# SHARP COHOMOLOGY

## Sharp cohomology theories: a road map

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September 17, 2011

# Paradigma

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$H^1$ is just an avatar of Pic !

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Actually, this subspace is clearly independent from the chosen compactification !

# $H^1$ versus $H^1_\sharp$ of schemes

For any scheme $X$ we get a smooth hypercovering $\widetilde{X}$ and we can see that

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That is $T_{Hodge}, T_{\mathrm{dR}}, T_\ell, T_{\mathrm{crys}}$ applied to the 1-motive $\mathrm{Pic}^+(X)$ yield $H^1(X, \mathbb{Z}(1)), H^1(X_{\text{ét}}, \mathbb{Z}_\ell(1)), H^1_{\mathrm{dR}}(X), H^1_{\mathrm{crys}}(X)$.

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- For $X$ proper we discover that $\mathrm{Pic}^+(X)$ is just the semiabelian quotient of $\mathrm{Pic}^0(X)$. Thus we may regard $H^1(X)$ as a quotient of a refined $H^1_\sharp(X)$.
- For $X$ smooth just consider the formal completion at zero $\mathrm{Inf}^0_Y(\overline{X})$ of

$$\ker H^1(\overline{X}, \mathcal{O}_{\overline{X}}) \to H^1(X, \mathcal{O}_X)$$

Thus $\mathrm{Div}^0_Y(\overline{X}) \times \mathrm{Inf}^0_Y(\overline{X})$ is a formal group and

$$\mathrm{Pic}^+_a(X) := [\mathrm{Div}^0_Y(\overline{X}) \times \mathrm{Inf}^0_Y(\overline{X}) \to \mathrm{Pic}^0(\overline{X})] \rightsquigarrow H^1_\sharp(X)$$

so that $H^1(X)$ is a subobject of $H^1_\sharp(X)$.

# Sharp cohomology

The sharp (singular, de Rham, etc.) cohomology

$$(X, Z) \rightsquigarrow H^*_\sharp(X, Z)$$

is at least a contravariant functor from pairs $(X, Z)$ with $Z \subseteq X$ closed to formal groups (formal Hodge structures, etc.) which is provided with a long exact sequence of the triples.

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- $f^* : H^*_\sharp(X, Z) \to H^*_\sharp(X', Z')$ for a morphism $f : X' \to X$ such that $f|_{Z'} : Z' \to Z$ and
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By the way, ordinary singular cohomology Hodge structure

$$H^*(X, Z) = H^*_\sharp(X, Z)_{\text{ét}}$$

is the étale structure associated to the formal Hodge structure.

# Formal Hodge structures of level $\leq n$

A formal Hodge structure of level $\leq n$ is given by

- $H := H_{\mathbb{Z}} \times H^0$ a formal group over $\mathbb{C}$ such that $H_{\mathbb{Z}}$ is the underlying group of a level $\leq n$ mixed Hodge structure $H_{\text{ét}} = (H_{\mathbb{Z}}, W_*, F^*_{Hodge})$,
- $V := V_n \to \cdots \to V_1$ a diagram given by composable linear mappings of finite dimensional $\mathbb{C}$-vector spaces,
- an augmentation map $v : H \to V$
- a subdiagram $V^0 \subset V$ such that $V/V^0 \cong H_{\mathbb{C}}/F_{Hodge}$ yielding a commutative diagram

$$\begin{array}{ccccc} H_{\mathbb{Z}} & \xrightarrow{c} & H_{\mathbb{C}}/F^n_{Hodge} & \twoheadrightarrow \cdots \twoheadrightarrow & H_{\mathbb{C}}/F^1_{Hodge} \\ \downarrow & & \uparrow & & \uparrow \\ H & \xrightarrow{v} & V_n & \to \cdots \longrightarrow & V_1 \end{array}$$

