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TWISTED DERIVED CATEGORIES AND K3 SURFACES

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A GLIMPSE AT DERIVED CATEGORIES

X smooth projective. Consider the abelian category $\mathbf{Coh}(X)$.

$$\Downarrow$$

Pass to the category of bounded complexes and require that any quasi-isomorphism is invertible.

$$\Downarrow$$

We get the bounded derived category $\mathrm{D}^{\mathrm{b}}(X)$.

Not all functors with geometric meaning are exact in $\mathbf{Coh}(X)$.

$$\Downarrow$$

Procedure to produce from them exact functors in $\mathrm{D}^{\mathrm{b}}(X)$ (not abelian but triangulated). We get *left and right derived functors*.

All “geometric functors” can be derived.

What are derived categories?

Let X be a smooth variety.

As a first step we define the abelian category $C(X) := C(\mathbf{Coh}(X))$ whose objects are complexes of sheaves in $\mathbf{Coh}(X)$

$$\ldots \xrightarrow{d^{i-2}} \mathcal{E}^{i-1} \xrightarrow{d^{i-1}} \mathcal{E}^{i} \xrightarrow{d^{i}} \mathcal{E}^{i+1} \xrightarrow{d^{i+1}} \ldots$$

and whose morphisms are morphisms of complexes, i.e. sets of vertical arrows $\{f^i\}_{i\in\mathbb{Z}}$ as in the following diagram

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{d_1} & \mathcal{E}^{i-1} & \xrightarrow{d_1} & \mathcal{E}^{i} & \xrightarrow{d_1} & \mathcal{E}^{i+1} & \xrightarrow{d_1} & \ldots \\ & & \downarrow f^{i-1} & & \downarrow f^{i} & & \downarrow f^{i+1} & & \\ \ldots & \xrightarrow{d_2} & \mathcal{F}^{i-1} & \xrightarrow{d_2} & \mathcal{F}^{i} & \xrightarrow{d_2} & \mathcal{F}^{i+1} & \xrightarrow{d_2} & \ldots \end{array}$$

and such that, for any $i \in \mathbb{Z}$,

$$f^i \circ d_1 = d_2 \circ f^{i-1}.$$

Given a complex $\mathcal{E}^\bullet$, its *i-th cohomology sheaf* is the sheaf

$$\mathcal{H}^i(\mathcal{E}^\bullet) := \frac{\ker(d^i)}{\operatorname{Im}(d^{i-1})}.$$

A morphism of complexes $f^\bullet : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ induces, for any $i \in \mathbb{Z}$, a morphism of sheaves

$$\mathcal{H}^i(f^\bullet) : \mathcal{H}^i(\mathcal{E}^\bullet) \to \mathcal{H}^i(\mathcal{F}^\bullet).$$

Among the morphisms in $C(X)$ we have a special class Qis whose elements are the *quasi-isomorphisms*, i.e. morphisms of complexes $f^\bullet$ such that, for any $i \in \mathbb{Z}$, $\mathcal{H}^i(f^\bullet)$ is an isomorphism.

We say that two morphisms of complexes

$$f^\bullet, g^\bullet : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$$

are *homotopically equivalent* ($f^\bullet \sim g^\bullet$) if there exists a collection of morphisms

$$\delta^i : \mathcal{E}^i \to \mathcal{F}^{i-1},$$

for $i \in \mathbb{Z}$, such that

$$f^i - g^i = \delta^{i+1} \circ d^i_{\mathcal{E}} + d^{i-1}_{\mathcal{F}} \circ \delta^i.$$

Observe that it makes sense to consider sums and differences of morphisms of complexes because $C(X)$ is abelian.

The *homotopy category* Kom(X) is the category whose objects are complexes and whose morphisms are such that

$$\mathrm{Mor}_{\mathrm{Kom}(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \mathrm{Mor}_{C(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)/\sim .$$

In Kom(X) the class Qis is a localizing class. Hence we can localize Kom(X) with respect to Qis getting the *derived category of coherent sheaves* D(X). Such a localization does not change the objects in Kom(X) but it makes the quasi-isomorphisms invertible. In particular, there exists a functor

$$Q_X : C(X) \longrightarrow \mathrm{D}(X)$$

such that:

(1) Q_X(quasi-isom) = isom;

(2) for any category T and any functor $P : C(X) \to T$ such that P(quasi-isom) = isom, there exists a functor $R : \mathrm{D}(X) \to T$ so that $T = R \circ Q_X$.

