

# Outline

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## 1 The geometric setting

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## 2 3-folds

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- Bridgeland stability conditions
- Irrationality

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# Aim

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The aim of the talk is to propose a ‘categorical’ treatment for some fundamental (often unknown) geometric properties of smooth (complex) hypersurfaces of degree 3

$$Y \subseteq \mathbb{P}^{n+1}.$$

We will study **cubic 3-fold** ( $n = 3$ ) and **cubic 4-fold** ( $n = 4$ ).

For example:

- Rationality/irrationality of those varieties;
- Torelli type theorems;
- Geometric description of the Fano varieties of lines of those cubics.

# The definition

Let  $\mathbf{A}$  be an abelian category (e.g.,  $\mathbf{mod}\text{-}R$ , right  $R$ -modules,  $R$  an ass. ring with unity, and  $\mathbf{Coh}(X)$ ).

Define  $C^b(\mathbf{A})$  to be the (abelian) **category of bounded complexes** of objects in  $\mathbf{A}$ . In particular:

- Objects:

$$M^\bullet := \{ \dots \rightarrow M^{p-1} \xrightarrow{d^{p-1}} M^p \xrightarrow{d^p} M^{p+1} \rightarrow \dots \}$$

- Morphisms: sets of arrows  $f^\bullet := \{f^i\}_{i \in \mathbb{Z}}$  making commutative the following diagram

$$\begin{array}{ccccccccc} \dots & \xrightarrow{d_{M^\bullet}^{i-2}} & M^{i-1} & \xrightarrow{d_{M^\bullet}^{i-1}} & M^i & \xrightarrow{d_{M^\bullet}^i} & M^{i+1} & \xrightarrow{d_{M^\bullet}^{i+1}} & \dots \\ & & \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} & & \\ \dots & \xrightarrow{d_{L^\bullet}^{i-2}} & L^{i-1} & \xrightarrow{d_{L^\bullet}^{i-1}} & L^i & \xrightarrow{d_{L^\bullet}^i} & L^{i+1} & \xrightarrow{d_{L^\bullet}^{i+1}} & \dots \end{array}$$

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For a complex  $M^\bullet \in C^b(\mathbf{A})$ , its  $i$ -th cohomology is

$$H^i(M^\bullet) := \frac{\ker(d^i)}{\operatorname{im}(d^{i-1})} \in \mathbf{A}.$$

A morphism of complexes is a **quasi-isomorphism** (qis) if it induces isomorphisms on cohomology.

## Definition

The **bounded derived category**  $D^b(\mathbf{A})$  of the abelian category  $\mathbf{A}$  is such that:

- Objects:  $\operatorname{Ob}(C^b(\mathbf{A})) = \operatorname{Ob}(D^b(\mathbf{A}))$ ;
- Morphisms: (very) roughly speaking, obtained 'by inverting qis in  $C^b(\mathbf{A})$ '.

# Semi-orthogonal decompositions

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Suppose we have a sequence of full triangulated subcategories  $\mathbf{T}_1, \dots, \mathbf{T}_n \subseteq D^b(X) := D^b(\mathbf{Coh}(X))$ , where  $X$  is smooth projective, such that:

- $\mathrm{Hom}_{D^b(X)}(\mathbf{T}_i, \mathbf{T}_j) = 0$ , for  $i > j$ ,
- For all  $K \in D^b(X)$ , there exists a chain of morphisms in  $D^b(X)$

$$0 = K_n \rightarrow K_{n-1} \rightarrow \dots \rightarrow K_1 \rightarrow K_0 = K$$

with  $\mathrm{cone}(K_i \rightarrow K_{i-1}) \in \mathbf{T}_i$ , for all  $i = 1, \dots, n$ .

This is a **semi-orthogonal** decomposition of  $D^b(X)$ :

$$D^b(X) = \langle \mathbf{T}_1, \dots, \mathbf{T}_n \rangle.$$



# Derived categories and Fano varieties

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## Theorem (Bondal–Orlov)

Let  $X$  be a smooth projective complex Fano variety and assume that  $Y$  is a smooth projective variety such that

$$D^b(X) \cong D^b(Y).$$

Then  $X \cong Y$ .

