A tour on Bridgeland stability

Paolo Stellari

UNIVERSITÀ
DEGLI STUDI
DI MILANO

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Outline

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1. Moduli spaces and stability
   - Curves
   - Stability
   - Recasting
2. Geometry out of stability
   - Fourier–Mukai transforms
   - Varying stability

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Thus we take a different perspective: we present Bridgeland stability conditions as emerging from the quest of a general approach to the geometry of moduli spaces.

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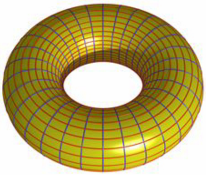

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Then X is called Fermat cubic curve.

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Roughly, they are supported at points, with multiplicity.

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It is called slope.

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- Non semi-stable sheaves can then be filtered by semi-stable ones.

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We are using that the category is abelian.

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Observe that:

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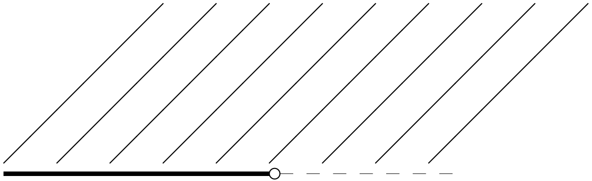
$$Z(\mathcal{E}) \in \mathbb{R}_{>0} e^{(0,1]\sqrt{-1}\pi}.$$



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Such a filtration is actually unique.

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- The morphisms are slightly complicated: they are a localization of the usual morphisms of complexes.

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- The morphisms are slightly complicated: they are a localization of the usual morphisms of complexes.
  But we do not need to understand them properly here...

Fourier–Mukai functors 1

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Definition

A functor isomorphic to one as above is called Fourier–Mukai functor. And $\mathcal{F}$ is its Fourier–Mukai kernel.

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2. Mukai: Used by Mukai to study moduli spaces on abelian varieties (i.e. higher dimensional analogues of elliptic curves).

3. Hodge: ‘Categorification’ of the usual notion of correspondence.

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But, in general,

1. FM functors do not send sheaves to sheaves.
2. FM functors do not preserve stability, in the sense we explained before.

Outline

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Suppose that X carries many different types of stability (stability conditions) and that all these stability conditions are nicely parametrized by a geometric object S.

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Then one may start with a moduli space M of μ-stable sheaves and begin changing stability inside S.

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- There might be regions (chambers) of S where M does not change even if stability is changing.
- But passing through a different region (wall) of S, all sheaves in M get destabilized and M has to be replaced by a different moduli space M' of stable sheaves.

We call this wall–crossing phenomenon.

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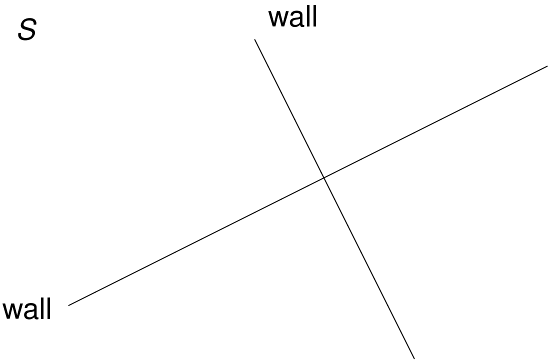

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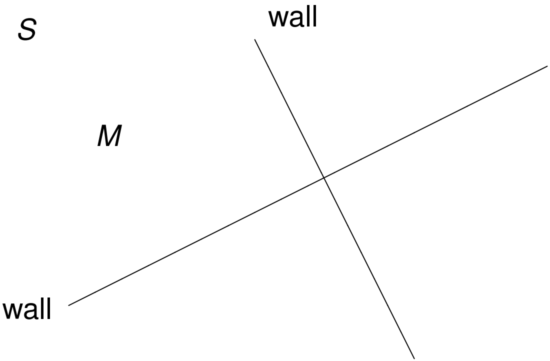