The subcategory $\mathsf{D}^{\mathsf{b}}(X)$ of $\mathsf{D}(X)$ whose objects are complexes with finitely many sheaves different from 0 is the *bounded derived category of coherent sheaves on X*.

In the same way, we write $\mathsf{D}^{+}(X)$ and $\mathsf{D}^{-}(X)$ for the subcategories of $\mathsf{D}(X)$ whose complexes are bounded (i.e. they are definitely zero) at the right, respectively the left, hand side.

Remark. These categories are not abelian but they are *triangulated*.

$$\Downarrow$$

Short ex. seq's $\Leftrightarrow$ Distinguished triangles

A functor $F : \mathsf{D}^{\mathsf{b}}(X) \to \mathsf{D}^{\mathsf{b}}(Y)$ is *exact* if it sends distinguished triangles to distinguished triangles.

Remark. For the rest of this talk, all the equivalences will be supposed to be exact in this sense.

What about functors?

Let X and Y be smooth projective varieties.

Remark. A functor $F : \mathbf{Coh}(X) \to \mathbf{Coh}(Y)$ naturally extends to a functor

$$\mathrm{Kom}(F) : \mathrm{Kom}(X) \to \mathrm{Kom}(Y)$$

but, in general, not to a functor

$$\mathrm{D}(F) : \mathrm{D}(X) \to \mathrm{D}(Y).$$

Suppose that F is left-exact (i.e. it preserves the short exact sequences in $\mathbf{Coh}(X)$ on the left hand side).

The *right derived functor* of F, if it exists, is the functor

$$\mathbf{R}F : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$$

uniquely (up to a unique isomorphism) determined by the properties:

(i) $\mathbf{R}F$ is exact (in the sense of triangulated categories);

(ii) there exists a morphism of functors $Q_Y \circ \mathsf{Kom}(F) \to \mathbf{R}F \circ Q_X$;

(iii) suppose that $G : \mathsf{D}^{\mathsf{b}}(X) \to \mathsf{D}^{\mathsf{b}}(Y)$ is an exact functor. Then any morphism of functors $Q_Y \circ \mathsf{Kom}(F) \to G \circ Q_X$ factorizes over a morphism $\mathbf{R}F \to G$.

One can give a similar definition for a right-exact functor $G : \mathbf{Coh}(X) \to \mathbf{Coh}(Y)$ getting the *left derived functor* $\mathbf{L}G : \mathsf{D}^{\mathsf{b}}(X) \to \mathsf{D}^{\mathsf{b}}(Y)$.

TWISTED DERIVED CATEGORIES

Let X be a smooth projective variety. The group

$$\mathrm{Br}(X) := H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}}$$

is the *Brauer group* of X.

Example. A *K3 surface* is a simply connected complex smooth projective surface with trivial canonical bundle. In this case

$$\mathrm{Br}(X) \cong \mathrm{Hom}(T(X), \mathbb{Q}/\mathbb{Z}).$$

Due to this description, for any $\alpha \in \mathrm{Br}(X)$ we put

$$T(X, \alpha) := \ker(\alpha) \subseteq T(X).$$

Definition. A pair (X, α) where X is a smooth projective variety and $\alpha \in \mathrm{Br}(X)$ is a *twisted variety*.

Any $\alpha \in \mathrm{Br}(X)$ is determined by some $B \in H^2(X, \mathbb{Q})$ and vice-versa. In this case we write $\alpha_B := \alpha$.

Any $B \in H^2(X, \mathbb{Q})$ is called *B-field*.

Let (X, α) be a twisted variety. $\alpha \in \mathrm{Br}(X)$ can be represented by a Čech 2-cocycle

$$\{\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$$

on an analytic open cover $X = \bigcup_{i \in I} U_i$.