Thus, if  $Y$  is a cubic hypersurface as above, then  $D^b(Y)$  is a too strong invariant.

## Question

Does some ‘piece’ in a semi-orthogonal decomposition of  $D^b(Y)$  behave nicely?

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# First properties

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Let  $Y \subseteq \mathbb{P}^4$  be a smooth cubic 3-fold. The following are classical results:

## Torelli Theorem (Clemens–Griffiths, Tyurin)

Let  $Y_1$  and  $Y_2$  be cubic 3-folds. Then  $Y_1 \cong Y_2$  if and only if the intermediate Jacobians  $(J(Y_1), \Theta_1)$  and  $(J(Y_2), \Theta_2)$  are isomorphic.

## Theorem (Clemens–Griffiths)

Cubic 3-folds are not rational.

Use that  $J(Y)$  does not decompose as direct sum of Jacobians of curves.

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Let  $Y \subseteq \mathbb{P}^4$  be a smooth cubic 3-fold.

## Theorem (Kuznetsov)

The derived category  $D^b(Y)$  has a semi-orthogonal decomposition

$$D^b(Y) = \langle \mathbf{T}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle.$$

The subcategory  $\mathbf{T}_Y$  is highly non-trivial and cannot be the derived category of a smooth projective variety.

Indeed the Serre functor  $S_{\mathbf{T}_Y}$  is such that  $S_{\mathbf{T}_Y}^3 \cong [5]$ . So  $\mathbf{T}_Y$  is a so called **Calabi–Yau category of fractional dimension  $\frac{5}{3}$** .

# Categorical Torelli

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## Question (Kuznetsov)

Given two cubic 3-folds  $Y_1$  and  $Y_2$ , is it true that  $Y_1 \cong Y_2$  if and only if  $\mathbf{T}_{Y_1} \cong \mathbf{T}_{Y_2}$ ?

## Theorem (Bernardara–Macrì–Mehrotra–S.)

The answer to the above question is positive.

**Idea:** realize the Fano variety of lines of  $Y_i$  as moduli space of stable objects according to a Bridgeland stability condition on  $\mathbf{T}_{Y_i}$ .

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A **stability condition** on a triangulated category  $\mathbf{T}$  is a pair  $\sigma = (Z, \mathcal{P})$  where

- $Z : K(\mathbf{T}) \rightarrow \mathbb{C}$  is a linear map called **central charge** (similar to the slope for sheaves);
- $\mathcal{P}(\phi) \subset \mathbf{T}$  are full additive subcategories for each  $\phi \in \mathbb{R}$  (**semistable objects** of phase  $\phi$ )

satisfying some compatibilities.

The minimal objects in  $\mathcal{P}(\phi)$  are called **stable objects**.

$\text{Stab}(\mathbf{T})$  is the space parametrizing stability conditions on  $\mathbf{T}$ .



# Some questions

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Let  $Y$  be a cubic 3-fold. As a consequence of the result of Bernardara–Macrì–Mehrotra–S. above, we have that

$$\mathrm{Stab}(\mathbf{D}^b(Y)) \neq \emptyset \neq \mathrm{Stab}(\mathbf{T}_Y).$$

The category  $\mathbf{T}_Y$  behaves almost as the derived category of a smooth complex curve  $C$ . The stability conditions on  $\mathbf{D}^b(C)$  are completely classified.

## Problem

Classify completely all the stability conditions in  $\mathrm{Stab}(\mathbf{T}_Y)$ .

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# Open question and new perspectives

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## Question

Does the category  $\mathbf{T}_Y$  encode the irrationality of  $Y$ ?

A new perspective in this direction is provided by the recent work of Ballard–Favero–Katzarkov:

- 1 **Idea:** the irrationality of  $Y$  should be related to the presence of gaps in the interval of integers corresponding to the ‘generation time’ of the objects in  $D^b(Y)$ .
- 2 This is related to a conjecture of Orlov. In this case: the dimension of the category  $D^b(Y)$  is  $3 = \dim(Y)$ .

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Let  $Y \subseteq \mathbb{P}^5$  be a smooth cubic 4-fold. Denote by  $H^2$  the self-intersection of the hyperplane class of  $Y$ .

The moduli space  $\mathcal{C}$  of smooth cubic 4-folds is a quasi-projective variety of dimension 20.

