Derived Torelli Theorem and Orientation

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Outline

1. **Motivations**
   - The setting
   - The geometric case
   - The derived case

Outline

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Aim of the talk

Let X be a smooth projective K3 surface.

Problem

Describe the group of autoequivalences of the tringulated category

$$\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}_{\mathbf{Coh}}(\mathcal{O}_X\text{-}\mathbf{Mod}),$$

of bounded complexes of $\mathcal{O}_X$-modules with coherent cohomologies.

More generally: Consider the derived category of twisted sheaves!

Outline

Motivations: geometry

Theorem (Torelli Theorem)

Let X and Y be K3 surfaces. Suppose that there exists a Hodge isometry

$$g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$$

which maps the class of an ample line bundle on X into the ample cone of Y. Then there exists a unique isomorphism $f : X \xrightarrow{\sim} Y$ such that $f_* = g$.

Lattice theory + Hodge structures + ample cone

Motivations: geometry

All K3 surfaces are diffeomorphic. Fix X and let $\Lambda := H^2(X, \mathbb{Z})$.

Problem: diffeomorphisms

Describe the group of diffeomorphisms of X.

Theorem (Borcea, Donaldson)

Consider the natural map

$$\rho : \mathrm{Diff}(X) \longrightarrow \mathrm{O}(H^2(X, \mathbb{Z})).$$

Then $\mathrm{im}\,(\rho) = \mathrm{O}_+(H^2(X, \mathbb{Z}))$, where $\mathrm{O}_+(H^2(X, \mathbb{Z}))$ is the group of orientation preserving Hodge isometries.

Outline

The statement

Derived Torelli Theorem (Orlov, Mukai)

Let X and Y be K3 surfaces. Then the following conditions are equivalent:

1. $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$;
2. there exists a Hodge isometry $f : \widetilde{H}(X, \mathbb{Z}) \to \widetilde{H}(Y, \mathbb{Z})$;
3. there exists a Hodge isometry $g : T(X) \to T(Y)$;
4. Y is isomorphic to a smooth compact 2-dimensional fine moduli space of stable sheaves on X.

Lattice theory + Hodge structures

Outline

Brauer groups

Definition

The Brauer group of X is

$$\mathrm{Br}\,(X) := H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}}$$

in the analytic topology.

If X is a K3 surface, the Universal Coefficient Theorem, yields a nice description of the Brauer group:

$$\mathrm{Br}\,(X) \cong \mathrm{Hom}(T(X), \mathbb{Q}/\mathbb{Z})$$

where $T(X) := \mathrm{NS}(X)^{\perp}$.

Twisted sheaves

Definition

A pair (X, α) where X is a smooth projective variety and $\alpha \in \mathrm{Br}(X)$ is a twisted variety.

Represent $\alpha \in \mathrm{Br}(X)$ as a Čech 2-cocycle

$$\{\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$$

on an analytic open cover $X = \bigcup_{i \in I} U_i$.

Remark

The following definitions depend on the choice of the cocycle representing $\alpha \in \mathrm{Br}(X)$.

Twisted sheaves

An α-twisted coherent sheaf $\mathcal{E}$ is a collection of pairs

$$(\{\mathcal{E}_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$$

where

- $\mathcal{E}_i$ is a coherent sheaf on the open subset U_i;
- $\varphi_{ij} : \mathcal{E}_j|_{U_i\cap U_j} \to \mathcal{E}_i|_{U_i\cap U_j}$ is an isomorphism

such that

1. $\varphi_{ii} = \mathrm{id}$,
2. $\varphi_{ji} = \varphi_{ij}^{-1}$ and
3. $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$.

Twisted sheaves

- In this way we get the abelian category **Coh**(X, α).
- Pass to the category of bounded complexes.
- Localize: require that any quasi-isomorphism is invertible.
- We get the bounded derived category $\mathrm{D}^{\mathrm{b}}(X, \alpha)$.

Motivations

M is a 2-dimensional, irreducible, smooth and projective coarse moduli space of stable sheaves on X.

Căldăraru: the obstruction

A special element $\alpha \in \mathrm{Br}(M)$ is the obstruction to the existence of a universal family on M.