Definition. An *α-twisted coherent sheaf* $\mathcal{E}$ is a collection of pairs $(\{\mathcal{E}_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$ where $\mathcal{E}_i$ is a coherent sheaf on the open subset U_i and

$$\varphi_{ij} : \mathcal{E}_j|_{U_i\cap U_j} \to \mathcal{E}_i|_{U_i\cap U_j}$$

is an isomorphism such that

(i) $\varphi_{ii} = \mathrm{id}$,

(ii) $\varphi_{ji} = \varphi_{ij}^{-1}$ and

(iii) $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$.

$$\Downarrow$$

We get the abelian category $\mathbf{Coh}(X, \alpha)$.

$$\Downarrow$$

Via the standard procedure we get the triangulated category

$$\mathrm{D}^{\mathrm{b}}(X, \alpha) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X, \alpha)).$$

FOURIER-MUKAI FUNCTORS

Orlov proved that any exact functor between the bounded derived categories of coherent sheaves of two smooth projective varieties which

(i) is fully faithful

(ii) admits a left adjoint

is a *Fourier-Mukai functor*.

$F : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ is of *Fourier-Mukai type* if there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$ and an isomorphism of functors

$$F \cong \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(-)),$$

where $p : X \times Y \to Y$ and $q : X \times Y \to X$ are the natural projections.

The complex $\mathcal{E}$ is called the *kernel* of F and a Fourier-Mukai functor with kernel $\mathcal{E}$ is denoted by $\Phi_{\mathcal{E}}$.

In recent years some attention was paid to the twisted case but a question remained open:

Are all equivalences between
the twisted derived categories
of smooth projective varieties
of Fourier-Mukai type?

This is known in some geometric cases involving K3 surfaces:

(1) moduli spaces of stable sheaves on K3 surfaces (Căldăraru, see later);

(2) K3 surfaces with large Picard number (H.-S.).

A complete solution to the previous question comes as an easy corollary of the following result:

Theorem. (C.-S.) Let (X, α) and (Y, β) be twisted varieties. Let $F : \mathrm{D}^{\mathrm{b}}(X, \alpha) \to \mathrm{D}^{\mathrm{b}}(Y, \beta)$ be an exact functor such that, for any $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}(X, \alpha)$,

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(Y,\beta)}(F(\mathcal{F}), F(\mathcal{G})[j]) = 0 \ \text{ if } j < 0.$$

Then there exist $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y, \alpha^{-1} \boxtimes \beta)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{E}}$. Moreover, $\mathcal{E}$ is uniquely determined up to isomorphism.

The previous result covers some interesting cases:

(1) full functors;

(2) (as a special case) equivalences.

This improves Orlov's result.

Why derived categories?

As an application we get the following twisted version of a result of Gabriel:

Proposition. **Let (X, α) and (Y, β) be twisted varieties. Then there exists an isomorphism $f : X \cong Y$ such that $f^*(\beta) = \alpha$ if and only if there exists an exact equivalence $\mathbf{Coh}(X, \alpha) \cong \mathbf{Coh}(Y, \beta)$.**

The abelian category of twisted coherent sheaves is a too strong invariant!

Needs:

1. Preserve deep geometric relationships (moduli spaces) (Mukai,. . .).

2. A good birational invariant $\Rightarrow$ Some kind of "Derived MMP" (Kawamata, Bridgeland, Chen,. . .).

3. Relevant for physics $\Rightarrow$ Mirror Symmetry (Kontsevich,. . .).

GEOMETRY OF K3 SURFACES

The main geometric result about K3 surfaces is the following classical theorem:

Theorem. (Torelli Theorem) **Let X and Y be K3 surfaces. Suppose that there exists a Hodge isometry**

$$g : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$

which maps the class of an ample line bundle on X into the ample cone of Y. Then there exists a unique isomorphism

$$f : X \cong Y$$

such that $f_* = g$.

$$\Downarrow$$

Lattice theory + Hodge structures
+ ample cone

DERIVED TORELLI THEOREM

Existence of equivalences: Orlov + Mukai

$$\Downarrow$$

Theorem. (Derived Torelli Theorem) **Let X and Y be K3 surfaces. Then the following conditions are equivalent:**

(i) $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$;

(ii) **there exists a Hodge isometry**

$$f : \widetilde{H}(X, \mathbb{Z}) \to \widetilde{H}(Y, \mathbb{Z});$$

(iii) **there exists a Hodge isometry**

$$g : T(X) \to T(Y);$$

(iv) Y **is isomorphic to a smooth compact 2-dimensional fine moduli space of stable sheaves on** X**.**

$$\Downarrow$$

Lattice theory + Hodge structures

TWISTED DERIVED TORELLI THEOREM

Theorem. (H.-S.) Let X and X' be two projective K3 surfaces endowed with B-fields $B \in H^2(X,\mathbb{Q})$ and $B' \in H^2(X',\mathbb{Q})$.