**Voisin:** Smooth cubic 4-folds  $Y$  containing a plane  $P$  form a divisor  $\mathcal{C}_8$  in  $\mathcal{C}$ .

Denote by  $T := \langle H^2, P \rangle$  the primitive sublattice (with respect to the intersection form) of  $H^4(Y, \mathbb{Z})$  generated by  $H^2$  and  $P$ . Then the intersection form is of type

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

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Projecting from  $P$  onto a disjoint  $\mathbb{P}^2$ , we get  $\pi_P : Y \dashrightarrow \mathbb{P}^2$ .  
Blowing up the plane inside  $Y$  gives a quadric fibration

$$\pi'_P : \tilde{Y} \rightarrow \mathbb{P}^2$$

whose fibres degenerate along a plane sextic  $C$ .

The double cover of  $\mathbb{P}^2$  ramified along  $C$  is a **K3 surface**  $S$  (i.e. a smooth complex projective simply connected surface with trivial canonical bundle).

The quadric fibration provides an element

$$\beta \in \mathrm{Br}(S) := H^2(S, \mathcal{O}_S^*)_{\mathrm{tor}}$$

in the **Brauer group** of  $S$ .

# Hodge theory

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Back to the case of  $Y$  any cubic 4-fold (not necessarily containing a plane). We have the following remarkable results:

- **Torelli theorem (Voisin):** Let  $Y_1$  and  $Y_2$  be two cubic 4-folds and assume that there exists a Hodge isometry

$$\phi : H^4(Y_1, \mathbb{Z}) \rightarrow H^4(Y_2, \mathbb{Z})$$

sending  $H_1^2$  to  $H_2^2$ . Then there exists an isomorphism  $f : Y_2 \cong Y_1$  such that  $\phi = f^*$ .

- **Surjectivity of the period map (Looijenga, Laza):** The period map surjects onto an explicitly described subset of the period domain.

# Hassett: constructing divisors

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Hassett proposed a very nice way to construct divisors in the moduli space  $\mathcal{C}$ .

For a positive integer  $d$ , define  $\mathcal{C}_d$  to be the set of all  $Y \in \mathcal{C}$  such that

- There is a rank-2 lattice  $K_d$  with  $\det(K_d) = d$ .
- There is a primitive embedding  $K_d \hookrightarrow H^4(Y, \mathbb{Z})$ .
- There is  $h^2 \in K_d$  mapped to  $H^2$ .

**Hassett:**  $\mathcal{C}_d$  is an irreducible divisor as soon as  $d > 6$  and  $d \equiv 0, 2 \pmod{6}$ .



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# The semi-orthogonal decomposition

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## Theorem (Kuznetsov)

The derived category  $D^b(Y)$  has a semi-orthogonal decomposition

$$D^b(Y) = \langle \mathbf{T}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle.$$

## Theorem (Kuznetsov)

The triangulated category  $\mathbf{T}_Y$  is a 2-Calabi–Yau category.

Recall that a triangulated category  $\mathbf{T}$  is a **2-Calabi–Yau category** if  $\mathbf{T}$  has a Serre functor which is isomorphic to the shift by 2.

# Which 2-Calabi–Yau category?

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## Theorem (Kuznetsov)

Let  $Y$  be a cubic 4-fold containing a plane and such that the plane sextic  $C$  is smooth. Then there exists an exact equivalence

$$\mathbf{T}_Y \cong D^b(S, \beta)$$

## Remark

If  $Y$  is generic with the above properties (i.e.  $H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y) = \langle H^2, P \rangle$ ), then there is no smooth projective K3 surface  $S'$  such that

$$\mathbf{T}_Y \cong D^b(S').$$

# Twisted sheaves

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Represent  $\beta \in \text{Br}(S)$  as a Čech 2-cocycle

$$\{\beta_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$$

on an analytic open cover  $S = \bigcup_{i \in I} U_i$ .