Proposition (Mukai, Căldăraru)

Let X be a K3 surface and let M be a coarse moduli space of stable sheaves on X as above. Then $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(M, \alpha^{-1})$ (via the twisted universal/quasi-universal family.

The twisted derived case

Twisted Derived Torelli Theorem (Huybrechts-S.)

Let X and X' be two projective K3 surfaces endowed with B-fields $B \in H^2(X, \mathbb{Q})$ and $B' \in H^2(X', \mathbb{Q})$.

1. If $\Phi : \mathrm{D}^{\mathrm{b}}(X, \alpha_B) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha_{B'})$ is an equivalence, then there exists a naturally defined Hodge isometry $\Phi_*^{B,B'} : \widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X', B', \mathbb{Z})$.
2. Suppose there exists a Hodge isometry $g : \widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X', B', \mathbb{Z})$ that preserves the natural orientation of the four positive directions. Then there exists an equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X, \alpha_B) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha_{B'})$ such that $\Phi_*^{B,B'} = g$.

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Lattice structure

Using the cup product, we get the Mukai pairing on $H^*(X,\mathbb{Z})$:

$$\langle\alpha,\beta\rangle := -\alpha_1\cdot\beta_3+\alpha_2\cdot\beta_2-\alpha_3\cdot\beta_1,$$

for every $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $\beta=(\beta_1,\beta_2,\beta_3)$ in $H^*(X,\mathbb{Z})$.

$H^*(X,\mathbb{Z})$ endowed with the Mukai pairing is called Mukai lattice and we write $\widetilde{H}(X,\mathbb{Z})$ for it.

The Hodge structure

- Any $\alpha \in \mathrm{Br}(X)$ is determined by some $B \in H^2(X, \mathbb{Q})$ and vice-versa.

Actually α is determined by $B \in T(X)^\vee \otimes \mathbb{Q}/\mathbb{Z}$!

- More precisely, given $\alpha \in \mathrm{Br}(X)$, there exists $B \in H^2(X, \mathbb{Q})$ such that
$$\alpha = \exp(B).$$

- Let $H^{2,0}(X) = \langle \sigma \rangle$ and let B be a B-field on X.
$$\varphi = \exp(B) \cdot \sigma = \sigma + B \wedge \sigma \in H^2(X, \mathbb{C}) \oplus H^4(X, \mathbb{C})$$
is a generalized Calabi-Yau structure.

Hitchin, Huybrechts: complete classification of generalized CY structures on K3 surfaces.

The Hodge structure

Definition

Let X be a K3 surface with a B-field $B \in H^2(X, \mathbb{Q})$. We denote by $\widetilde{H}(X, B, \mathbb{Z})$ the weight-two Hodge structure on $H^*(X, \mathbb{Z})$ with

$$\widetilde{H}^{2,0}(X, B) := \exp(B)\left(H^{2,0}(X)\right)$$

and $\widetilde{H}^{1,1}(X, B)$ its orthogonal complement with respect to the Mukai pairing.

This gives a generalization of the notion of Picard lattice and transcendental lattice.

Orientation

Let σ_X be a generator of $H^{2,0}(X)$ and let ω be a Kähler class. Then

$$\langle \mathrm{Re}(\sigma_X), \mathrm{Im}(\sigma_X), 1 - \omega^2/2, \omega \rangle$$

is a positive four-space in $\widetilde{H}(X, \mathbb{R})$.

Remark

1. It comes, by the choice of the basis, with a natural orientation.
2. It is easy to see that this orientation is independent of the choice of σ_X and ω.
3. The isometry $j := \mathrm{id}_{H^0 \oplus H^4} \oplus (-\mathrm{id})_{H^2}$ is not orientation preserving.

Outline

Fourier-Mukai functors

Definition

$F : \mathrm{D}^{\mathrm{b}}(X, \alpha) \to \mathrm{D}^{\mathrm{b}}(Y, \beta)$ is of Fourier-Mukai type if there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y, \alpha^{-1} \boxtimes \beta)$ and an isomorphism of functors

$$F \cong \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(-)),$$

where $p : X \times Y \to Y$ and $q : X \times Y \to X$ are the natural projections.

The complex $\mathcal{E}$ is called the kernel of F and a Fourier-Mukai functor with kernel $\mathcal{E}$ is denoted by $\Phi_{\mathcal{E}}$.