(i) If $\Phi : \mathrm{D}^{\mathrm{b}}(X,\alpha_B) \cong \mathrm{D}^{\mathrm{b}}(X',\alpha_{B'})$ is an equivalence, then there exists a naturally defined Hodge isometry

$$\Phi_*^{B,B'} : \widetilde{H}(X,B,\mathbb{Z}) \cong \widetilde{H}(X',B',\mathbb{Z}).$$

(ii) Suppose there exists a Hodge isometry

$$g : \widetilde{H}(X,B,\mathbb{Z}) \cong \widetilde{H}(X',B',\mathbb{Z})$$

that preserves the natural orientation of the four positive directions. Then there exists an equivalence

$$\Phi : \mathrm{D}^{\mathrm{b}}(X,\alpha_B) \cong \mathrm{D}^{\mathrm{b}}(X',\alpha_{B'})$$

such that $\Phi_*^{B,B'} = g$.

The lattice structure

Using the cup product, we get the *Mukai pairing* on $H^*(X, \mathbb{Z})$:

$$\langle \alpha, \beta \rangle := -\alpha_1 \cdot \beta_3 + \alpha_2 \cdot \beta_2 - \alpha_3 \cdot \beta_1,$$

for every $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ in $H^*(X, \mathbb{Z})$.

$H^*(X, \mathbb{Z})$ endowed with the Mukai pairing is called *Mukai lattice* and we write $\widetilde{H}(X, \mathbb{Z})$ for it.

Hodge structure

Let $H^{2,0}(X) = \langle \sigma \rangle$ and let B be a B-field on X.

$\varphi = \exp(B) \cdot \sigma = \sigma + B \wedge \sigma \in H^2(X, \mathbb{C}) \oplus H^4(X, \mathbb{C})$

is a *generalized Calabi-Yau structure* (Hitchin and Huybrechts).

Definition. Let X be a K3 surface with a B-field $B \in H^2(X, \mathbb{Q})$. We denote by $\widetilde{H}(X, B, \mathbb{Z})$ the weight-two Hodge structure on $H^*(X, \mathbb{Z})$ with

$$\widetilde{H}^{2,0}(X, B) := \exp(B)\left(H^{2,0}(X)\right)$$

and $\widetilde{H}^{1,1}(X, B)$ its orthogonal complement with respect to the Mukai pairing.

Definition. The *generalized (or twisted) Picard group* is

$$\mathrm{Pic}(X, \varphi) := \{\beta \in H^*(X, \mathbb{Z}) : \langle \beta, \varphi \rangle = 0\}$$

and the *generalized (or twisted) transcendental lattice*

$$T(X, \varphi) := \mathrm{Pic}(X, \varphi)^{\perp} \subset H^*(X, \mathbb{Z}),$$

where the orthogonal complement is taken with respect to the Mukai pairing.

Orientation

If X is a K3 surface, σ_X is a generator of $H^{2,0}(X)$ and ω is a Kähler class, then

$$\langle \mathrm{Re}(\sigma_X), \mathrm{Im}(\sigma_X), 1-\omega^2/2, \omega\rangle$$

is a positive four-space in $\widetilde{H}(X,\mathbb{R})$.

$$\Downarrow$$

It comes, by the choice of the basis, with a natural orientation.

Remark. It is easy to see that this orientation is independent of the choice of σ_X and ω.

Example. The isometry

$$j \in \mathrm{O}(\widetilde{H}(X,\mathbb{Z}), \sigma_X)$$

defined by

(i) $j|_{H^0(X,\mathbb{Z})\oplus H^4(X,\mathbb{Z})} = id_{H^0(X,\mathbb{Z})\oplus H^4(X,\mathbb{Z})}$;

(ii) $j|_{H^2(X,\mathbb{Z})} = -id_{H^2(X,\mathbb{Z})}$.

is not orientation preserving!