# Formal Hodge structures

FHS is the abelian category obtained by taking $\mathrm{Colim}_n \mathrm{FHS}_n$ where $\mathrm{FHS}_n$ are level $\leq n$ formal Hodge structures; note that we have a forgetful (faithful exact) functor $(H, V) \rightsquigarrow H_{\mathbb{Z}} \times H^0 \times \widehat{V}^0$ from FHS to $\mathrm{Fgrp}_{\mathbb{C}}$ formal groups

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$\mathrm{FHS}_{\text{ét}}$ is the full subcategory of FHS of étale structures, *i.e.*, for $(H, V) \in \mathrm{FHS}_n$ let

$$(H, V)_{\text{ét}} := (H_{\mathbb{Z}}, V/V^0) \cong (H_{\mathbb{Z}}, H_{\mathbb{C}}/F_{Hodge})$$

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Denote $(H, V)_{\times} := (H, V/V^0)$. We have a canonical extension

$$0 \to (H, V)_{\text{ét}} \to (H, V)_{\times} \to (H^0, 0) \to 0$$

# Sharp versus ordinary cohomologies

In this framework, for $(X, Z)$ with $\dim X = n$ over $\mathbb{C}$ we may seek for $H^*_\sharp(X, Z) = (H^*(X, Z) \times H^0, V) \in \mathrm{FHS}_n$ along with the canonical extension

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For example, we have that $H^1_\sharp(X)$, $H^1_{\sharp-\mathrm{dR}}(X)$, etc. is sitting in an extension

$$0 \to H^1(X) \to H^1_\sharp(X)/V(\mathrm{Pic}\,) \to V(\mathrm{Alb}\,)^\vee \to 0$$

where

- $V(\mathrm{Pic}\,) :=$ the Lie algebra $V^0$ of the vector group given by the maximal additive subgroup of $\mathrm{Pic}^0$
- $V(\mathrm{Alb}\,)^\vee :=$ the connected formal group $H^0 = \mathrm{Inf}$ whose Lie algebra is just dual of the maximal additive subgroup of Faltings-Wüstholz Alb

# Special formal Hodge structures

Note that we also have that $(H^0, V)$ is a connected structure associated to any $(H, V)$ but it is not a substructure, in general.

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FHS$^s$ are the special structures, *i.e.*, say that $(H, V)$ is special if $(H^0, V^0) := (H, V)^0$ is a substructure of $(H, V)$ or, equivalently, $(H, V)_{\text{ét}}$ is a quotient of $(H, V)$, so that we have an extension

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This is the largest subcategory of FHS such that MHS = FHS$_{\text{ét}}$ into FHS$^s$ has a left adjoint and FHS$^0$= VSP into FHS$^s$ has a right adjoint.

# Formal Hodge structures versus 1-motives

Deligne's Hodge realization for 1-motives with torsion can be further extended to an equivalence with graded polarizable (twisted) formal Hodge structures of level $\leq 1$

$$T_{\oint} : \text{Laumon 1-motives} \xrightarrow{\simeq} \text{FHS}_1^p$$

where $T_{\oint}([F \xrightarrow{u} G]) := (T_{\oint}(F), \text{Lie}\,(G))$ where $T_{\oint}(F)_{\text{ét}}$ is the underlying abelian group to $T_{Hodge}([F \xrightarrow{u} G]_{\text{ét}})$ and $T_{\oint}(F)^0 = F^0$.

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$$\begin{array}{ccc} \text{Deligne 1-motives} & \xrightarrow{T_{Hodge}} & \text{MHS}_1^p \\ \uparrow\downarrow & & \uparrow\downarrow \\ \text{Laumon 1-motives} & \xrightarrow{T_{\oint}} & \text{FHS}_1^p \end{array}$$

and

$$H^1_{\sharp}(X) := T_{\oint}(\text{Pic}_a^+(X))$$

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For $(H, V) \in \mathrm{FHS}_1$, similarly, we get the sharp envelope $(H, V)^\sharp \in \mathrm{FHS}_1$. Note that if $(H, V)$ is étale, *i.e.*, $H^0 = V^0 = 0$, we get $(H, V)^\sharp \cong (H_\mathbb{Z}, H_\mathbb{C}/F^0_{Hodge})^\sharp = (H_\mathbb{Z}, H_\mathbb{C})$.