Fourier–Mukai equivalences

Theorem (Orlov)

Any exact functor $F : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ which

1. is fully faithful
2. admits a left adjoint

is a Fourier-Mukai functor.

Remark (Bondal-Van den Bergh)

Item (2) is automatic!

Fourier–Mukai equivalences

Theorem. (Canonanco-S.)

Let (X,α) and (Y,β) be twisted varieties. Let

$$F:\mathrm{D}^{\mathrm{b}}(X,\alpha)\to\mathrm{D}^{\mathrm{b}}(Y,\beta)$$

be an exact functor such that, for any $\mathcal{F},\mathcal{G}\in\mathbf{Coh}(X,\alpha)$,

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(Y,\beta)}(F(\mathcal{F}),F(\mathcal{G})[j])=0\ \text{ if } j<0.$$

Then there exist $\mathcal{E}\in\mathrm{D}^{\mathrm{b}}(X\times Y,\alpha^{-1}\boxtimes\beta)$ and an isomorphism of functors $F\cong\Phi_{\mathcal{E}}$. Moreover, $\mathcal{E}$ is uniquely determined up to isomorphism.

Any equivalence is of Fourier–Mukai type.

The Chern character

Take $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X, \alpha)$ and its image $\mathcal{E} = [\mathcal{E}]$ in the Grothendieck group

$$K(X, \alpha) := K(\mathrm{D}^{\mathrm{b}}(X, \alpha)).$$

One defines a twisted Chern character

$$\mathrm{ch}^{B}(\mathcal{E}) \in H^{*}(X, \mathbb{Z})$$

depending on the choice of a B-field representing the element $\alpha \in \mathrm{Br}(X)$.

Remark

The notion of $\mathrm{c}_1(\mathcal{E})$ is well-defined.

A commutative diagram

Using the Chern character one gets the commutative diagram:

$$
\begin{array}{ccc}
\mathrm{D}^{\mathrm{b}}(X,\alpha) & \xrightarrow[\Phi]{} & \mathrm{D}^{\mathrm{b}}(Y,\beta) \\
\Big\downarrow{\scriptstyle [-]} & & \Big\downarrow{\scriptstyle [-]} \\
K(X,\alpha) & \longrightarrow & K(Y,\beta) \\
\Big\downarrow{\scriptstyle \mathrm{ch}^{B}(-)\cdot\sqrt{\mathrm{td}(X)}} & & \Big\downarrow{\scriptstyle \mathrm{ch}^{B'}(-)\cdot\sqrt{\mathrm{td}(Y)}} \\
\widetilde{H}(X,B,\mathbb{Z}) & \xrightarrow{\Phi_*^{B,B'}} & \widetilde{H}(Y,B',\mathbb{Z}).
\end{array}
$$

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Orienatation

The orientation preserving requirement is missing in item (i) of the Twisted Derived Torelli Theorem.

Proposition (Huybrechts-S.)

Any known twisted or untwisted equivalence is orientation preserving.

Conjecture (Szendröi, Huybrechts-S.)

Let X and X' be two algebraic K3 surfaces with B-fields B and B'. If $\Phi : \mathrm{D}^{\mathrm{b}}(X, \alpha_B) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha_{B'})$ is an equivalence then $\Phi_*^{B,B'} : \widetilde{H}(X, B, \mathbb{Z}) \to \widetilde{H}(X', B', \mathbb{Z})$ preserves the natural orientation of the four positive directions.

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Evidence and problem

Proposition (Huybrechts-S.)

Let X and X' be K3 surfaces with large Picard number and let $B \in H^2(X, \mathbb{Q})$ and $B' \in H^2(X', \mathbb{Q})$. Then the following are equivalent:

1. There exists an exact equivalence
$$\Phi : \mathrm{D^b}(X, \alpha_B) \xrightarrow{\sim} \mathrm{D^b}(X', \alpha_{B'}).$$
2. There exists an orientation preserving Hodge isometry
$$g : \widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X', B', \mathbb{Z}).$$

Problem: Is this always true?

In the untwisted case, this is always true.

Outline

The ‘final’ Twisted Derived Torelli Theorem

Theorem (Huybrechts-Macrì-S.)