Main ingredients in the proof

Proposition. **Consider** $\alpha_B \in \mathrm{Br}(X)$. **Then there exists a map**

$$\mathrm{ch}^B : K(X, \alpha_B) \longrightarrow H^*(X, \mathbb{Q})$$

such that:

(i) ch^B **is additive, i.e.**

$$\mathrm{ch}^B(E_1 \oplus E_2) = \mathrm{ch}^B(E_1) + \mathrm{ch}^B(E_2).$$

(ii) **If** $B = \mathrm{c}_1(L) \in H^2(X, \mathbb{Z})$, **then** $\mathrm{ch}^B(E) = \exp(\mathrm{c}_1(L)) \cdot \mathrm{ch}(E)$. **(Note that with this assumption** α **is trivial and an** α**-twisted sheaf is just an ordinary sheaf.)**

(iii) **For two choices** $\alpha_1 := \alpha_{B_1}$, $\alpha_2 := \alpha_{B_2}$ **and** $E_i \in K(X, \alpha_i)$ **one has**

$$\mathrm{ch}^{B_1}(E_1) \cdot \mathrm{ch}^{B_2}(E_2) = \mathrm{ch}^{B_1+B_2}(E_1 \otimes E_2).$$

(iv) **For any** $E \in K(X, \alpha)$ **one has** $\mathrm{ch}^B(E) \in \exp(B)\left(\bigoplus H^{p,p}(X)\right)$.

Twisted Mukai vector to induce isometries on cohomology.

$$\begin{array}{ccc}
\mathsf{D}^{\mathsf{b}}(X,\alpha_B) & \longrightarrow & \mathsf{D}^{\mathsf{b}}(X',\alpha_{B'}) \\
\downarrow{\scriptstyle [\ ]} & & \downarrow{\scriptstyle [\ ]} \\
\mathsf{K}(X,\alpha) & \longrightarrow & K(X',\alpha_{B'}) \\
\downarrow{\scriptstyle v^B} & & \downarrow{\scriptstyle v^{B'}} \\
\widetilde{H}(X,B,\mathbb{Z}) & \longrightarrow & \widetilde{H}(X',B',\mathbb{Z})
\end{array}$$

Moduli spaces of twisted sheaves (Yoshioka, Lieblich,...) + the study of some special equivalences.

AN EXAMPLE

M is a 2-dimensional, irreducible, smooth and projective moduli space of stable sheaves on a K3 surface X (M is a K3 surface).

Mukai proved that there exists an embedding

$$\varphi : T(X) \hookrightarrow T(M)$$

which preserves the Hodge and lattice structures.

$$\Downarrow$$

We have the short exact sequence

$$0 \longrightarrow T(X) \xrightarrow{\varphi} T(M) \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

$$\Downarrow$$

Apply $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ to get

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathrm{Br}(M) \xrightarrow{\varphi^\vee} \mathrm{Br}(X) \longrightarrow 0.$$

Căldăraru: A special generator $\alpha \in \mathrm{Br}(M)$ of the kernel of $\varphi^\vee$ is the obstruction to the existence of a universal family on M.

Theorem. (Căldăraru) Let X be a K3 surface and let M be a coarse moduli space of stable sheaves on X as above. Then

(i) $\mathrm{D^b}(X) \cong \mathrm{D^b}(M, \alpha^{-1})$ **(via the twisted universal/quasi-universal family);**

(ii) **there is a Hodge isometry**

$$T(X) \cong T(M, \alpha^{-1}).$$

Such a result fits perfectly with the Derived Torelli Theorem.

Căldăraru stated the following conjecture:

Conjecture. (Căldăraru) Let (X, α) and (Y, β) be twisted K3 surfaces. Then the following two conditions are equivalent:

(i) $\mathsf{D}^{\mathsf{b}}(X, \alpha) \cong \mathsf{D}^{\mathsf{b}}(Y, \beta)$**;**

(ii) **there exists a Hodge isometry** $T(X, \alpha) \cong T(Y, \beta)$**.**

$$\Downarrow$$

Evidence: Work of Donagi and Pantev about elliptic fibrations.