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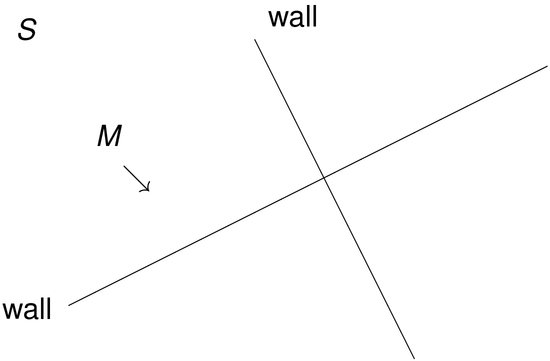

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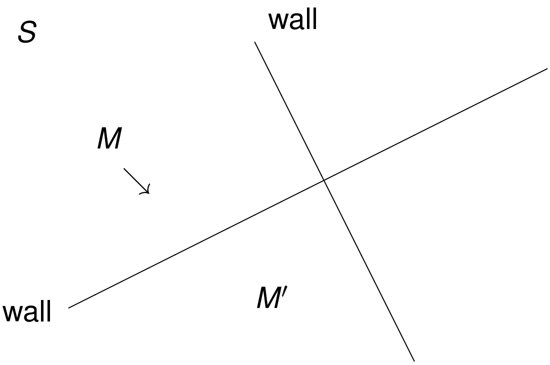

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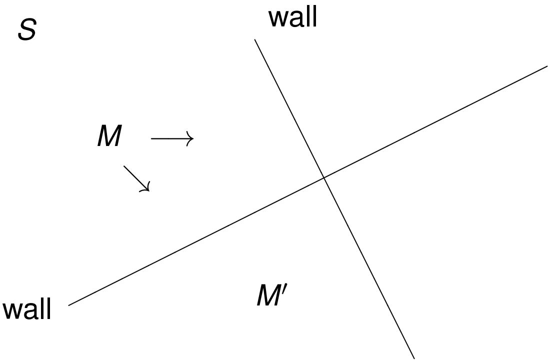

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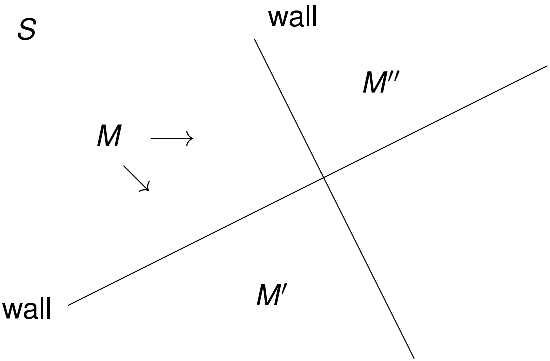

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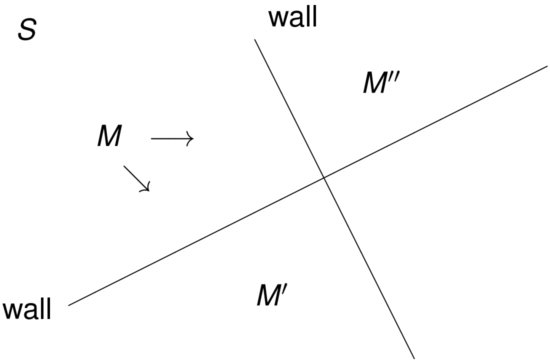

During this process, we might get M' and M'' birational to M: this means that M and M' (or M'') are isomorphic just along open subsets.

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This, in turn, is related to variations of GIT quotients: Thaddeus, Matsuki–Wentworth, ...

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By varying stability, can we get all birational models of M?

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Again, variations of the usual stability cannot be sufficient to get such a complete picture.

Outline

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Change perspective on stability!

Bridgeland definition (very roughly)

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1. **A** is an abelian category (...with some technical assumptions...);

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3. Kontsevich–Soibelman: support property (ensuring that, if we have one stability condition, then we get an entire open subset).

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Hence, in this setup, we can apply our two methods.

