A Twisted Derived Torelli Theorem for K3 Surfaces

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Based on (math.AG/0602399) and on joint works with A. Canonaco (math.AG/0605229), D. Huybrechts (math.AG/0409030, math.AG/0411541) and D. Huybrechts-E. Macrì (math.AG/0608430)
Brauer groups

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If $X$ is a curve, then $H^2(X, \mathcal{O}_X) = H^3(X, \mathbb{Z}) = 0$. Hence $\text{Br}(X) = \{0\}$
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$$\text{Br}(X) \cong \text{Hom}(T(X), \mathbb{Q}/\mathbb{Z}).$$
Due to the previous remark, for any $\alpha \in \text{Br}(X)$ we put

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Any $B \in H^2(X, \mathbb{Q})$ is called B-field.
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**Definition**

A pair $(X, \alpha)$ where $X$ is a smooth projective variety and $\alpha \in \text{Br}(X)$ is a **twisted variety**.
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A pair \((X, \alpha)\) where \(X\) is a smooth projective variety and \(\alpha \in \text{Br}(X)\) is a twisted variety.

Represent \(\alpha \in \text{Br}(X)\) as a Čech 2-cocycle

\[
\{ \alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*) \}
\]

on an analytic open cover \(X = \bigcup_{i \in I} U_i\).
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We get **left and right derived functors**.

All “geometric functors” can be derived.
Why twists?

There are two order of problems which requires twists.

**Mirror Symmetry (Kontsevich)**
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Moduli spaces (Mukai)

If $X$ is a K3 surface and $M$ is a fine moduli space of stable sheaves on $X$ with suitable properties, then $M$ is a K3 surface. There exists an equivalence $\Phi : D^b(X) \to D^b(M)$ induced by the universal family (Mukai). There is a Hodge isometry $T(X) \cong T(M)$ of the transcendental lattices.
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  \[0 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow \text{Br}(M) \xrightarrow{\varphi^\vee} \text{Br}(X) \rightarrow 0.\]
Căldăraru’s results

The obstruction to the existence of a universal family on $M$ is the special generator $\alpha \in \text{Br}(M)$ of the kernel of $\phi$.

Theorem

1. $\mathcal{D}b(X) \sim = \mathcal{D}b(M, \alpha - 1)(\text{via the twisted universal/quasi-universal family});$

2. There is a Hodge isometry $T(X) \sim = T(M, \alpha - 1).$
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Conjecture

Let $(X, \alpha)$ and $(Y, \beta)$ be twisted K3 surfaces. Then the following two conditions are equivalent:
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Evidence: Work of Donagi and Pantev about elliptic fibrations.
Fourier-Mukai functors
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**Definition**

$F: \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is of **Fourier-Mukai type** if there exists $\mathcal{E} \in \mathcal{D}^b(X \times Y)$ and an isomorphism of functors

$$F \cong \mathcal{R}p^*_L(\mathcal{E} \otimes q^*(-)),$$

where $p: X \times Y \to Y$ and $q: X \times Y \to X$ are the natural projections.
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where $\rho : X \times Y \to Y$ and $q : X \times Y \to X$ are the natural projections.

The complex $\mathcal{E}$ is called the **kernel** of $F$ and a Fourier-Mukai functor with kernel $\mathcal{E}$ is denoted by $\Phi_\mathcal{E}$. 
Orlov’s result

Theorem (Orlov)
Any exact functor $F: \text{D}^b(X) \to \text{D}^b(Y)$ which is fully faithful admits a left adjoint is a Fourier-Mukai functor.

Remark (Bondal, Van den Bergh)
Item (2) is automatic!
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Twisted case

Question

Are all equivalences between the twisted derived categories of smooth projective varieties of Fourier-Mukai type?

This is known in some geometric cases involving K3 surfaces:

- moduli spaces of stable sheaves on K3 surfaces (C˘ald˘araru);
- K3 surfaces with large Picard number (H.-S.).
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The main theorem

Theorem. (C.-S.)

Let \((X, \alpha)\) and \((Y, \beta)\) be twisted varieties. Let \(F: \text{Db}(X, \alpha) \to \text{Db}(Y, \beta)\) be an exact functor such that, for any \(F, G \in \text{Coh}(X, \alpha)\),

\[
\text{Hom}_{\text{Db}(Y, \beta)}(F(F), F(G)[j]) = 0 \quad \text{if} \quad j < 0.
\]

Then there exist \(E \in \text{Db}(X \times Y, \alpha^{-1} \boxtimes \beta)\) and an isomorphism of functors

\[
F \sim \Phi^E.
\]

Moreover, \(E\) is uniquely determined up to isomorphism.
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be an exact functor.