We constructed a twisted K3 surface (X, α) such that

$$T(X, \alpha) \cong T(X, \alpha^2),$$

but the two twisted Hodge structures $\widetilde{H}(X, B, \mathbb{Z})$ and $\widetilde{H}(X, 2B, \mathbb{Z})$ are not Hodge isometric.

$$\Downarrow$$

Due to the Twisted Derived Torelli Theorem, there is no twisted Fourier-Mukai transform

$$\mathsf{D}^{\mathsf{b}}(X, \alpha) \cong \mathsf{D}^{\mathsf{b}}(X, \alpha^2).$$

$$\Downarrow$$

One implication in Căldăraru's conjecture is false.

ORIENTATION

The orientation preserving requirement is missing in item (i) of the Twisted Derived Torelli Theorem.

On the other hand we know the following result:

Proposition. **(H.-S.)** **Any known twisted or untwisted equivalence is orientation preserving.**

This encourages us to formulate the following:

Conjecture. **Let X and X' be two algebraic K3 surfaces with B-fields B and B'. If**

$$\Phi : \mathrm{D}^{\mathrm{b}}(X, \alpha_B) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha_{B'})$$

is a Fourier-Mukai transform, then

$$\Phi_*^{B,B'} : \widetilde{H}(X, B, \mathbb{Z}) \to \widetilde{H}(X', B', \mathbb{Z})$$

preserves the natural orientation of the four positive directions.

Recently, we established the previous conjecture for generic twisted K3 surfaces. In this case, indeed, we get a precise description of the group of autoequivalences of the twisted derived category.

Theorem. (H.-M.-S.) For a generic twisted K3 surface (X, α_B) there exists a short exact sequence

$$1 \to \mathbb{Z}[2] \to \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X, \alpha_B)) \xrightarrow{\varphi} \mathrm{O}_+ \to 1,$$

where O_+ is the group of the Hodge isometries of $\widetilde{H}(X, B, \mathbb{Z})$ preserving the orientation.

More precisely, in collaboration with Huybrechts and Macrì we proved *Bridgeland's Conjecture* (involving the space parametrizing stability conditions) for generic twisted K3 surfaces.

CONSEQUENCES

Number of FM-partners: Classical problem in the untwisted case.

Proposition. (H.-S.) **Any twisted K3 surface (X, α) admits only finitely many Fourier-Mukai partners up to isomorphisms.**

$$\text{Untwisted} \neq \text{Twisted}$$
$$\Downarrow$$

Proposition. (H.-S.) **For any positive integer N there exist N pairwise non-isomorphic twisted K3 surfaces**

$$(X_1, \alpha_1), \ldots, (X_N, \alpha_N)$$

of Picard number 20 **and such that the twisted derived categories** $\mathrm{D}^{\mathrm{b}}(X_i, \alpha_i)$**, are all Fourier-Mukai equivalent.**

KUMMER SURFACES

Hosono, Lian, Oguiso and Yau

$\Downarrow$

(A) *given two abelian surfaces* A *and* B*,*

$$\mathsf{D}^{\mathsf{b}}(A) \cong \mathsf{D}^{\mathsf{b}}(B)$$

if and only if

$$\mathsf{D}^{\mathsf{b}}(\mathsf{Km}(A)) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{Km}(B)).$$

The argument: they notice that, due to the geometric construction of the Kummer surfaces Km(A) and Km(B), the transcendental lattices of A and B are Hodge isometric if and only if the transcendental lattices of Km(A) and Km(B) are Hodge isometric. Then, they apply the Derived Torelli Theorem.

Evident that (A) can be reformulated in the following way:

(B) *given two abelian surfaces A and B,*

$$\mathrm{D^b}(\mathrm{Km}(A)) \cong \mathrm{D^b}(\mathrm{Km}(B))$$

if and only if there exists a Hodge isometry between the transcendental lattices of A and B.

Due to a result of Mukai, (A) and (B) are equivalent to the following statement:

(C) *given two abelian surfaces A and B,*

$$\mathrm{D^b}(A) \cong \mathrm{D^b}(B)$$

if and only if

$$\mathrm{Km}(A) \cong \mathrm{Km}(B).$$

We want to apply the Twisted Derived Torelli Theorem to generalize (B) in the twisted case.