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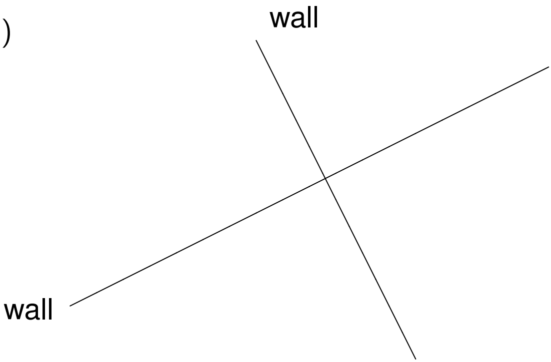

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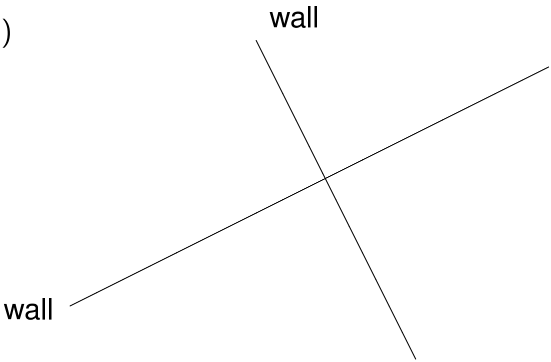

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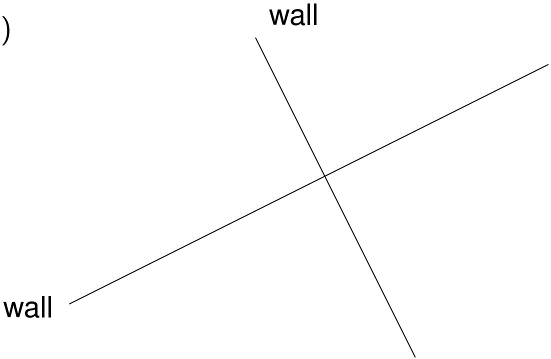

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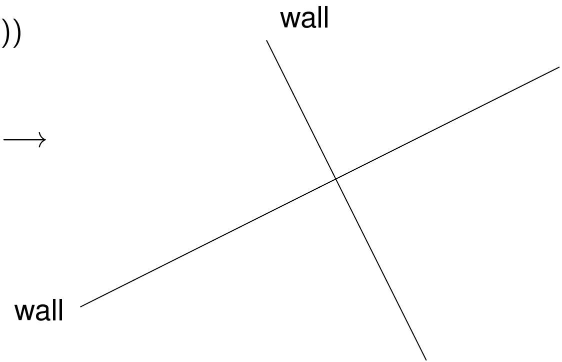

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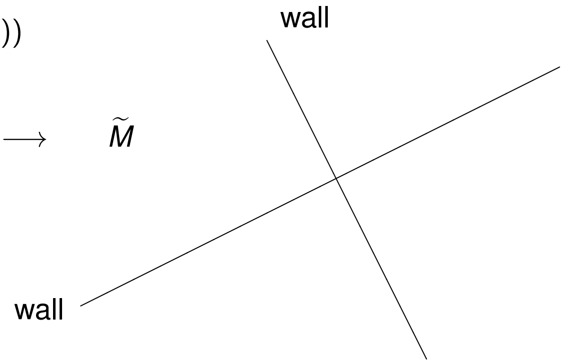

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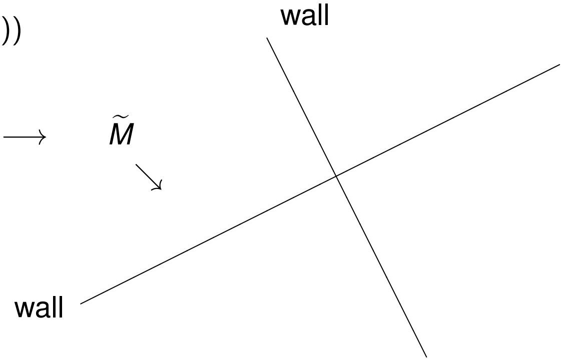