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Actually

$$T_\oint(M)^\sharp \cong T_\oint(M^\sharp) = (T_\oint(F), T_\sharp(M))$$

This is a sharp version of the de Rham comparison theorem.

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Example: the Cartier dual of $M = [\widehat{A} \to A]$ for an abelian variety $A$ is the universal $\mathbb{G}_a$-extension $\mathrm{Pic}^{0,\natural}(A)$ of the dual $\mathrm{Pic}^0(A)$.

# Nori representation of sharp cohomology

For $k \hookrightarrow \mathbb{C}$ let ${}^{D}(Sch_k)^{op}$ be the following graph: objects are triples $(X, Y, i)$ where $X \in Sch_k$ and $Y \subseteq X$ is closed and $i$ is an integer, the arrows are as follows

a) $f^{op} : (X', Y', i) \to (X, Y, i)$ for any morphism $f : X \to X'$ such that $f\mid_Y : Y \to Y'$ and

b) $\delta^{op} : (Y, Z, i-1) \to (X, Y, i)$ for any $Z \subseteq Y \subseteq X$ closed in $X$.

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b) $\delta^{op} : (Y, Z, i-1) \to (X, Y, i)$ for any $Z \subseteq Y \subseteq X$ closed in $X$.

We get a canonical representation

$$H^*_\sharp : {}^{D}(Sch_k)^{op} \to \overrightarrow{Fgrp}$$

given by $(X, Y, i) \rightsquigarrow H^i_\sharp(X, Y)$, forgetting the (formal) Hodge structure of the singular sharp cohomology of the pair $(X_{\mathrm{an}}, Y_{\mathrm{an}})$, *i.e.*, by the contravariant functoriality and the long exact sequence of the triple.

# Sharp motives via Nori's theorem

Given a representation $T : D \to \mathcal{A}$ of any (small) graph $D$ into a suitable abelian category $\mathcal{A}$ there exists an abelian category $\mathcal{C}(T)$, a forgetful (faithful, exact) functor $F_T : \mathcal{C}(T) \to \mathcal{A}$ and $\widetilde{T} : D \to \mathcal{C}(T)$ such that $F_T \circ \widetilde{T} = T$ universally, *i.e.*, $\mathcal{C}(T)$ is initial (up to isomorphisms of functors) with respect to all these factorizations of the representation $T$.

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For $T = H^*_\sharp$ and $\mathcal{A} = \overrightarrow{Fgrp}$ call effective cohomological sharp mixed motives the resulting abelian categories

$$\mathrm{ECM}^\sharp := \mathcal{C}(H^*_\sharp)$$

Note that this is just a speculation!

# Scholium

Existence of the sharp cohomology $H^*_\sharp$ functor such that

$$H^*_\sharp(X, Y) \to H^*_\sharp(X, Z) \to H^*_\sharp(Y, Z) \to H^{*+1}_\sharp(X, Y)$$

is exact for $Z \subseteq Y \subseteq X$ closed in $X$

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$$\begin{array}{ccc} \mathrm{ECM}^\sharp & \xrightarrow{R_\oint} & \mathrm{FHS}^p \\ \uparrow & & \uparrow \\ \mathrm{ECM} & \xrightarrow{R_{Hodge}} & \mathrm{MHS}^p \end{array}$$

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such that

$$\begin{array}{ccc} \text{Laumon 1-motives} & \longrightarrow & \mathrm{ECM}^\sharp \\ \uparrow\downarrow & & \uparrow\downarrow \\ \text{Deligne 1-motives} & \longrightarrow & \mathrm{ECM} \end{array}$$

# Thanks!

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