Let X be a K3 surface and let $B \in H^2(X, \mathbb{Q})$ be such that (X, α_B) is generic. Then there exists a short exact sequence

$$1 \to \mathbb{Z}[2] \to \mathrm{Aut}\left(\mathrm{D}^{\mathrm{b}}(X, \alpha_B)\right) \xrightarrow{\varphi} \mathrm{O}_+ \to 1,$$

where O_+ is the group of the Hodge isometries of $\widetilde{H}(X, B, \mathbb{Z})$ preserving the orientation.

A twisted K3 surface (X, α) is generic meaning that it is generic in the moduli space of twisted K3 surfaces.

Remarks

Properties

A generic K3 surface (X, α) is, in the rest of this talk, characterized by the fact that:

1. The underlying K3 surface X is generic and projective (i.e. Picard number 1).
2. The twist α is generic (i.e. restrictions on the order of α).

Warning

We have no idea about how to use the generic case to prove the conjecture for any twisted K3 surface (X, α).

Remarks

The equivalent formulation would be:

Take an equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X, \alpha) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y, \beta)$, of generic twisted K3 surfaces (X, α) and (Y, β). Then the natural map Φ_* induced on cohomology is an orientation preserving Hodge isometry.

Remark

We proved Bridgeland's Conjecture for generic twisted K3 surfaces.

Outline

The genericity hypothesis

Definition

An object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X, \alpha)$ is spherical if

$$\mathrm{Hom}\,(\mathcal{E}, \mathcal{E}[i]) \cong \begin{cases} \mathbb{C} & \text{if } i \in \{0, 2\} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition

There exists a dense subset of twisted K3 surfaces (X, α) such that $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ does not contain spherical objects.

In the untwisted case, $\mathrm{D}^{\mathrm{b}}(X)$ always contains spherical objects.

Stability conditions: Bridgeland

For simplicity, we restrict ourselves to the case of stability conditions on derived categories $\mathrm{D}^{\mathrm{b}}(X, \alpha)$! Any triangulated category would fit.

A stability condition on $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ is a pair $\sigma = (Z, \mathcal{P})$ where

- $Z : \mathcal{N}(X, \alpha) \otimes \mathbb{C} \to \mathbb{C}$ is a linear map (the central charge; here $\mathcal{N}(X, \alpha)$ is the sublattice of $\widetilde{H}(X, \mathbb{Z})$ orthogonal to the generalized complex structure)
- $\mathcal{P}(\phi) \subset \mathrm{D}^{\mathrm{b}}(X, \alpha)$ are full additive subcategories for each $\phi \in \mathbb{R}$

satisfying the following conditions:

Stability conditions: Bridgeland

(a) If $0 \neq \mathcal{E} \in \mathcal{P}(\phi)$, then $Z(\mathcal{E}) = m(\mathcal{E}) \exp(i\pi\phi)$ for some $m(\mathcal{E}) \in \mathbb{R}_{>0}$.

(b) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all ϕ.

(c) $\mathrm{Hom}\,(\mathcal{E}_1, \mathcal{E}_2) = 0$ for all $\mathcal{E}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \phi_2$.

(d) Any $0 \neq \mathcal{E} \in \mathrm{D^b}(X, \alpha)$ admits a Harder–Narasimhan filtration given by a collection of distinguished triangles

$$\mathcal{E}_{i-1} \to \mathcal{E}_i \to \mathcal{A}_i$$

with $\mathcal{E}_0 = 0$ and $\mathcal{E}_n = \mathcal{E}$ such that $\mathcal{A}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \ldots > \phi_n$.

Stability conditions: Bridgeland

To exhibit a stability condition on $\mathrm{D}^{\mathrm{b}}(X, \alpha)$, it is enough to give

- a bounded t-structure on $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ with heart $\mathbf{A}$;
- a group homomorphism $Z : K(\mathbf{A}) \to \mathbb{C}$ such that $Z(\mathcal{E}) \in \mathbb{H}$, for all $0 \neq \mathcal{E} \in \mathbf{A}$, and with the Harder–Narasimhan property ($\mathbb{H} := \{z \in \mathbb{C} : z = |z| \exp(i\pi\phi), 0 < \phi \leq 1\}$).