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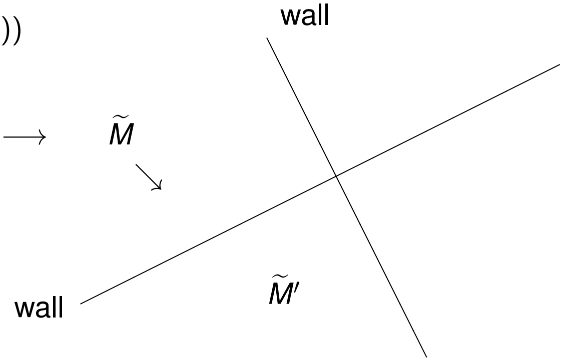

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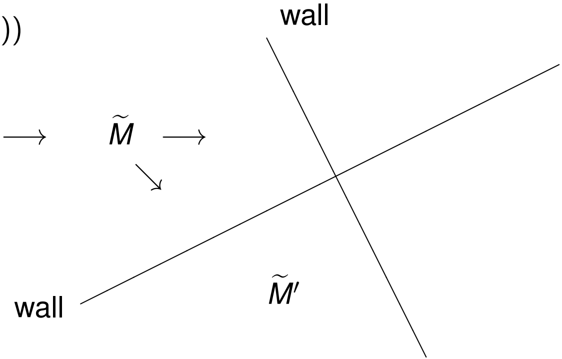

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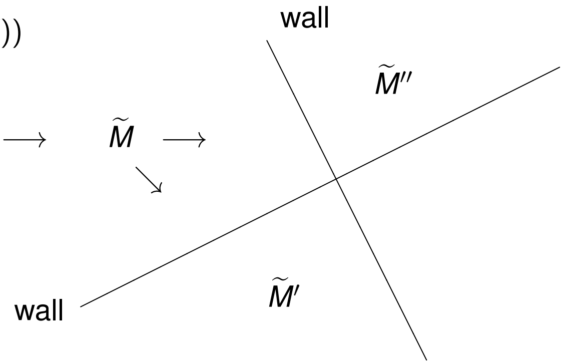

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- Arcara–Bertram–Coskun–Huizenga: Hilbert scheme of points on the projective plane (i.e. stable sheaves with very special topological invariants);

Wall crossing 4

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- Bayer–Macrì: Moduli spaces of (Gieseker) stable sheaves on K3 surfaces (e.g. zero locus in $\mathbb{P}^3$ of $x_0^4 + x_1^4 + x_2^4 + x_3^4$);

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- Nuer: Moduli spaces of (Gieseker) stable sheaves on Enriques surfaces (i.e. quotients of special K3 surfaces under the action of a free involution).

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- $\dim(X) = 1$ (Bridgeland): $\mathrm{Stab}(\mathrm{D}^b(X)) \neq \emptyset$. Namely, μ-stability is THE example.
- $\dim(X) = 2$ (Bridgeland and others): one can describe connected components of $\mathrm{Stab}(\mathrm{D}^b(X)) \neq \emptyset$.

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An example of Calabi–Yau 3-folds (not covered by our result) is the Fermat quintic, i.e. zero locus in $\mathbb{P}^4$ of the homogeneous polynomial

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For an example of the last set of CY's, one can take the product $E \times E \times E$, where E is an elliptic curve, and quotient by the diagonal action of $\mathbb{Z}/3\mathbb{Z}$.

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Macrì–Mehrotra–S.

There is a closed embedding

$$\mathrm{Stab}(\mathrm{D}^b(A))^G \hookrightarrow \mathrm{Stab}(\mathrm{D}^b(Y)),$$

where Y is a crepant resolution of A/G.

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- Generic ppav: Maciocia–Piyaratne (special case of our result).

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This is certainly a difficult problem. But it could work in several interesting cases: special Hilbert schemes on $\mathbb{P}^3$.