Definition. Let (X_1, α_1) and (X_2, α_2) be twisted K3 or abelian surfaces.

(i) They are *D-equivalent* if there exists a twisted Fourier-Mukai transform

$$\Phi : \mathrm{D}^{\mathrm{b}}(X_1, \alpha_1) \to \mathrm{D}^{\mathrm{b}}(X_2, \alpha_2).$$

(ii) They are *T-equivalent* if there exist $B_i \in H^2(X_i, \mathbb{Q})$ such that $\alpha_i = \alpha_{B_i}$ and a Hodge isometry

$$\varphi : T(X_1, B_1) \to T(X_2, B_2).$$

Theorem. (S.) Let A_1 and A_2 be abelian surfaces. Then the following two conditions are equivalent:

(i) there exist $\alpha_1 \in \mathrm{Br}(\mathrm{Km}(A_1))$ and $\alpha_2 \in \mathrm{Br}(\mathrm{Km}(A_2))$ such that $(\mathrm{Km}(A_1), \alpha_1)$ and $(\mathrm{Km}(A_2), \alpha_2)$ are D-equivalent;

(ii) there exist $\beta_1 \in \mathrm{Br}(A_1)$ and $\beta_2 \in \mathrm{Br}(A_2)$ such that (A_1, β_1) and (A_2, β_2) are T-equivalent.

Furthermore, if one of these two equivalent conditions holds true, then A_1 and A_2 are isogenous.

Remarks

(i) If $\alpha_j \in \mathsf{Br}(\mathsf{Km}(A_j))$ is non-trivial for any $j \in \{1,2\}$, then the existence of an equivalence

$$\mathsf{D}^{\mathsf{b}}(\mathsf{Km}(A_1),\alpha_1) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{Km}(A_2),\alpha_2)$$

does not imply that $\mathsf{Km}(A_1) \cong \mathsf{Km}(A_2)$. This is one of the main differences with the untwisted case.

(ii) We would expect (ii) in the previous theorem to be equivalent to the existence of a Fourier-Mukai transform

$$\mathsf{D}^{\mathsf{b}}(A_1,\beta_1) \cong \mathsf{D}^{\mathsf{b}}(A_2,\beta_2),$$

where $\beta_i \in \mathsf{Br}(A_i)$. Unfortunately this is not the case.

(iii) One can prove directly that if two abelian varieties A_1 and A_2 are such that $\mathsf{D}^{\mathsf{b}}(A_1) \cong \mathsf{D}^{\mathsf{b}}(A_2)$) then

$$A_1 \times \widehat{A_1} \cong A_2 \times \widehat{A_2}$$

(due Orlov's results) and A_1 is isogenous to A_2.

On the other hand, one cannot expect that if A_1 and A_2 are isogenous abelian varieties then $\mathsf{D}^{\mathsf{b}}(A_1) \cong \mathsf{D}^{\mathsf{b}}(A_2)$: elliptic curves!

The number of Kummer structures

By the previous theorem, we have a surjective map

$$\Psi : \{\text{Tw ab surf}\}/\cong \longrightarrow \{\text{Tw Kum surf}\}/\cong .$$

The main result of Hosono, Lian, Oguiso and Yau proves that the preimage of $[(\mathrm{Km}(A), 1)]$ is finite, for any abelian surface A and $1 \in \mathrm{Br}(A)$ the trivial class.

$+$

The cardinality of the preimages of Ψ can be arbitrarily large.

$\Downarrow$

This answers an old question of Shioda.

This picture can be completely generalized to the twisted case.

Proposition. (S.) (i) For any twisted Kummer surface $(\mathrm{Km}(A), \alpha)$**, the preimage**

$$\Psi^{-1}([(\mathrm{Km}(A), \alpha)])$$

is finite.

(ii) For positive integers N **and** n**, there exists a twisted Kummer surface** $(\mathrm{Km}(A), \alpha)$ **with** α **of order** n **in** $\mathrm{Br}(\mathrm{Km}(A))$ **and such that**

$$|\Psi^{-1}([(\mathrm{Km}(A), \alpha)])| \geq N.$$

$$\Downarrow$$

On a twisted K3 surface we can put just a finite number of non-isomorphic *twisted Kummer structures*.