Moreover, \(E\) is uniquely determined up to isomorphism.
The main theorem

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Let \((X, \alpha)\) and \((Y, \beta)\) be twisted varieties. Then there exists an isomorphism \(f : X \cong Y\) such that \(f^*(\beta) = \alpha\) if and only if there exists an exact equivalence \(\text{Coh}(X, \alpha) \cong \text{Coh}(Y, \beta)\).
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Theorem (Torelli Theorem)

Let $X$ and $Y$ be K3 surfaces. Suppose that there exists a Hodge isometry $g: H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ which maps the class of an ample line bundle on $X$ into the ample cone of $Y$. Then there exists a unique isomorphism $f: X \cong Y$ such that $f^* = g$.
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Lattice theory + Hodge structures + ample cone
Derived case

Let $X$ and $Y$ be K3 surfaces. Then the following conditions are equivalent:

1. $\mathbb{D}b(X) \cong \mathbb{D}b(Y)$;
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3. there exists a Hodge isometry $g: T(X) \rightarrow T(Y)$;
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Lattice theory + Hodge structures
Twisted derived case

Let $X$ and $X'$ be two projective K3 surfaces endowed with $B$-fields $B \in H^2(X, \mathbb{Q})$ and $B' \in H^2(X', \mathbb{Q})$.

1. If $\Phi : D_b(X, \alpha B) \sim \cong D_b(X', \alpha B')$ is an equivalence, then there exists a naturally defined Hodge isometry $\Phi_{B, B'}^* : \tilde{H}(X, B, Z) \sim \cong \tilde{H}(X', B', Z)$.

2. Suppose there exists a Hodge isometry $g : \tilde{H}(X, B, Z) \sim \cong \tilde{H}(X', B', Z)$ that preserves the natural orientation of the four positive directions. Then there exists an equivalence $\Phi : D_b(X, \alpha B) \sim \cong D_b(X', \alpha B')$ such that $\Phi_{B, B'}^* = g$.

There is something missing!
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Using the cup product, we get the Mukai pairing on $H^\ast(X,Z)$:
$$\langle \alpha, \beta \rangle := -\alpha_1 \cdot \beta_3 + \alpha_2 \cdot \beta_2 - \alpha_3 \cdot \beta_1,$$
for every $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ in $H^\ast(X,Z)$.

$H^\ast(X,Z)$ endowed with the Mukai pairing is called the Mukai lattice and we write $\tilde{H}(X,Z)$ for it.
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**Definition**

Let $X$ be a K3 surface with a B-field $B \in H^2(X, \mathbb{Q})$. We denote by $\tilde{H}(X, B, \mathbb{Z})$ the weight-two Hodge structure on $H^*(X, \mathbb{Z})$ with

$$\tilde{H}^{2,0}(X, B) := \exp(B) \left( H^{2,0}(X) \right)$$

and $\tilde{H}^{1,1}(X, B)$ its orthogonal complement with respect to the Mukai pairing.
Let $X$ be a K3 surface, $\sigma_X$ be a generator of $H^2(X)$, and $\omega$ be a Kähler class. Then $\langle \text{Re}(\sigma_X), \text{Im}(\sigma_X), 1 - \omega^2/2, \omega \rangle$ is a positive four-space in $\tilde{H}(X, \mathbb{R})$.

Remark: It comes, by the choice of the basis, with a natural orientation.

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The orientation preserving requirement is missing in item (i) of the Twisted Derived Torelli Theorem.

Proposition (H.-S.)

Any known twisted or untwisted equivalence is orientation preserving.

Conjecture

Let \( X \) and \( X' \) be two algebraic K3 surfaces with B-fields \( B \) and \( B' \).

If \( \Phi : \text{Db}(X, \alpha_B) \cong \text{Db}(X', \alpha_{B'}) \) is a Fourier-Mukai transform, then \( \Phi_{B, B'}^* : \tilde{H}(X, B, Z) \to \tilde{H}(X', B', Z) \) preserves the natural orientation of the four positive directions.
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Theorem (H.-M.-S.)

For a generic twisted K3 surface \((X, \alpha_B)\) there exists a short exact sequence

\[
1 \rightarrow \mathbb{Z}[2] \rightarrow \mathrm{Aut}(D^b(X, \alpha_B)) \rightarrow \mathcal{O}_+ \rightarrow 1,
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where \(\mathcal{O}_+\) is the group of the Hodge isometries of \(\tilde{H}(X, B, \mathbb{Z})\) preserving the orientation.

We proved Bridgeland's Conjecture for generic twisted K3 surfaces.
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Căldăraru’s conjecture is false
Lemma

If $\Phi : D^b(X, \alpha) \cong D^b(X', \alpha')$ is an equivalence, then there is a Hodge isometry $T(X, \alpha) \cong T(X', \alpha')$. 