All stability conditions are assumed to be locally-finite. Hence every object in $\mathcal{P}(\phi)$ has a finite Jordan–Hölder filtration. $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X, \alpha))$ is the set of locally finite stability conditions.

The group $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X, \alpha))$ acts on $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X, \alpha))$.

Stability conditions: Bridgeland

Bridgeland's main results are the following:

1. For each connected component $\Sigma \subseteq \mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ there is a linear subspace $V(\Sigma) \subseteq (\mathcal{N}(X,\alpha) \otimes \mathbb{C})^{\vee}$ with a well-defined linear topology such that the natural map

$$\mathcal{Z} : \Sigma \longrightarrow V(\Sigma), \qquad (Z,\mathcal{P}) \longmapsto Z$$

is a local homeomorphism.

2. The manifold $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ is finite dimensional.

Stability conditions: examples

Denote by

$$P(X,\alpha) \subset \mathcal{N}(X,\alpha) \otimes \mathbb{C}$$

the open subset of vectors φ such that real part and imaginary part of φ generate a positive plane in $\mathcal{N}(X,\alpha) \otimes \mathbb{R}$.

As in the untwisted case, $P(X,\alpha)$ has two connected components.

We shall denote by $P^{+}(X,\alpha)$ the one that contains

$$\varphi = \exp(B + i\omega)$$

with B rational B-field and ω rational Kähler class.

Stability conditions: examples

Fix $\varphi \in P^+(X,\alpha)$. We define:

1. The function $Z_\varphi(\mathcal{E}) := \langle v^\alpha(\mathcal{E}), \varphi \rangle$.
2. The full additive subcategory $\mathcal{T}(\varphi) \subset \mathbf{Coh}(X,\alpha)$ of the non-trivial objects $\mathcal{E} \in \mathrm{D^b}(X,\alpha)$ such that every non-trivial torsion free quotient $\mathcal{E} \twoheadrightarrow \mathcal{E}'$ satisfies $\mathrm{im}(Z_\varphi(\mathcal{E}')) > 0$.
3. The full additive subcategory $\mathcal{F}(\varphi) \subset \mathbf{Coh}(X,\alpha)$ of the non-trivial objects $\mathcal{E} \in \mathrm{D^b}(X,\alpha)$ which are torsion free and are such that every non-zero subsheaf $\mathcal{E}'$ satisfies $\mathrm{im}(Z_\varphi(\mathcal{E}')) \leq 0$.

Stability conditions: examples

Define the abelian category

$$\mathcal{A}(\varphi) := \left\{ \mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X, \alpha) : \begin{array}{l} \bullet\ \mathcal{H}^{i}(\mathcal{E}) = 0 \text{ for } i \notin \{-1, 0\}, \\ \bullet\ \mathcal{H}^{-1}(\mathcal{E}) \in \mathcal{F}(\varphi), \\ \bullet\ \mathcal{H}^{0}(\mathcal{E}) \in \mathcal{T}(\varphi) \end{array} \right\}.$$

For good choices of B and ω, the pair $(Z_{\varphi}, \mathcal{A}(\varphi))$ defines a stability condition.

Stability conditions: the connected component

Following Bridgeland one defines a connected component

$$\mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X,\alpha)) \subseteq \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$$

containing the stability conditions previously described.

Theorem (Huybrechts-Macrì-S.)

If (X,α) is a generic twisted K3 surface, then $\mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ is the unique connected component of maximal dimension in $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$. Such a component is also simply-connected.

Concluding the proof

1. There exists a continuous map

$$\pi : \mathrm{Stab}^{\dagger}(\mathrm{D^b}(X,\alpha)) \to P^{+}(X,\alpha)$$

which is a covering map.

2. Since $\mathrm{Stab}^{\dagger}(\mathrm{D^b}(X,\alpha))$ is the unique connected component of maximal dimension, the action of $\mathrm{Aut}\,(\mathrm{D^b}(X,\alpha))$ on $\mathrm{Stab}\,(\mathrm{D^b}(X,\alpha))$ preserves it.

3. This action corresponds to the action of deck transformations, and $P^{+}(X,\alpha)$ corresponds to the positive directions. One gets that any $\Phi \in \mathrm{Aut}\,(\mathrm{D^b}(X,\alpha))$ is orientation preserving.