A  $\beta$ -twisted coherent sheaf  $\mathcal{E}$  is a collection of pairs  $(\{\mathcal{E}_i\}_{i \in I}, \{\varphi_{ij}\}_{i,j \in I})$  where

- $\mathcal{E}_i$  is a coherent sheaf on the open subset  $U_i$ ;
- $\varphi_{ij} : \mathcal{E}_j|_{U_i \cap U_j} \rightarrow \mathcal{E}_i|_{U_i \cap U_j}$  is an isomorphism

such that

- 1  $\varphi_{ii} = \text{id}$  and  $\varphi_{ji} = \varphi_{ij}^{-1}$ ;
- 2  $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \beta_{ijk} \cdot \text{id}$ .

In this way we get the abelian category  $\mathbf{Coh}(S, \beta)$ .

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## Theorem (Bernardara–Macrì–Mehrotra–S.)

Given a cubic fourfold  $Y$  containing a plane  $P$  and such that  $C$  is smooth, there exist only finitely many isomorphism classes of cubic 4-folds  $Y_1 = Y, Y_2, \dots, Y_n$  containing a plane and with smooth plane sextics such that  $\mathbf{T}_Y \cong \mathbf{T}_{Y_j}$ , with  $j \in \{1, \dots, n\}$ . Moreover, if  $Y$  is generic, then  $n = 1$ .

## Questions

- 1 Can we prove a similar result for any possible cubic 4-fold (with a plane or not)?
- 2 Can the number  $n$  be arbitrarily large?

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For a cubic 4-fold  $Y$ , we denote by  $F(Y)$  the Fano variety of lines contained in  $Y$ .

## Theorem (Beauville–Donagi)

- 1  $F(Y)$  is a irreducible holomorphic symplectic manifold of dimension 4 (i.e. a simply connected, Kähler manifold such that  $H^{2,0}(F(Y))$  is generated by a non-degenerate 2-form).
- 2  $F(Y)$  is deformation equivalent to  $\text{Hilb}^2(S)$ , the Hilbert scheme of length-2 0-dimensional subschemes on a K3 surface  $S$ .



# Hassett's results

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## Theorem (Hassett)

Assume that  $d = 2(n^2 + n + 1)$  for  $n \geq 2$ . Then the generic cubic 4-fold  $Y$  contained in  $\mathcal{C}_d$  is such that  $F(Y) \cong \text{Hilb}^2(S)$  for some K3 surface  $S$ .

## Question (Hassett)

Are there other  $d$ 's such that the generic points in  $\mathcal{C}_d$  have the same property for some K3 surface?

When there is a plane, the twist cannot be avoided...

# The answer when there is a plane

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## Theorem (Macrì–S.)

If  $Y$  is a generic cubic fourfold containing a plane, then  $F(Y)$  is isomorphic to a moduli space of stable objects in the derived category  $D^b(S, \beta)$  of bounded complexes of  $\beta$ -twisted coherent sheaves on  $S$ .

## Theorem (Macrì–S.)

For all cubic fourfolds  $Y$  containing a plane, the Fano variety  $F(Y)$  is birational to a smooth projective moduli space of twisted sheaves on a K3 surface. Moreover, if  $Y$  is generic, then such a birational map is either an isomorphism or a Mukai flop.

# Outline

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and cubic hy-  
persurfaces

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Derived categories  
Bridgeland stability  
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Irrationality

4-folds

Geometry  
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theorem  
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Rationality

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- Geometry
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- Bridgeland stability conditions
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## 3 4-folds

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# Hodge theoretical results

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Rationality

**Beauville–Donagi, Morin:** They provide examples of rational cubic 4-folds (Pfaffian cubic 4-folds).

**Hassett:** Using lattice and Hodge theory, he constructs countably many divisors in  $\mathcal{C}_8$  consisting of rational cubic 4-folds.

The way he defines these families is by showing that the quadric fibration mentioned above has a section.

Notice that the presence of such a section implies that the Brauer class  $\beta$  in  $\text{Br}(S)$  is automatically trivial.

# The categorical approach

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## Conjecture (Kuznetsov)

A cubic 4-fold  $Y$  is rational if and only if there exists a K3 surface  $S'$  and an exact equivalence  $\mathbf{T}_Y \cong D^b(S')$ .

The conjecture is verified by Beauville–Donagi–Morin's and Hassett's examples.

The generic cubic 4-fold with a plane is such that there are no K3 surfaces  $S'$  with the property above.

## Problem

Use categorical methods to prove that the generic cubic 4-fold with a plane is not rational.