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- Take $(X, \alpha)$ such that $T(X, \alpha) \cong T(X, \alpha^2)$ but $\tilde{H}(X, B, \mathbb{Z}) \cong \tilde{H}(X, 2B, \mathbb{Z})$. 
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- No twisted Fourier-Mukai transforms $D^b(X, \alpha) \cong D^b(X, \alpha^2)$. 

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- No twisted Fourier-Mukai transforms $D^b(X, \alpha) \cong D^b(X, \alpha^2)$.
- One implication in Căldăraru’s conjecture is false.
Proposition (H.-S.)

Any twisted K3 surface \((X, \alpha)\) admits only finitely many Fourier-Mukai partners up to isomorphisms.

Proposition (H.-S.)

For any positive integer \(N\) there exist \(N\) pairwise non-isomorphic twisted K3 surfaces \((X_1, \alpha_1), \ldots, (X_N, \alpha_N)\) of Picard number 20 and such that the twisted derived categories \(D^b((X_i, \alpha_i))\) are all Fourier-Mukai equivalent.
Number of Fourier-Mukai partners

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Any twisted K3 surface $(X, \alpha)$ admits only finitely many Fourier-Mukai partners up to isomorphisms.

Untwisted $\neq$ Twisted!!

Proposition (H.-S.)
For any positive integer $N$ there exist $N$ pairwise non-isomorphic twisted K3 surfaces

$$(X_1, \alpha_1), \ldots, (X_N, \alpha_N)$$

of Picard number 20 and such that the twisted derived categories $\mathcal{D}^b(X_i, \alpha_i)$, are all Fourier-Mukai equivalent.
The untwisted case: HLOY

Given two abelian surfaces $A$ and $B$, $D_b(A) \cong D_b(B)$ if and only if $D_b(Km(A)) \cong D_b(Km(B))$. The argument: they notice that, due to the geometric construction of the Kummer surfaces $Km(A)$ and $Km(B)$, the transcendental lattices of $A$ and $B$ are Hodge isometric if and only if the transcendental lattices of $Km(A)$ and $Km(B)$ are Hodge isometric. Then, they apply the Derived Torelli Theorem.
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The twisted case

Definition

Let \((X_1, \alpha_1)\) and \((X_2, \alpha_2)\) be twisted K3 or abelian surfaces. They are \(D\)-equivalent if there exists a twisted Fourier-Mukai transform \(\Phi:\mathcal{D}(X_1, \alpha_1) \rightarrow \mathcal{D}(X_2, \alpha_2)\).

They are \(T\)-equivalent if there exist \(B_i \in H_2(X_i, \mathbb{Q})\) such that \(\alpha_i = \alpha_{B_i}\) and a Hodge isometry \(\phi: T(X_1, \alpha_{B_1}) \rightarrow T(X_2, \alpha_{B_2})\).
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[Theorem (S.)]

Let $A_1$ and $A_2$ be abelian surfaces. Then the following two conditions are equivalent:

1. There exist $\alpha_1 \in \text{Br}(Km(A_1))$ and $\alpha_2 \in \text{Br}(Km(A_2))$ such that $(Km(A_1), \alpha_1)$ and $(Km(A_2), \alpha_2)$ are $D$-equivalent;

2. There exist $\beta_1 \in \text{Br}(A_1)$ and $\beta_2 \in \text{Br}(A_2)$ such that $(A_1, \beta_1)$ and $(A_2, \beta_2)$ are $T$-equivalent.

Furthermore, if one of these two equivalent conditions holds true, then $A_1$ and $A_2$ are isogenous.

Analogue of the second statement!

There are no twisted analogues of the first and third statement!
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By the previous theorem, we have a surjective map $\Psi: \{\text{Tw ab surf}\}/\sim \rightarrow \{\text{Tw Kum surf}\}/\sim$. The main result of Hosono, Lian, Oguiso and Yau proves that the preimage of $[\left(\text{Km}(A), 1\right)]$ is finite, for any abelian surface $A$ and $1 \in \text{Br}(A)$ the trivial class. The cardinality of the preimages of $\Psi$ can be arbitrarily large. This answers an old question of Shioda.
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This picture can be completely generalized to the twisted case.

Proposition (S.)

(i) For any twisted Kummer surface $(\text{Km}(A), \alpha)$, the preimage $\Psi^{-1}(\{(\text{Km}(A), \alpha)\})$ is finite.

(ii) For positive integers $N$ and $n$, there exists a twisted Kummer surface $(\text{Km}(A), \alpha)$ with $\alpha$ of order $n$ in $\text{Br}(\text{Km}(A))$ such that $|\Psi^{-1}(\{(\text{Km}(A), \alpha)\})| \geq N$.

On a twisted K3 surface we can put just a finite number of non-isomorphic twisted Kummer structures.
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